J. Appl. & Pure Math. Vol. **6**(2024), No. 3 - 4, pp. 191 - 200
https://doi.org/10.23091/japm.2024.191

GENERALIZATION OF THE BUZANO'S INEQUALITY AND NUMERICAL RADIUS INEQUALITIES

VUK STOJILJKOVIĆ* AND MEHMET GÜRDAL

Abstract. Motivated by the previously reported results, this work attempts to provide fresh refinements to both vector and numerical radius inequalities by providing a refinement to the well known Buzano's inequality which as a consequence yielded another refinement of the Cauchy-Schwartz (CS) inequality. Utilizing the new refinements of the Buzano's and Cauchy-Schwartz inequalities, we proceed to obtain various vector and numerical radius type inequalities. Methods used in the paper are standard for the operator theory inequality topics.

AMS Mathematics Subject Classification : 26D05, 26D07, 26D20.
Key words and phrases : Numerical radius, norm, inequalities.

1. Introduction

Let $\mathfrak{B}(\mathfrak{H})$ denote the $\mathfrak{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathfrak{H}$ with inner product $\langle .,. \rangle$. For $\mathfrak{Q} \in \mathfrak{B}(\mathfrak{H})$, let $\|\mathfrak{Q}\|$ denote the the usual operator norm of $\mathfrak{Q}$. Numerical radius is defined as usual, as a supremum on the unit sphere of the quadratic form. Various properties of such mapping are well known, interested readers are referenced to [1, 4, 5, 6, 8, 13, 21, 23, 24, 27, 28, 29] for more information. It is well known that $w(.)$ forms a norm on $\mathfrak{B}(\mathfrak{H})$ which is equivalent with the usual operator norm

$$\|\mathfrak{Q}\| = \sup_{\|x\|=1} \langle \mathfrak{Q}x, \mathfrak{Q}x \rangle^{\frac{1}{2}},$$

for $\mathfrak{Q} \in \mathfrak{B}(\mathfrak{H})$. In particular, the following sharp inequality holds

$$\frac{1}{2} \|\mathfrak{Q}\| \le w(\mathfrak{Q}) \le \|\mathfrak{Q}\|. \tag{1}$$

Received April 30, 2024. Revised May 25, 2024. Accepted June 7, 2024. *Corresponding author.

In [22], Kittaneh substantially improved the upper bound in (1) by showing that if $\mathfrak{Q} \in \mathfrak{B}(\mathfrak{H})$, then

$$w(\mathfrak{Q}) \leq \frac{1}{2} \left\| |\mathfrak{Q}| + |\mathfrak{Q}^*| \right\| \leq \frac{1}{2} \left(\|\mathfrak{Q}\| + \left\|\mathfrak{Q}^2\right\|^{\frac{1}{2}} \right). \tag{2}$$

Furthermore, Kittaneh et al. [12] obtained inequalities for one operator which are as follows:

$$w^{\hbar}(\mathfrak{J}) \leq \frac{1}{2} \left\| |\mathfrak{J}|^{2\hbar s} + |\mathfrak{J}^*|^{2\hbar(1-s)} \right\| \tag{3}$$

and

$$w^{2\hbar}(\mathfrak{J}) \leq \left\| s\,|\mathfrak{J}|^{2\hbar} + (1-s)\,|\mathfrak{J}^*|^{2\hbar} \right\|, \tag{4}$$

where $\mathfrak{J} \in \mathfrak{B}(\mathfrak{H})$, $0 \leq s \leq 1$, and $\hbar \geq 1$.

Recently, Stojiljković and Dragomir obtained the following refinement with respect to the mapping Δ, which is as follows.

$$\begin{aligned} w(\mathfrak{Q})^k &\leq \frac{\Delta(\kappa)}{2} \left\| |\mathfrak{Q}|^{2\hbar k} + |\mathfrak{Q}^*|^{2k(1-\hbar)} \right\| + \frac{\Delta(1-\kappa)}{2} w^{\frac{k}{2}}(\mathfrak{Q}) \left\| |\mathfrak{Q}|^{\hbar k} + |\mathfrak{Q}^*|^{k(1-\hbar)} \right\| \\ &\leq \frac{1}{2} \left\| |\mathfrak{Q}|^{2\hbar k} + |\mathfrak{Q}^*|^{2(1-\hbar)} \right\|, \end{aligned}$$

for $k \geq 2$, $\hbar \in [0,1]$, $\mathfrak{Q} \in \mathfrak{B}(\mathfrak{H})$ and Δ as defined in [30]. For further information, consult the paper from which the result is given [30].

