Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 585–593
https://doi.org/10.4134/CKMS.c230279
pISSN: 1225-1763 / eISSN: 2234-3024

ON $\eta-$GENERALIZED DERIVATIONS IN RINGS WITH JORDAN INVOLUTION

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Abstract. Let $\mathscr{K}$ be a ring. An additive map $\mathfrak{u}^\diamond \to \mathfrak{u}$ is called Jordan involution on $\mathscr{K}$ if $(\mathfrak{u}^\diamond)^\diamond = \mathfrak{u}$ and $(\mathfrak{u}\mathfrak{v}+\mathfrak{v}\mathfrak{u})^\diamond = \mathfrak{u}^\diamond\mathfrak{v}^\diamond+\mathfrak{v}^\diamond\mathfrak{u}^\diamond$ for all $\mathfrak{u},\mathfrak{v} \in \mathscr{K}$. If Θ is a (non-zero) $\eta-$generalized derivation on $\mathscr{K}$ associated with a derivation Ω on $\mathscr{K}$, then it is shown that $\Theta(\mathfrak{u}) = \gamma\mathfrak{u}$ for all $\mathfrak{u} \in \mathscr{K}$ such that $\gamma \in \Xi$ and $\gamma^2 = 1$, whenever Θ possesses $[\Theta(\mathfrak{u}), \Theta(\mathfrak{u}^\diamond)] = [\mathfrak{u}, \mathfrak{u}^\diamond]$ for all $\mathfrak{u} \in \mathscr{K}$.

1. Introduction and Preliminaries

Consider that $\mathscr{K}$, everywhere in this paper, is a prime ring with center given by $\mathscr{Z}(\mathscr{K})$. In addition, the right Martindale ring of quotients is denoted by $\mathscr{Q}$, with $\Xi = \mathscr{Z}(\mathscr{Q})$, known as the extended centroid of $\mathscr{K}$. An additive map Ω on $\mathscr{K}$ is called a derivation if $\Omega(\mathfrak{u}\mathfrak{v}) = \Omega(\mathfrak{u})\mathfrak{v} + \mathfrak{u}\Omega(\mathfrak{v})$ is true for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. The map $\Theta : \mathscr{K} \to \mathscr{K}$ is said to be a generalized derivation if there exists a derivation $\Omega : \mathscr{K} \to \mathscr{K}$ such that $\Theta(\mathfrak{u}\mathfrak{v}) = \Theta(\mathfrak{u})\mathfrak{v} + \mathfrak{u}\Omega(\mathfrak{v})$. For an element $\mathfrak{b} \in \mathscr{K}$, the inner derivation $I_\mathfrak{b} : \mathscr{K} \to \mathscr{K}$ is defined as $I_\mathfrak{b}(k_0) = k_0 I_\mathfrak{b} - I_\mathfrak{b} k_0$ for all $k_0 \in \mathscr{K}$. It is clear from the definition that every generalized derivation is associated with a derivation. If a derivation Ω is inner, the generalized derivation Θ, which is associated to f is an inner generalized (generalized inner) derivation. In [12], Koşan and Lee proposed a new type of derivation, called an $\eta-$generalized derivation. Literally, they stated that an additive map $\Theta : \mathscr{K} \to \mathscr{Q}$ is termed as a left $\eta-$generalized derivation associative with an additive mapping Ω from $\mathscr{K}$ to $\mathscr{Q}$, if $\Theta(\mathfrak{u}\mathfrak{v}) = \Theta(\mathfrak{u})\mathfrak{v} + \eta\mathfrak{u}\Omega(\mathfrak{v})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$ and $\eta \in \mathscr{Q}$.

An additive map $\mathfrak{u}^* \mapsto \mathfrak{u}$ is called an involution if $(\mathfrak{u}^*)^* = \mathfrak{u}$ and $(\mathfrak{u}\mathfrak{v})^* = \mathfrak{v}^*\mathfrak{u}^*$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. In light of the definition of an involution, here we would like to introduce the following concept of a Jordan involution, which is first discussed by Yood [22] and also generalizes the concept of involution, as

Received October 18, 2023; Accepted February 1, 2024.
2010 *Mathematics Subject Classification.* 16N60, 16W10, 16W25.
Key words and phrases. Prime ring, Jordan involution, $\eta-$generalized derivation, strong commutativity preserving (SCP) map.

