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HYERS-ULAM STABILITY OF BABBAGE EQUATION

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Abstract. In this paper, we study the Hyers-Ulam stability of the classical iterative functional equation $f^n(x) = x$, the Babbage equation, using strictly monotonic approximate solutions on a real interval.

1. Introduction

In the eighteen century, Charles Babbage [2] was the first mathematician who investigated the existence of solutions $f : X \to X$ of the iterative functional equation

$$f^n(x) = x \text{ for all } x \in X, \tag{1}$$

where X is a non-empty set, n is a natural number, $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in X$. Equation (1) is named after him as the *Babbage equation*. The solutions of (1) are called periodic functions or nth iterative roots of the identity function. It is known that every continuous solution f of (1) on a real interval I is either the identity function ($f(x) = x$ for all $x \in I$) or a strictly decreasing involution ($f^2(x) = x$ for all $x \in I$) and n is even (see [6, Section 11.7]). Let F be a self-map on I. The following is a generalized nonlinear iterative equation of (1):

$$f^n(x) = F(x) \text{ for all } x \in I. \tag{2}$$

A solution f of (2) is known as an *iterative root* of F on I of order n. The existence, non-existence, and uniqueness of solutions of (2) were well studied for continuous monotone and non-monotone functions (see [6,8,9,14] and references therein).

On the other hand, in 1940, S. M. Ulam (see [12] and [3–5]) proposed a problem on the stability of Cauchy's functional equation $f(x \cdot y) = f(x) * f(y)$ between two groups during a talk before a Mathematical Colloquium at the University of Wisconsin as follows:

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Given a group $(G_1, \cdot)$ *and a metric group* $(G_2, *)$ *with metric* d *and a positive number* ϵ*, does there exist a* $\delta > 0$ *such that, if a function* $g : G_1 \to G_2$ *satisfies* $d(g(x\cdot y), g(x)*g(y)) < \delta$ *for all* $x, y \in G_1$*, then there is a function* $f : G_1 \to G_2$ *such that* $f(x \cdot y) = f(x) * f(y)$ *and* $d(f(x), g(x)) < \epsilon$ *for all* $x \in G_1$*?*

In the next year, D. H. Hyers [4] answered Ulam's problem partially when G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized and studied widely for linear and nonlinear iterative equations (see [1,3,5,11] and [13]).

As in [1], we say the equation (2) has the Hyers-Ulam stability if for $\delta > 0$ and for every $g : I \to I$ such that

$$|g^n(x) - F(x)| \leq \delta \text{ for all } x \in I, \tag{3}$$

there exists a solution $f : I \to I$ of (2), which satisfies

$$|g(x) - f(x)| \leq \varepsilon_\delta \text{ for all } x \in I,$$

where the constant $\varepsilon_\delta > 0$ depends only on δ. A function g in (3) is called the δ-approximate solution of $f^n = F$.

In [7], Li et al. discussed the stability of (2) for a class of continuous piecewise monotone functions F with $f^n = F$ on the range of F. In 2021, the results on stability of (2) were obtained for a class of continuous non-monotone functions F, which is strictly monotone in its range, and the equation has stability in its range [8]. Recently, in [10], this problem has been studied for strictly increasing continuous functions F with the assumption that either $F(x) < x$ or $F(x) > x$ for all x in the interior of I. Moreover, no study exists on the Hyers-Ulam stability of (2) for $F(x) = x$ on I.

In this paper, we study the Hyers-Ulam stability of (1) whenever g is a monotonic (increasing or decreasing) approximate solution using the monotonicity of g and properties of continuous solutions of (1).

2. Stability results

The following theorems discuss the Hyers-Ulam stability of (1) for odd and even $n \in \mathbb{N}$, respectively.

Theorem 2.1. *Let* n *be odd. Suppose* g *is a strictly increasing continuous self-map on* I *such that*

$$|g^n(x) - x| \leq \delta \quad \textit{for all } x \in I \tag{4}$$

for some constant $\delta > 0$*. Then there exists a strictly increasing solution* f *of* (1) *such that*

$$|g(x) - f(x)| \leq \delta \quad \textit{for all } x \in I.$$

Proof. Since n is odd, $f(x) = x$ is the solution of (1). Let $x \in I$. Suppose $g(x) \leq x$, as g is strictly increasing on I,