However, Berezin transforms have been extensively applied in the field of reproducing kernel Hilbert spaces, addressing various problems. In their study of the boundedness of operators on reproducing kernel Hilbert spaces, Chalendar et al. [9] looked at more generic operators and investigated the Berezin symbols of their unitary orbits. Bhunia et al. [7] introduced a new norm, the $\hbar$-Berezin norm, for the space of all bounded linear operators defined on a reproducing kernel Hilbert space, which extends the Berezin radius and Berezin norm. For further information on Berezin transforms, refer to [14, 15, 16, 17, 18, 19, 20, 31].

2. Preliminaries

We begin our section with stating one of the most influential inequalities.

Lemma 2.1. *(Hölder Mc-Carty inequality [25]). Let $\mathfrak{Q}$ be a positive bounded linear operator and let x be any unit vector, then the following inequalities holds:*

$$\langle \mathfrak{Q}x, x\rangle^r \leq \langle \mathfrak{Q}^r x, x\rangle \text{ for } r \geq 1, \tag{5}$$

$$\langle \mathfrak{Q}^r x, x\rangle \leq \langle \mathfrak{Q}x, x\rangle^r \text{ for } 0 < r \leq 1. \tag{6}$$

The next result is the well-known Buzano's inequality.

Lemma 2.2. *Let $x, y, \mathfrak{e} \in \mathfrak{H}$ with $\|\mathfrak{e}\| = 1$. Then we have*

$$|\langle x, \mathfrak{e}\rangle\langle \mathfrak{e}, y\rangle| \leq \frac{1}{2}(\|x\|\,\|y\| + |\langle x, y\rangle|). \tag{7}$$

Lemma 2.3. *([2]) Let $\mathfrak{f}$ be a non-negative convex function on $[0,+\infty)$ and $\mathfrak{Q},\mathfrak{B}\in\mathfrak{B}(\mathfrak{H})$ be positive operators. Then*

$$\left\|\mathfrak{f}\left(\frac{\mathfrak{Q}+\mathfrak{B}}{2}\right)\right\| \leq \left\|\frac{\mathfrak{f}(\mathfrak{Q})+\mathfrak{f}(\mathfrak{B})}{2}\right\|. \tag{8}$$

Lemma 2.4. *([11]) Let $\mathfrak{Q},\mathfrak{B}\in\mathfrak{B}(\mathfrak{H})$ and $r\geq 1$, then*

$$w^r(\mathfrak{B}^*\mathfrak{Q}) \leq \frac{1}{2}\left\||\mathfrak{Q}|^{2r}+|\mathfrak{B}|^{2r}\right\|. \tag{9}$$

Lemma 2.5. *([10, eq. 31]) Let $\mathfrak{Q},\mathfrak{B},\mathfrak{C},\mathfrak{D}\in\mathfrak{B}(\mathfrak{H})$. Then we have*

$$\|\mathfrak{D}\mathfrak{C}\mathfrak{B}\mathfrak{Q}\|^2 \leq \left\|\mathfrak{Q}^*\,|\mathfrak{B}|^2\,\mathfrak{Q}\right\|\left\|\mathfrak{D}\,|\mathfrak{C}^*|^2\,\mathfrak{D}^*\right\|. \tag{10}$$

Lemma 2.6. *([30])Let $\mathfrak{l},\mathfrak{z}\in\mathfrak{H}$ and let $k\geq 1$. Let J be a set such that $(0,1)\subset J$. Let Δ be a mapping such that $\Delta: J\to\mathbb{R}^+$, such that the following holds $\Delta(\kappa)+\Delta(1-\kappa)=1$. Then the following inequality holds*

$$|\langle\mathfrak{l},\mathfrak{z}\rangle|^{2k} \leq \Delta(\kappa)\,\|\mathfrak{l}\|^{2k}\,\|\mathfrak{z}\|^{2k}+\Delta(1-\kappa)|\langle\mathfrak{l},\mathfrak{z}\rangle|^k\,\|\mathfrak{l}\|^k\,\|\mathfrak{z}\|^k \leq \|\mathfrak{l}\|^{2k}\,\|\mathfrak{z}\|^{2k}. \tag{11}$$

3. Main results

Our initial outcome is a Lemma that plays a crucial role in shaping the subsequent results.