Definition. An (additive) map $\diamond : \mathscr{K} \to \mathscr{K}$ is called a *Jordan involution* if $(\mathfrak{u}^\diamond)^\diamond = \mathfrak{u}$ and $(\mathfrak{u}\mathfrak{v} + \mathfrak{v}\mathfrak{u})^\diamond = \mathfrak{u}^\diamond\mathfrak{v}^\diamond + \mathfrak{v}^\diamond\mathfrak{u}^\diamond$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$.

It is obvious that every involution is a Jordan involution, but the reverse does not necessarily hold true. For instance:

Example 1.1. Let $\mathscr{K} = \left\{ \begin{pmatrix} \mathtt{k}_1 & \mathtt{k}_2 \\ \mathtt{k}_3 & \mathtt{k}_4 \end{pmatrix} \,\middle|\, \mathtt{k}_1, \mathtt{k}_2, \mathtt{k}_3, \mathtt{k}_4 \in \mathbb{C} \right\}$. Of course $\mathscr{K}$ is an algebra under usual matrix addition and matrix multiplication. Define $\diamond : \mathscr{K} \longrightarrow \mathscr{K}$ as

$$\begin{pmatrix} \mathtt{k}_1 & \mathtt{k}_2 \\ \mathtt{k}_3 & \mathtt{k}_4 \end{pmatrix}^\diamond = \begin{pmatrix} \overline{\mathtt{k}_1} & \overline{\mathtt{k}_2} \\ \overline{\mathtt{k}_3} & \overline{\mathtt{k}_4} \end{pmatrix}.$$

Obviously, $\diamond$ is a Jordan involution but not an involution.

In light of the above definition, for any ring $\mathscr{K}$ (2-torsion free) with Jordan involution $\diamond$, the following remarks are useful:

Remark 1.2. We say a Jordan involution $\diamond$ is $\diamond-$symmetric and $\diamond-$skew symmetric if $\mathfrak{u}^\diamond = \mathfrak{u}$ and $\mathfrak{u}^\diamond = -\mathfrak{u}$ for all $\mathfrak{u} \in \mathscr{K}$.

Remark 1.3. If an element $\mathfrak{u} \in \mathscr{K}$ is $\diamond-$commuting, i.e., $\mathfrak{u}\mathfrak{u}^\diamond = \mathfrak{u}^\diamond\mathfrak{u}$, then $\mathfrak{u}\mathfrak{u}^\diamond$ is $\diamond-$symmetric.

Remark 1.4. For any $\mathfrak{u} \in \mathscr{K}$, $\mathfrak{u} + \mathfrak{u}^\diamond$ and $\mathfrak{u} - \mathfrak{u}^\diamond$ are $\diamond-$ symmetric and $\diamond-$skew symmetric element of $\mathscr{K}$.

Remark 1.5. Every element of $\mathscr{K}$ can be written as sum of $\diamond-$ symmetric and $\diamond-$skew symmetric elements of $\mathscr{K}$ uniquely.

Note. We denote $J_{\mathscr{H}}$ and $J_{\mathscr{S}}$ for set of all $\diamond-$ symmetric and $\diamond-$skew symmetric elements of $\mathscr{K}$, respectively.

Remark 1.6. For any $\mathfrak{u} \in \mathscr{K}$, $(\mathfrak{u}\beta)^\diamond = -\mathfrak{u}^\diamond\beta$, where $\beta \in J_{\mathscr{S}}$.

Proof. We have

$$2(\mathfrak{u}\beta)^\diamond = (\mathfrak{u}\beta + \beta\mathfrak{u})^\diamond = -\beta\mathfrak{u}^\diamond - \mathfrak{u}^\diamond\beta = -2\mathfrak{u}^\diamond\beta.$$

Therefore $(\mathfrak{u}\beta)^\diamond = -\mathfrak{u}^\diamond\beta$. □

Similarly, one can easily deduce that

Remark 1.7. For any $\mathfrak{u} \in \mathscr{K}$, $(\mathfrak{u}\alpha)^\diamond = \mathfrak{u}^\diamond\alpha$, where $\alpha \in J_{\mathscr{H}}$.

Remark 1.8. If $\Omega(\alpha) = 0$, where $\alpha \neq 0$ is a central $\diamond-$symmetric element of $\mathscr{K}$, then $\Omega(z) = 0$ for all $z \in \mathscr{Z}(\mathscr{K})$.