$$g^n(x) \leq g(x) \leq x.$$

From (4), since $0 \le x - g(x)$,

$$0 \le |x - g(x)| \le |x - g^n(x)| \le \delta.$$

Suppose that $g(x) \ge x$. Then $x \le g(x) \le g^n(x)$. This implies by (4),

$$0 \le |g(x) - x| \le |g^n(x) - x| \le \delta.$$

Thus

$$|g(x) - x| \le \delta \text{ for all } x \in I.$$

□

For odd n, suppose that g is a strictly decreasing homeomorphism (strictly decreasing, continuous and onto) on a bounded interval $[a, b]$ such that

$$|g^n(x) - x| \le \delta \text{ for all } x \in [a, b]$$

for some $\delta > 0$. Then

$$|g^n(a) - a| = b - a \le \delta$$

by the fact that g^n is strictly decreasing and onto. Since the identity function is the continuous solution of (1),

$$|g(x) - x| \le |g(b) - b| = b - a \le \delta \text{ for all } x \in [a, b].$$

For even n, it is known from [6, Section 11.2C] that the problem of finding solutions of $f^n(x) = x$ on I is reduced to solve the equation $f^2(x) = x$. Suppose f is a continuous decreasing function such that $f^2(x) = x$ for all $x \in I$, then f is homeomorphism. Also, f maps $I \cap (-\infty, c]$ onto $I \cap [c, \infty)$, where c is the unique fixed point of f in the interior of I. Moreover, the interval I is either open or closed.

Theorem 2.2. *Let I be an open or closed interval. Suppose that $g : I \to I$ is a strictly decreasing homeomorphism such that*

$$|g^2(x) - x| \le \delta \quad \text{for all } x \in I$$

for some constant $\delta > 0$. Then there exists a strictly decreasing homeomorphism f such that $f^2(x) = x$ for all $x \in I$ and

$$|f(x) - g(x)| \le \delta \quad \text{for all } x \in I.$$

Proof. Since g is a strictly decreasing homeomorphism, g has unique fixed c in the interior of I. Let $I_1 := I \cap (-\infty, c]$ and $I_2 := I \cap [c, \infty)$. Define $f : I \to I$ by

$$f(x) := \begin{cases} g(x), & \text{if } x \in I_1, \\ g^{-1}(x), & \text{if } x \in I_2. \end{cases} \tag{5}$$

Clearly, f is a strictly decreasing homeomorphism on I by the monotonicity and continuity of g and g^{-1}. Let $x \in I_1$. Since $g(x) \in I_2$,

$$f^2(x) = f(g(x)) = g^{-1}(g(x)) = x.$$

Now, let $x \in I_2$. Since $g^{-1}(x) \in I_1$,

$$f^2(x) = f(g^{-1}(x)) = g(g^{-1}(x)) = x.$$

This implies

$$f^2(x) = x \text{ for all } x \in I.$$

To prove the stability, let $x \in I_1$. Then $|f(x) - g(x)| = 0$ by (5). For $x \in I_2$, we have $x = g(y)$ for some $y \in I_1$. Since g is a strictly decreasing homeomorphism,

$$\begin{aligned} |f(x) - g(x)| &= |g^{-1}(x) - g(x)| \\ &= |g^{-1}(g(y)) - g(g(y))| \\ &= |y - g^2(y)| \\ &\leq \delta. \end{aligned}$$

Thus,

$$|f(x) - g(x)| \leq \delta \text{ for all } x \in I. \qquad \square$$

3. Illustrate examples

Now, we illustrate our results with the following examples.

Example 3.1. Let $g : [0, 1] \to [0, 1]$ be defined by

$$g(x) := \begin{cases} 1 - \frac{x}{2}, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{15}{16} - \frac{x}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 3(1 - x), & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Clearly, g is a strictly decreasing homeomorphism on $[0, 1]$ and $g\left(\frac{3}{4}\right) = \frac{3}{4}$ (see Figure 1). Also, we have

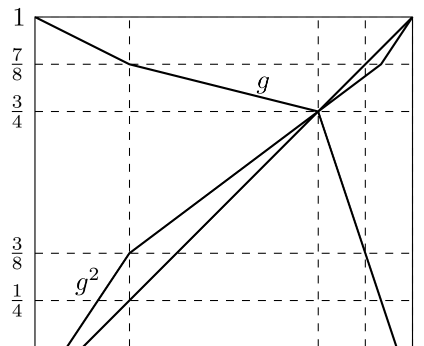

FIGURE 1. A decreasing $\frac{1}{8}$-approximate solution g of $f^2(x) = x$ on $[0, 1]$

$$g^2(x) = \begin{cases} \frac{3x}{2}, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{3x}{4} + \frac{3}{16}, & \text{if } x \in \left[\frac{1}{4}, \frac{11}{12}\right], \\ \frac{3x-1}{2}, & \text{if } x \in \left[\frac{11}{12}, 1\right], \end{cases}$$

and

$$|g^2(x) - x| \leq \left| g^2\left(\frac{1}{4}\right) - \frac{1}{4} \right| = \frac{1}{8} \text{ for all } x \in [0,1].$$