Lemma 3.1. *Let $\mathfrak{l},\mathfrak{z},\mathfrak{e}\in\mathcal{H}$ such that $\|\mathfrak{e}\|=1$ and let $k\geq 1$. Let J be a set such that $(0,1)\subset J\subset\mathbb{R}$. Let Δ be a mapping such that $\Delta: J\to\mathbb{R}^+$, such that the following holds $\Delta(\kappa)+\Delta(1-\kappa)=1$. Then the following inequality holds*

$$\begin{aligned}&|\langle\mathfrak{l},\mathfrak{e}\rangle\langle\mathfrak{e},\mathfrak{z}\rangle|^k \qquad (12)\\ &\leq \min\Big\{\frac{\Delta(\kappa)+1}{2}\,\|\mathfrak{l}\|^k\,\|\mathfrak{z}\|^k+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{l},\mathfrak{z}\rangle|^k,\\ &\frac{\Delta(1-\kappa)+1}{2}\,\|\mathfrak{l}\|^k\,\|\mathfrak{z}\|^k+\frac{\Delta(\kappa)}{2}|\langle\mathfrak{l},\mathfrak{z}\rangle|^k\Big\}.\end{aligned}$$

Proof. Consider the following,

$$\begin{aligned}|\langle\mathfrak{l},\mathfrak{e}\rangle\langle\mathfrak{e},\mathfrak{z}\rangle| &= \Delta(\kappa)\,|\langle\mathfrak{l},\mathfrak{e}\rangle\langle\mathfrak{e},\mathfrak{z}\rangle|+\Delta(1-\kappa)\,|\langle\mathfrak{l},\mathfrak{e}\rangle\langle\mathfrak{e},\mathfrak{z}\rangle|\\ &\leq \Delta(\kappa)\,\|\mathfrak{l}\|\,\|\mathfrak{z}\|+\frac{\Delta(1-\kappa)}{2}\left(\|\mathfrak{l}\|\,\|\mathfrak{z}\|+|\langle\mathfrak{l},\mathfrak{z}\rangle|\right)\\ &= \frac{\Delta(\kappa)+1}{2}\,\|\mathfrak{l}\|\,\|\mathfrak{z}\|+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{l},\mathfrak{z}\rangle|.\end{aligned}$$

Now using the fact that $\mathfrak{f}(x)=x^k$ is convex for $k\geq 1$, we get

$$|\langle\mathfrak{l},\mathfrak{e}\rangle\langle\mathfrak{e},\mathfrak{z}\rangle|^k \leq \frac{\Delta(\kappa)+1}{2}\,\|\mathfrak{l}\|^k\,\|\mathfrak{z}\|^k+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{l},\mathfrak{z}\rangle|^k.$$

Obtaining the other inequality using the same technique, we obtain the desired inequality. □

Corollary 3.2. *The following new inequality of (12) type can be easily obtained by setting* $\Delta(\kappa)=\frac{1}{2}\left(\frac{1}{2}+\kappa\right)$

$$|\langle \mathfrak{l},\mathfrak{e}\rangle\langle \mathfrak{e},\mathfrak{z}\rangle|^k \le \min\left\{\left(\frac{5}{8}+\frac{\kappa}{4}\right)\|\mathfrak{l}\|^k\|\mathfrak{z}\|^k+\left(\frac{3}{8}-\frac{\kappa}{4}\right)|\langle \mathfrak{l},\mathfrak{z}\rangle|^k,\right. \tag{13}$$

$$\left.\left(\frac{7}{8}-\frac{\kappa}{4}\right)\|\mathfrak{l}\|^k\|\mathfrak{z}\|^k+\left(\frac{1}{8}+\frac{\kappa}{4}\right)|\langle \mathfrak{l},\mathfrak{z}\rangle|^k\right\}.$$

Corollary 3.3. *The following refinement of the CS inequality is evident from (12):*

$$|\langle \mathfrak{l},\mathfrak{z}\rangle|^k \le \frac{\Delta(\kappa)+1}{2}\|\mathfrak{l}\|^k\|\mathfrak{z}\|^k+\frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{l},\mathfrak{z}\rangle|^k \le \|\mathfrak{l}\|^k\|\mathfrak{z}\|^k. \tag{14}$$

Proof. Since both inequalities hold from the proof of (12), we can without the loss of generality consider that the one shown in the proof holds, then using the CS inequality is obtained. □

The following inequalities are corollaries of (14).