2. Characterizations of $\eta-$generalized derivations having Jordan involution

The work of this section is motivated by the study of strong commutative preserving (SCP) mappings. Before presenting our results, let's recall some known definitions and related theories established by eminent ring theorists for the sake of completeness. It was Bell and Mason [4] who introduced SCP maps and they stated that for any subset S of $\mathscr{K}$, a map ξ is known as SCP on $\mathscr{S}$ if $[\xi(\mathfrak{u}), \xi(\mathfrak{v})] = [\mathfrak{u}, \mathfrak{v}]$ holds for all $\mathfrak{u}, \mathfrak{v} \in S$. Since then, many impressive literary works have been done concerning SCP on certain types of derivations, generalized derivations, automorphisms on prime and semiprime rings, see [2, 3, 7, 9, 13, 14, 16, 18, 21] and references therein.

It was Brešar et al. [6] who initiated research on additive maps in rings with involution to analyze skew-symmetric elements in prime rings. The SCP maps on skew symmetric elements of prime rings with involution were characterized by Lin and Liu [15]. The above mentioned result for non additive case was later improved by Liu and Liau [18]. Interestingly, Ali et al. [1] established the following: Let $\mathscr{K}$ be a prime ring with involution of the second kind such that $char(\mathscr{K}) \neq 2$. Let ϕ be a non-zero derivation of $\mathscr{K}$ such that $[\phi(\mathfrak{u}), \phi(\mathfrak{u}^*)] - [\mathfrak{u}, \mathfrak{u}^*] = 0$ for all $\mathfrak{u} \in \mathscr{K}$, then $\mathscr{K}$ is commutative. Recent extensions of the aforementioned result for different additive maps like generalized derivations, endomorphisms etcetera can be seen in [1, 8, 10, 20].

Inspired by the present ongoing contribution in this direction, it is unsurprising to discuss SCP $\eta-$generalized derivations on prime rings with Jordan involution. Additionally, we will elaborate the said problem in the setting of $\eta-$generalized derivations together with any mapping on prime ring.

It is essential to demonstrate the following lemmas before demonstrating the main theorem:

Note: Now onwards, $\mathscr{K}$ is a non-commutative prime ring with $char(\mathscr{K}) \neq 2$ and Jordan involution, unless otherwise stated, and Θ is a (non-zero) η-generalized derivation on $\mathscr{K}$ associated with a derivation Ω on $\mathscr{K}$.

Lemma 2.1. *$\mathscr{K}$ is commutative if and only if $[\mathfrak{u}, \mathfrak{u}^\diamond] \in \mathscr{Z}(\mathscr{K})$ for all $\mathfrak{u} \in \mathscr{K}$.*

Proof. Since $\diamond$ is additive, therefore [5, Proposition 3.1] assured that $[\mathfrak{u}, \mathfrak{u}^\diamond] = 0$ for all $\mathfrak{u} \in \mathscr{K}$. Again additivity of $\diamond$ together with [5, Theorem 3.2] gives, for all $\mathfrak{u} \in \mathscr{K}$

$$\mathfrak{u}^\diamond = \gamma\mathfrak{u} + \mu(\mathfrak{u}), \tag{1}$$

where $\gamma \in \Xi$ and μ is an additive map from $\mathscr{K}$ to Ξ. Taking $\mathfrak{u}$ as $\mathfrak{u}^\diamond$ in last expression gives

$$\mathfrak{u} = \gamma\mathfrak{u}^\diamond + \mu(\mathfrak{u}^\diamond) \tag{2}$$

for all $\mathfrak{u} \in \mathscr{K}$. Considering Lemma 2.1, we made it

$$\begin{aligned} \mathfrak{u} &= \gamma(\gamma\mathfrak{u} + \mu(\mathfrak{u})) + \mu(\mathfrak{u}^\diamond) \\ &= \gamma^2\mathfrak{u} + \gamma\mu(\mathfrak{u}) + \mu(\mathfrak{u}^\diamond) \end{aligned} \tag{3}$$

for all $\mathfrak{u} \in \mathscr{K}$. On commuting (3) with $\mathfrak{v}$, we found

$$[\mathfrak{u}, \mathfrak{v}] = \gamma^2[\mathfrak{u}, \mathfrak{v}] \tag{4}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. However, on commuting (1) with $\mathfrak{v}$, we prevail

$$[\mathfrak{u}^\diamond, \mathfrak{v}] = \gamma[\mathfrak{u}, \mathfrak{v}] \tag{5}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Further, multiplying (5) on left side by $\gamma \in \Xi$ and using (4), we get

$$\gamma[\mathfrak{u}^\diamond, \mathfrak{v}] = \gamma^2[\mathfrak{u}, \mathfrak{v}] = [\mathfrak{u}, \mathfrak{v}] \tag{6}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Now for $0 \neq \beta$, a central $\diamond-$skew symmetric element of $\mathscr{K}$, take $\mathfrak{u}$ as $\mathfrak{u}^\diamond\beta$ in (6). Application of Remark 1.6 yields