This implies that g satisfies all the assumptions of Theorem 2.2. Therefore the function f defined in (5) (see Figure 2),

$$f(x) = \begin{cases} 1 - \frac{x}{2}, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{15}{16} - \frac{x}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ \frac{15}{4} - 4x, & \text{if } x \in \left[\frac{3}{4}, \frac{7}{8}\right], \\ 2(1-x), & \text{if } x \in \left[\frac{7}{8}, 1\right], \end{cases}$$

is a continuous solution of $f^2(x) = x$ on $[0,1]$ and

$$|f(x) - g(x)| \leq \left| f\left(\frac{7}{8}\right) - g\left(\frac{7}{8}\right) \right| = \frac{1}{8} \text{ for all } x \in [0,1].$$

Example 3.2. Let $n = 3$ and consider a function $g : [0,1] \to [0,1]$ defined by

$$g(x) := \begin{cases} \frac{x}{2}, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{5x}{2} - \frac{4}{8}, & \text{if } x \in [\frac{1}{4}, \frac{1}{2}], \\ \frac{x}{2} + \frac{1}{2}, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

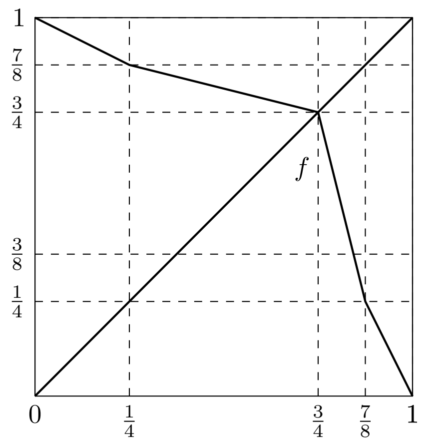

FIGURE 2. A decreasing solution f of $f^2(x) = x$ on $[0,1]$

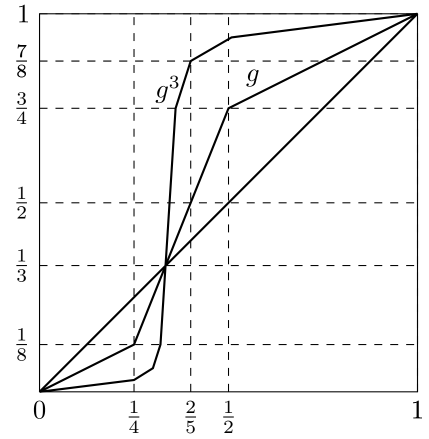

FIGURE 3. An increasing $\frac{19}{40}$-approximate solution g of $f^3(x) = x$ on $[0, 1]$

Clearly, g is continuous and strictly increasing on $[0, 1]$ (see Figure 3). Also, we have

$$g^3(x) = \begin{cases} \frac{x}{8}, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{5x-1}{8}, & \text{if } x \in [\frac{1}{4}, \frac{3}{10}], \\ \frac{25x-7}{8}, & \text{if } x \in [\frac{3}{10}, \frac{8}{25}], \\ \frac{125x-39}{8}, & \text{if } x \in [\frac{8}{25}, \frac{9}{25}], \\ \frac{25x-3}{8}, & \text{if } x \in [\frac{9}{25}, \frac{2}{5}], \\ \frac{5x+5}{8}, & \text{if } x \in [\frac{2}{5}, \frac{1}{2}], \\ \frac{x+7}{8}, & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$\left|g^3(x) - x\right| \le \left|g^3\left(\frac{2}{5}\right) - \frac{2}{5}\right| = \frac{19}{40} \text{ for all } x \in [0, 1].$$

This implies that g satisfies all the conditions of Theorem 2.1 and then

$$|g(x) - x| \le \left|g\left(\frac{1}{2}\right) - \frac{1}{2}\right| = \frac{1}{4} \le \frac{19}{40} \text{ for all } x \in [0, 1].$$

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