Theorem 3.4. *Let* $\mathfrak{Q},\mathfrak{B},\mathfrak{C},\mathfrak{D}\in\mathfrak{B}(\mathfrak{H})$ *and* Δ *such that (14) holds. Then for* $x,y\in\mathfrak{H}$ *and* $k\ge 1$ *we have the inequality*

$$|\langle \mathfrak{D}\mathfrak{C}\mathfrak{B}\mathfrak{Q}x,y\rangle|^k \le \frac{\Delta(\kappa)+1}{2}\sqrt[\frac{k}{2}]{\langle \mathfrak{Q}^*|\mathfrak{B}|^2\mathfrak{Q}x,x\rangle}\sqrt[\frac{k}{2}]{\langle \mathfrak{D}|\mathfrak{C}^*|^2\mathfrak{D}^*y,y\rangle} \tag{15}$$

$$+\frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{D}\mathfrak{C}\mathfrak{B}\mathfrak{Q}x,y\rangle|^k \le \sqrt[\frac{k}{2}]{\langle \mathfrak{Q}^*|\mathfrak{B}|^2\mathfrak{Q}x,x\rangle}\sqrt[\frac{k}{2}]{\langle \mathfrak{D}|\mathfrak{C}^*|^2\mathfrak{D}^*y,y\rangle}.$$

Proof. Using (14) and setting $\mathfrak{l}=\mathfrak{B}\mathfrak{Q}x$, $\mathfrak{z}=\mathfrak{C}^*\mathfrak{D}^*y$, then simplifying

$$\|\mathfrak{B}\mathfrak{Q}x\|^2=\langle \mathfrak{B}\mathfrak{Q}x,\mathfrak{B}\mathfrak{Q}x\rangle=\langle \mathfrak{Q}^*\mathfrak{B}^*\mathfrak{B}\mathfrak{Q}x,x\rangle=\left\langle \mathfrak{Q}^*|\mathfrak{B}|^2\mathfrak{Q}x,x\right\rangle$$

$$\|\mathfrak{C}^*\mathfrak{D}^*y\|^2=\left\langle \mathfrak{D}|\mathfrak{C}^*|^2\mathfrak{D}^*y,y\right\rangle$$

we obtain the desired inequality. □

Corollary 3.5. *Setting* $k=2$ *we obtain the refinement of the inequality given by Dragomir in his paper* [10, eq. 9].

In particular, the obtained inequality refines various inequalities as particular cases.

Remark 3.1. The Furuta inequality for $\hbar,\beta\ge 0$ with $\hbar+\beta\ge 1$ is a particular case of (15),

$$|\langle \mathfrak{T}|\mathfrak{T}|^{\hbar+\beta-1}x,y\rangle|^k \le \frac{\Delta(\kappa)+1}{2}\langle |\mathfrak{T}|^{2\hbar}x,x\rangle^{\frac{k}{2}}\langle |\mathfrak{T}^*|^{2\beta}y,y\rangle^{\frac{k}{2}} \tag{16}$$

$$+\frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}|\mathfrak{T}|^{\hbar+\beta-1}x,y\rangle|^k \le \langle |\mathfrak{T}|^{2\hbar}x,x\rangle^{\frac{k}{2}}\langle |\mathfrak{T}^*|^{2\beta}y,y\rangle^{\frac{k}{2}}.$$

The following result gives a sharper estimate to the one given by Dragomir in [10]. For any operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$ and any $\hbar, \beta \geq 1$ we have the inequality

$$|\langle \mathfrak{T}|\mathfrak{T}|^{\beta-1}\mathfrak{T}|\mathfrak{T}|^{\hbar-1}x, y\rangle|^k \leq \frac{\Delta(\kappa)+1}{2}\langle |\mathfrak{T}|^{2\hbar}x, x\rangle^{\frac{k}{2}}\langle |\mathfrak{T}^*|^{2\beta}y, y\rangle^{\frac{k}{2}} \tag{17}$$

$$+\frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}|\mathfrak{T}|^{\beta-1}\mathfrak{T}|\mathfrak{T}|^{\hbar-1}x, y\rangle|^k \leq \langle |\mathfrak{T}|^{2\hbar}x, x\rangle^{\frac{k}{2}}\langle |\mathfrak{T}^*|^{2\beta}y, y\rangle^{\frac{k}{2}}.$$