$$-\gamma[\mathfrak{u}^\diamond, \mathfrak{v}]\beta = [\mathfrak{u}, \mathfrak{v}]\beta \tag{7}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Multiply (6) by β and combine with (7), we infer that $[\mathfrak{u}, \mathfrak{v}]\beta = 0$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Since $0 \neq \beta$ is central, so we have $\mathfrak{u}\mathfrak{v} = \mathfrak{v}\mathfrak{u}$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. This gives the required result. □

Lemma 2.2. *If $[a\mathfrak{u}, a\mathfrak{u}^\diamond] - [\mathfrak{u}, \mathfrak{u}^\diamond] = 0$ for all $\mathfrak{u} \in \mathscr{K}$, then either $\mathscr{K}$ is commutative or there exists $\gamma \in \Xi$ such that $\gamma^2 = 1$.*

Proof. For any $\mathfrak{u} \in \mathscr{K}$, we have

$$[a\mathfrak{u}, a\mathfrak{u}^\diamond] - [\mathfrak{u}, \mathfrak{u}^\diamond] = 0. \tag{8}$$

It may also be written as

$$a^2[\mathfrak{u}, \mathfrak{u}^\diamond] + a[\mathfrak{u}, a]\mathfrak{u}^\diamond + a[a, \mathfrak{u}^\diamond]\mathfrak{u} - [\mathfrak{u}, \mathfrak{u}^\diamond] = 0 \tag{9}$$

for all $\mathfrak{u} \in \mathscr{K}$. Substitute $\mathfrak{u}$ by $\mathfrak{u} + \beta$ in the preceding equation, where β is a non-zero $\diamond-$skew symmetric element of $\mathscr{K}$, we get

$$a^2[\mathfrak{u}, \mathfrak{u}^\diamond] + a[\mathfrak{u}, a]\mathfrak{u}^\diamond - a[\mathfrak{u}, a]\beta + a[a, \mathfrak{u}^\diamond]\mathfrak{u} + a[a, \mathfrak{u}^\diamond]\beta - [\mathfrak{u}, \mathfrak{u}^\diamond] = 0 \tag{10}$$

for all $\mathfrak{u} \in \mathscr{K}$ and $\beta \in J_{\mathscr{S}}$. In view of (9), we have

$$a[\gamma, \mathfrak{u}]\beta + a[a, \mathfrak{u}^\diamond]\beta = a[a, \mathfrak{u} + \mathfrak{u}^\diamond]\beta = 0 \tag{11}$$

for all $\mathfrak{u} \in \mathscr{K}$ and $\beta \in J_{\mathscr{S}}$. Since $\beta \neq 0$, we have

$$a[a, \mathfrak{u} + \mathfrak{u}^\diamond] = 0 \tag{12}$$

for all $\mathfrak{u} \in \mathscr{K}$. In view of Remark 1.5, we can write $\mathfrak{u} = \alpha' + \beta'$, where α' and β' are $\diamond-$symmetric and $\diamond-$skew symmetric elements of $\mathscr{K}$, respectively. Thus

$$a[a, \alpha'] = 0. \tag{13}$$

Substitute $\beta\beta'$ for α' in the above expression, where β is a $\diamond-$symmetric element and $0 \neq \beta'$ is a central $\diamond-$skew symmetric element of $\mathscr{K}$, we get

$$a[a, \beta] = 0 \tag{14}$$

since β' is non-zero. From Remark 1.5, observe that $a[a, \alpha] + a[a, \beta] = a[a, \alpha + \beta] = a[a, 2\mathfrak{u}] = 2a[a, \mathfrak{u}] = 0$. Therefore, $a = \gamma(say) \in \Xi$, and hence $(\gamma^2 - 1)[\mathfrak{u}, \mathfrak{u}^\diamond] = 0$ for all $\mathfrak{u} \in \mathscr{K}$. Primeness of $\mathscr{K}$ leads to $\gamma^2 = 1$ or $[\mathfrak{u}, \mathfrak{u}^\diamond] = 0$ for any $\mathfrak{u} \in \mathscr{K}$. Specifically in the latter case, using Lemma 2.1, we notice that $\mathscr{K}$ is commutative. □