Setting $\hbar = \beta = 1$, $k = 2$ in (17) we obtain the following inequality, which is sharper than the one given by Dragomir in his paper [10].

$$\begin{aligned}|\langle \mathfrak{T}^2x, y\rangle|^2 &\leq \frac{\Delta(\kappa)+1}{2}\langle |\mathfrak{T}|^2x, x\rangle\langle |\mathfrak{T}^*|^2y, y\rangle + \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}^2x, y\rangle|^2 \\ &\leq \langle |\mathfrak{T}|^2x, x\rangle\langle |\mathfrak{T}^*|^2y, y\rangle.\end{aligned} \tag{18}$$

4. Numerical radius inequalities

We begin the section by stating the first result concerning the norm variant of the CS inequality.

Theorem 4.1. *The conditions are as those in (15),*

$$\|\mathfrak{D}\mathfrak{C}\mathfrak{B}\mathfrak{Q}\|^k \leq \frac{\Delta(\kappa)+1}{2}\left\|\mathfrak{Q}^*|\mathfrak{B}|^2\mathfrak{Q}\right\|^{\frac{k}{2}}\left\|\mathfrak{D}|\mathfrak{C}^*|^2\mathfrak{D}^*\right\|^{\frac{k}{2}} \tag{19}$$

$$+\frac{\Delta(1-\kappa)}{2}\|\mathfrak{D}\mathfrak{C}\mathfrak{B}\mathfrak{Q}\|^k \leq \left\|\mathfrak{Q}^*|\mathfrak{B}|^2\mathfrak{Q}\right\|^{\frac{k}{2}}\left\|\mathfrak{D}|\mathfrak{C}^*|^2\mathfrak{D}^*\right\|^{\frac{k}{2}}.$$

Proof. The proof is omitted as it is similar to the one given in [30] by Stojiljković and Dragomir. □

Corollary 4.2. *Our inequality presents a refinement of the Dragomir's result established in* [10, eq. (31)].

Theorem 4.3. *The conditions are as the ones in (15) with an exception that $k \geq 2$, then the following inequality holds*

$$|\langle \mathfrak{T}x, x\rangle\langle x, \mathfrak{S}^*x\rangle|^k \leq \frac{\Delta(\kappa)+1}{8}\langle (|\mathfrak{T}|^{2k} + |\mathfrak{S}^*|^{2k})x, x\rangle \tag{20}$$

$$+\frac{\Delta(\kappa)+1}{4}|\langle |\mathfrak{T}|^2x, |\mathfrak{S}^*|^2x\rangle|^{\frac{k}{2}} + \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k.$$

In particular, we obtain

$$\begin{aligned}&w^{2k}(\mathfrak{T}) \\ &\leq \frac{\Delta(\kappa)+1}{8}\left\||\mathfrak{T}|^{2k} + |\mathfrak{T}^*|^{2k}\right\| + \frac{\Delta(\kappa)+1}{4}w^{\frac{k}{2}}(|\mathfrak{T}^*|^2|\mathfrak{T}|^2) + \frac{\Delta(1-\kappa)}{2}w^k(\mathfrak{T}^2) \\ &\leq \frac{\Delta(\kappa)+1}{4}\left\||\mathfrak{T}|^{2k} + |\mathfrak{T}^*|^{2k}\right\| + \frac{\Delta(1-\kappa)}{2}w^k(\mathfrak{T}^2).\end{aligned}$$