In a more general context, Deng and Ashraf [9] studied SCP maps as follows: Let $\mathscr{K}$ be a semiprime ring. If $\mathscr{K}$ admits a mapping ψ and a derivation Ω on $\mathscr{K}$ such that $[\psi(\mathfrak{u}), \Omega(\mathfrak{v})] = [\mathfrak{u}, \mathfrak{v}]$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$, then $\mathscr{K}$ is commutative. This result has been elaborated on different types of derivations like generalized derivations, skew derivations, etc (see [17], [16]). In this line of investigation, we establish the following result for η-generalized derivations which might be of some independent interest:

Theorem 2.3. *If a (non-zero) map ψ on $\mathscr{K}$ possesses $[\psi(\mathfrak{u}), \Theta(\mathfrak{v})] = [\mathfrak{u}, \mathfrak{v}]$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$, then there exist $\gamma(\neq 0) \in \Xi$ and an (additive) map $\mu : \mathscr{K} \to \Xi$ such that $\Theta(\mathfrak{u}) = \gamma\mathfrak{u}$, $\psi(\mathfrak{u}) = \gamma^{-1}\mathfrak{u} + \mu(\mathfrak{u})$ for some $\mathfrak{u} \in \mathscr{K}$.*

Proof. Take into account that if $\Omega = 0$ or $\eta = 0$, then Θ turns to a centralizer, i.e., $\Theta(\mathfrak{u}) = a\mathfrak{u}$, for any $\mathfrak{u} \in \mathscr{K}$. In light of [16, Theorem 1.1], the conclusion follows. Thus, we use $\eta \neq 0$ and $\Omega \neq 0$ in the rest of the proof. Every $\eta-$generalized derivation assumes the form $\Theta(\mathfrak{u}) = a\mathfrak{u} + \eta\Omega(\mathfrak{u})$ for certain $a \in \mathscr{Q}$ and related derivation Ω of $\mathscr{K}$ due to Koşan and Lee [12, Theorem 2.3]. By the supposition

$$[\psi(\mathfrak{u}), a\mathfrak{v} + \eta\Omega(\mathfrak{v})] = [\mathfrak{u}, \mathfrak{v}] \tag{15}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. By replacing $\mathfrak{v}$ by $\mathfrak{v}z$, we have

$$\big(a\mathfrak{v} + \eta\Omega(\mathfrak{v})\big)[\psi(\mathfrak{u}), z] + [\psi(\mathfrak{u}), \eta\mathfrak{v}\Omega(z)] = \mathfrak{v}[\mathfrak{u}, z] \tag{16}$$

for all $\mathfrak{u}, \mathfrak{v}, z \in \mathscr{K}$.

Let us first assume that Ω is not an inner derivation of $\mathscr{K}$. In light of (16) and Kharchenko's in [11], we notice that

$$\big(a\mathfrak{v} + \eta\mathfrak{v}'\big)[\psi(\mathfrak{u}), z] + [\psi(\mathfrak{u}), \eta\mathfrak{v}z'] = \mathfrak{v}[\mathfrak{u}, z] \tag{17}$$

for all $\mathfrak{u}, \mathfrak{v}, z, \mathfrak{v}', z' \in \mathscr{K}$. Particularly, for $\mathfrak{v} = 0$ we have

$$\eta\mathfrak{v}'[\psi(\mathfrak{u}), z] = 0 \tag{18}$$

for all $\mathfrak{u}, z, \mathfrak{v}' \in \mathscr{K}$. And hence $\psi(\mathfrak{u}) \in \mathscr{Z}(\mathscr{K})$, as $\mathscr{K}$ is prime and $\eta \neq 0$, for any $\mathfrak{u} \in \mathscr{K}$. At the same time, considering (15) and $\psi(\mathfrak{u}) \in \mathscr{Z}(\mathscr{K})$, we got a contradiction $[\mathfrak{u}, \mathfrak{v}] = 0$ for any $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Now in case $q \in \mathscr{Q}$ whenever $\Omega(\mathfrak{u}) = [\mathfrak{u}, q]$ for all $\mathfrak{u} \in \mathscr{K}$. So let's rewrite (17) as follows

$$\big((a - \eta q)\mathfrak{v} + \eta\mathfrak{v}q\big)[\psi(\mathfrak{u}), z] + [\psi(\mathfrak{u}), \eta\mathfrak{v}zq - \eta\mathfrak{v}qz] = \mathfrak{v}[\mathfrak{u}, z], \tag{19}$$