Proof. Without the loss of generality, let us assume that one of the inequalities hold in (12). Setting $\mathfrak{l} = \mathfrak{T}x$, $\mathfrak{z} = \mathfrak{S}^*x$, $\mathfrak{e} = x$ in the Lemma 3.1, we obtain

$$\begin{aligned}
|\langle \mathfrak{T}x, x\rangle\langle x, \mathfrak{S}^*x\rangle|^k &\leq \frac{\Delta(\kappa)+1}{2}\left\|\mathfrak{T}x\right\|^k\left\|\mathfrak{S}^*x\right\|^k + \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k \\
&\leq \frac{\Delta(\kappa)+1}{4}\left(\left\||\mathfrak{T}|^2 x\right\|^{\frac{k}{2}}\left\||\mathfrak{S}^*|^2 x\right\|^{\frac{k}{2}} + |\langle |\mathfrak{T}|^2x, |\mathfrak{S}^*|^2x\rangle|^{\frac{k}{2}}\right) \\
&+ \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k \\
&\leq \frac{\Delta(\kappa)+1}{8}\left(\left\||\mathfrak{T}|^2 x\right\|^k + \left\||\mathfrak{S}^*|^2 x\right\|^k\right) \\
&+ \frac{\Delta(\kappa)+1}{4}|\langle |\mathfrak{T}|^2x, |\mathfrak{S}^*|^2x\rangle|^{\frac{k}{2}} + \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k \\
&= \frac{\Delta(\kappa)+1}{8}\left(\langle |\mathfrak{T}|^4x, x\rangle^{\frac{k}{2}} + \langle |\mathfrak{S}^*|^4x, x\rangle^{\frac{k}{2}}\right) \\
&+ \frac{\Delta(\kappa)+1}{4}|\langle |\mathfrak{T}|^2x, |\mathfrak{S}^*|^2x\rangle|^{\frac{k}{2}} + \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k \\
&\leq \frac{\Delta(\kappa)+1}{8}\langle (|\mathfrak{T}|^{2k} + |\mathfrak{S}^*|^{2k})x, x\rangle + \frac{\Delta(\kappa)+1}{4}|\langle |\mathfrak{T}|^2x, |\mathfrak{S}^*|^2x\rangle|^{\frac{k}{2}} \\
&+ \frac{\Delta(1-\kappa)}{2}|\langle \mathfrak{T}x, \mathfrak{S}^*x\rangle|^k.
\end{aligned}$$

Setting $\mathfrak{S}^* = \mathfrak{T}^*$ and utilizing (5) and (9) while taking $\sup_{\|x\|=1}$ we obtain the second inequality. □

Theorem 4.4. *The conditions are as the ones in (15) with an exception that $k \geq 1$, then the following inequality holds*

$$w^{2k}(\mathfrak{S}^*\mathfrak{T}) \leq \frac{\Delta(\kappa)}{4}\left\|(|\mathfrak{T}|^{4k} + |\mathfrak{S}|^{4k})\right\| \tag{21}$$

$$+\frac{\Delta(1-\kappa)}{2}\left\||\mathfrak{S}|^{2k} + |\mathfrak{T}|^{2k}\right\| w^k(\mathfrak{S}^*\mathfrak{T}) + \frac{\Delta(\kappa)}{2} w\left(|\mathfrak{S}|^{2k}|\mathfrak{T}|^{2k}\right).$$

Proof. We begin with setting $\mathfrak{l} = \mathfrak{T}x, \mathfrak{z} = \mathfrak{S}x$ in (11)

$$\begin{aligned}
|\langle \mathfrak{T}x, \mathfrak{S}x\rangle|^{2k} &\leq \Delta(\kappa)\left\|\mathfrak{T}x\right\|^{2k}\left\|\mathfrak{S}x\right\|^{2k} + \Delta(1-\kappa)|\langle \mathfrak{T}x, \mathfrak{S}x\rangle|^k\left\|\mathfrak{T}x\right\|^k\left\|\mathfrak{S}x\right\|^k \\
&\leq \frac{\Delta(1-\kappa)}{2}\langle (|\mathfrak{T}|^{2k} + |\mathfrak{S}|^{2k})x, x\rangle|\langle \mathfrak{T}x, \mathfrak{S}x\rangle|^k \\
&+ \frac{\Delta(\kappa)}{2}\left(\left\||\mathfrak{T}|^{2k} x\right\|\left\||\mathfrak{S}|^{2k} x\right\| + |\langle |\mathfrak{T}|^{2k}x, |\mathfrak{S}|^{2k}x\rangle|\right) \\
&\leq \frac{\Delta(1-\kappa)}{2}\langle (|\mathfrak{T}|^{2k} + |\mathfrak{S}|^{2k})x, x\rangle|\langle \mathfrak{T}x, \mathfrak{S}x\rangle|^k \\
&+ \frac{\Delta(\kappa)}{4}\langle (|\mathfrak{T}|^{4k} + |\mathfrak{S}|^{4k})x, x\rangle + \frac{\Delta(\kappa)}{2}|\langle |\mathfrak{S}|^{2k}|\mathfrak{T}|^{2k}x, x\rangle|.
\end{aligned}$$