that is

$$(20)\qquad \big((a-\eta q)\mathfrak{v}\psi(\mathfrak{u})+\eta\mathfrak{v}q\psi(\mathfrak{u})-\psi(\mathfrak{u})\eta\mathfrak{v}q-\mathfrak{v}\mathfrak{u}\big)z+\big(\eta q\mathfrak{v}-a\mathfrak{v}\big)z\psi(\mathfrak{u}) \\ +\psi(\mathfrak{u})\eta\mathfrak{v}zq-\eta\mathfrak{v}zq\psi(\mathfrak{u})+\mathfrak{v}z\mathfrak{u}=0$$

for all $\mathfrak{u},\mathfrak{v},z\in\mathscr{K}$. Let's now assume that there is $\mathfrak{v}$ in $\mathscr{K}$ with $\{\eta\mathfrak{v},\mathfrak{v}\}$ are linearly Ξ-independent. From (20) and [19, Theorem 2 (a)], we see that for any $\mathfrak{u}\in\mathscr{K}$, $\mathfrak{u}$ and $q\psi(\mathfrak{u})$ are $\Xi-$linear combinations of $\{1,q,\psi(\mathfrak{u})\}$. Alternatively, there exist $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\in\Xi$, depending on $\mathfrak{u}\in\mathscr{K}$ such that

$$(21)\qquad \mathfrak{u}=\alpha_1+\alpha_2\psi(\mathfrak{u})+\alpha_3 q$$

and

$$(22)\qquad q\psi(\mathfrak{u})=\beta_1+\beta_2\psi(\mathfrak{u})+\beta_3 q.$$

Notice that for $\alpha_2=0$, relation (21) infers that q commutes with $\mathfrak{u}\in\mathscr{K}$. In case $\alpha_2\neq 0$, by (21) we have that

$$(23)\qquad \psi(\mathfrak{u})=\alpha_2^{-1}(\mathfrak{u}-\alpha_1-\alpha_3 q).$$

Then, by using (23) in (22), it follows that

$$(24)\qquad \beta_1+\beta_3 q=\alpha_2^{-1}(q-\beta_2)(\mathfrak{u}-\alpha_1-\alpha_3 q).$$

We get $\alpha_2^{-1}(q-\beta_2)[\mathfrak{u},q]=0$, by commuting (24) with q, implies $[\mathfrak{u},q]=0$.This violates $\Omega\neq 0$. As a result, for all $\mathfrak{v}\in\mathscr{K}$, $\{\mathfrak{v},\eta\mathfrak{v}\}$ must be linearly Ξ-dependent. In such a circumstance, a typical argument suggests that $\eta\in\Xi$, implying that $\Theta(\mathfrak{u})=(a-\eta q)\mathfrak{u}+\mathfrak{u}(\eta q)$ for any $\mathfrak{u}\in\mathscr{K}$. As a result, Θ becomes a generalized derivation of $\mathscr{K}$, and the result derives from [16, Theorem 1.1]. □

We conclude this paper with our main result, which characterizes an η-generalized derivation on prime ring with Jordan involution. Particularly, we demonstrate the following.

Theorem 2.4. *If* $[\Theta(\mathfrak{u}),\Theta(\mathfrak{u}^\diamond)]=[\mathfrak{u},\mathfrak{u}^\diamond]$ *for all* $\mathfrak{u}\in\mathscr{K}$*, then* $\Theta(\mathfrak{u})=\gamma\mathfrak{u}$ *for all* $\mathfrak{u}\in\mathscr{K}$ *such that* $\gamma\in\Xi$ *and* $\gamma^2=1$.

Proof. By the given hypothesis, we have

$$(25)\qquad [\Theta(\mathfrak{u}),\Theta(\mathfrak{u}^\diamond)]-[\mathfrak{u},\mathfrak{u}^\diamond]=0 \text{ for every } \mathfrak{u}\in\mathscr{K}.$$

Taking $\mathfrak{u}$ as $\mathfrak{u}+\mathfrak{v}$ in (25) we get

$$(26)\qquad [\Theta(\mathfrak{u}),\Theta(\mathfrak{v}^\diamond)]+[\Theta(\mathfrak{v}),\Theta(\mathfrak{u}^\diamond)]-[\mathfrak{u},\mathfrak{v}^\diamond]-[\mathfrak{v},\mathfrak{u}^\diamond]=0$$

for all $\mathfrak{u},\mathfrak{v}\in\mathscr{K}$. Substitute $\mathfrak{v}\beta$ for $\mathfrak{v}$, where β is a non-zero central $\diamond-$skew symmetric element of $\mathscr{K}$, in above relation, we obtain

$$(27)\qquad 0=-[\Theta(\mathfrak{u}),\Theta(\mathfrak{v}^\diamond)]\beta-[\Theta(\mathfrak{u}),\eta\mathfrak{v}^\diamond]\Omega(\beta)+[\Theta(\mathfrak{v}),\Theta(\mathfrak{u}^\diamond)]\beta \\ +[\eta\mathfrak{v},\Theta(\mathfrak{u}^\diamond)]\Omega(\beta)+[\mathfrak{u},\mathfrak{v}^\diamond]\beta-[\mathfrak{v},\mathfrak{u}^\diamond]\beta$$

for all $\mathfrak{u},\mathfrak{v}\in\mathscr{K}$. Multiply (26) with β and add with (27) to get

$$(28)\qquad 2[\Theta(\mathfrak{v}),\Theta(\mathfrak{u}^\diamond)]\beta-2[\mathfrak{v},\mathfrak{u}^\diamond]\beta-[\Theta(\mathfrak{u}),\eta\mathfrak{v}^\diamond]\Omega(\beta)+[\eta\mathfrak{v},\Theta(\mathfrak{u}^\diamond)]\Omega(\beta)=0$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. Again substitute $\mathfrak{v}$ as $\mathfrak{v}\beta$ in (28), we get

$$0 = 2[\Theta(\mathfrak{v}), \Theta(\mathfrak{u}^\diamond)]\beta^2 + 2[\eta\mathfrak{v}, \Theta(\mathfrak{u}^\diamond)]\Omega(\beta)\beta - 2[\mathfrak{v}, \mathfrak{u}^\diamond]\beta^2 + [\Theta(\mathfrak{u}), \eta\mathfrak{v}^\diamond]\Omega(\beta)\beta + [\eta\mathfrak{v}, \Theta(\mathfrak{u}^\diamond)]\Omega(\beta)\beta \tag{29}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. In view of (28), we have

$$2[\eta\mathfrak{v}, \Theta(\mathfrak{u}^\diamond)]\Omega(\beta)\beta + 2[\Theta(\mathfrak{u}), \eta\mathfrak{v}^\diamond]\Omega(\beta)\beta = 0 \tag{30}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$ and $\beta \in J_{\mathscr{S}}$. Since $char(\mathscr{K}) \neq 2$ and β is non-zero, so $[\eta\mathfrak{v}, \Theta(\mathfrak{u}^\diamond)] + [\Theta(\mathfrak{u}), \eta\mathfrak{v}^\diamond] = [\Theta(\mathfrak{u}), \eta\mathfrak{v}^\diamond] - [\Theta(\mathfrak{u}^\diamond), \eta\mathfrak{v}] = 0$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$ or $\Omega(\beta)\beta = 0$. Observe that $\beta = 0$ also implies $\Omega(\beta) = 0$. Assume that $\Omega(\beta) \neq 0$, therefore we have

$$[\Theta(\mathfrak{u}), \eta\mathfrak{v}^\diamond] - [\Theta(\mathfrak{u}^\diamond), \eta\mathfrak{v}] = 0 \tag{31}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. With $\mathfrak{u} = \mathfrak{v} = \alpha' + \beta'$ in the above expression, we get

$$[\Theta(\beta'), \eta\alpha'] - [\Theta(\beta'), \eta\alpha'] = 0 \tag{32}$$

for all $\alpha' \in J_{\mathscr{H}}$ and $\beta' \in J_{\mathscr{S}}$. Replace β' by β, in (32), we get

$$[\Theta(\beta), \eta\alpha'] - [\Theta(\alpha'), \eta]\beta = 0 \tag{33}$$

for all $\alpha' \in J_{\mathscr{H}}$ and $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{K})$. In the last expression, if we replace α' with $\beta'\beta$, then we obtain

$$[\Theta(\beta), \eta\beta']\beta - [\Theta(\beta'), \eta]\beta^2 + \eta[\eta, \beta']\Omega(\beta)\beta = 0, \tag{34}$$

where β' is a $\diamond$−skew symmetric element and β is a non-zero central $\diamond$−skew symmetric element of $\mathscr{K}$, respectively. One can see from (31) that $[\Theta(\beta), \eta\beta'] = 0$ and $[\Theta(\beta'), \eta]\beta = 0$. This reduces (34) into

$$\eta[\eta, \beta']\Omega(\beta)\beta = 0. \tag{35}$$

This implies either $\eta[\eta, \beta'] = 0$ or $\Omega(\beta) = 0$. Suppose $\eta[\eta, \beta] = 0$ for any $\diamond$−skew symmetric element β of $\mathscr{K}$. Next, take $\beta' = \alpha_0\beta$, since β is non-zero, we get $\eta[\eta, \alpha_0] = 0$ for all $\alpha_0 \in J_{\mathscr{H}}$. An application of Remark 1.5 yields $\eta \in \Xi$. One can see from (34) that $\eta[\Theta(\beta), \beta'] = 0$. If $\eta = 0$ and it takes the form: $\Theta(\mathfrak{u}) = a\mathfrak{u}$, for some fixed element $a \in \mathscr{Q}$. Thus, by Lemma 2.2, we have the required conclusion.

So we assume $\eta \neq 0$ and $[\Theta(\beta), \beta'] = 0$ for all $\beta' \in J_{\mathscr{S}}$ and $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{K})$. Remark 1.5 in last relation yields $\Theta(\beta) \in \mathscr{Z}(\mathscr{K})$ when β' is taken as $\alpha_0\beta$. Next, take $\mathfrak{u} = \alpha' \in J_{\mathscr{H}}$ and $\mathfrak{v} = \beta' \in J_{\mathscr{S}}$ in (26), we get

$$[\Theta(\alpha'), \Theta(\alpha')] + [\alpha', \beta'] = 0. \tag{36}$$

Substitute $\alpha_0\beta$ for β' in above relation, we get

$$[\Theta(\alpha_0\beta), \Theta(\alpha')] + [\alpha', \alpha_0]\beta = 0 \tag{37}$$

for all $\alpha', \alpha_0 \in J_{\mathscr{H}}$ and $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{K})$. It follows from the hypothesis that

$$[\Theta(\alpha_0), \Theta(\alpha')]\beta + [\eta\alpha_0, \Theta(\alpha')]\Omega(\beta) + [\alpha', \alpha_0]\beta = 0. \tag{38}$$

For $\alpha_0 = \alpha'$, we have

$$\eta[\Theta(\alpha'), \alpha']\Omega(\beta) = 0. \tag{39}$$

Since $\Omega(\beta) \neq 0$ and $\eta \neq 0$, so we have $[\Theta(\alpha'), \alpha'] = 0$ for all $\alpha \in J_{\mathscr{H}}$. Since $\eta \in \Xi$, so we observe from (32) that

$$[\Theta(\beta'), \alpha'] - [\Theta(\alpha'), \beta'] = 0. \tag{40}$$

for all $\alpha' \in J_{\mathscr{H}}$ and $\beta' \in J_{\mathscr{S}}$. Replacement of β' by $\beta\alpha'$ in (40) and an application of $\Theta(\beta) \in \mathscr{Z}(\mathscr{K})$ and $\eta \neq 0$ yield $[\Omega(\alpha'), \alpha'] = 0$ for any $\diamond$-symmetric element α' of $\mathscr{K}$. After simple calculation, one can get $\Omega(\beta) = 0$. Finally, we suppose $\Omega(\beta) = 0$ for any non-zero central $\diamond$-skew symmetric element β of $\mathscr{K}$. Substitute $\mathfrak{v}$ by $\mathfrak{v}\beta$ in (26), we obtain

$$-[\Theta(\mathfrak{u}), \Theta(\mathfrak{v}^\diamond)]\beta + [\Theta(\mathfrak{v}), \Theta(\mathfrak{u}^\diamond)]\beta + [\mathfrak{u}, \mathfrak{v}^\diamond]\beta - [\mathfrak{v}, \mathfrak{u}^\diamond]\beta = 0 \tag{41}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$ and $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{K})$. Combination of (26) and (41) gives

$$([\Theta(\mathfrak{v}), \Theta(\mathfrak{u}^\diamond)] - [\mathfrak{v}, \mathfrak{u}^\diamond])\beta = 0 \tag{42}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$ and $\beta \in J_{\mathscr{S}} \cap \mathscr{Z}(\mathscr{K})$. This implies that

$$[\Theta(\mathfrak{v}), \Theta(\mathfrak{u}^\diamond)] - [\mathfrak{v}, \mathfrak{u}^\diamond] = 0 \tag{43}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. In particular

$$[\Theta(\mathfrak{u}), \Theta(\mathfrak{v})] - [\mathfrak{u}, \mathfrak{v}] = 0 \tag{44}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{K}$. In light of Theorem 2.3, Θ is of the form $\Theta(\mathfrak{u}) = \gamma\mathfrak{u}$, where $\gamma \in \Xi$ and $\gamma^2 = 1$. □

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