□

Corollary 4.5. *The conditions are the same as in (15) with an exception that $k \geq 1$, then the following inequality holds*

$$w^{2k}(\mathfrak{S}^*\mathfrak{T}) \leq \frac{\Delta(\kappa)}{4}\left\|(|\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k})\right\| \tag{22}$$

$$+\frac{\Delta(1-\kappa)}{2}\left\||\mathfrak{S}|^{2k}+|\mathfrak{T}|^{2k}\right\|w^k(\mathfrak{S}^*\mathfrak{T})+\frac{\Delta(\kappa)}{2}w\left(|\mathfrak{S}|^{2k}|\mathfrak{T}|^{2k}\right)$$

$$\leq \frac{\Delta(\kappa)}{2}\left\|(|\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k})\right\|+\frac{\Delta(1-\kappa)}{2}\left\||\mathfrak{S}|^{2k}+|\mathfrak{T}|^{2k}\right\|w^k(\mathfrak{S}^*\mathfrak{T})$$

$$\leq \frac{1}{2}\left\||\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}\right\|.$$

Proof. It is enough to utilize (9) on the third term to obtain the result. □

Remark 4.1. Setting $\Delta(\kappa)=\kappa$, $\kappa=\frac{\lambda}{1+\lambda}$, $\lambda \geq 0$ in (21) we obtain the inequality given by Nayak [26], namely we obtain Theorem 2.16. Specifying the following parameters $\Delta(\kappa)=\kappa$, $\kappa=\frac{\lambda}{1+\lambda}$, $\lambda \geq 0$ in (22) we obtain the inequality given by Nayak [26], namely we obtain Corollary 2.17. Choosing the following parameters $k=1$, $\Delta(\kappa)=\kappa$, $\kappa=\frac{\lambda}{1+\lambda}$, $\lambda \geq 0$ in (22) we obtain the refinement for the result given by Al-Dolat et al. [3], namely Theorem 2.6. It is interesting to note that our inequality generalizes the result given by Nayak in a way that the function Δ permits various combinations and that it generalizes and sharpens the inequality given by Al-Dolat et al. in both the inequality sense and in the sense that function Δ permits various options for the function Δ.

Theorem 4.6. *The conditions are as the ones in (15) with an exception that $k \geq 1$, then the following inequality holds*

$$w^k(\mathfrak{B}^*\mathfrak{Q}) \leq \frac{\Delta(\kappa)+1}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|+\frac{\Delta(1-\kappa)}{2}w^k(\mathfrak{B}^*\mathfrak{Q})$$

$$\leq \frac{1}{2}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|. \tag{23}$$

Proof. Setting $\mathfrak{B}=I$, $\mathfrak{C}=I$, $\mathfrak{D}^*=\mathfrak{B}$, $y=x$ in (15), we obtain the following

$$|\langle \mathfrak{B}^*\mathfrak{Q}x,x\rangle|^k \leq \frac{\Delta(\kappa)+1}{2}\langle|\mathfrak{Q}|^2x,x\rangle^{\frac{k}{2}}\langle|\mathfrak{B}|^2x,x\rangle^{\frac{k}{2}}+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{B}^*\mathfrak{Q}x,x\rangle|^k.$$

Using AG inequality, we obtain

$$\leq \frac{\Delta(\kappa)+1}{4}\left(\langle|\mathfrak{Q}|^2x,x\rangle^k+\langle|\mathfrak{B}|^2x,x\rangle^k\right)+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{B}^*\mathfrak{Q}x,x\rangle|^k.$$

Now using (5), we obtain

$$\leq \frac{\Delta(\kappa)+1}{4}\left(\langle|\mathfrak{Q}|^{2k}x,x\rangle+\langle|\mathfrak{B}|^{2k}x,x\rangle\right)+\frac{\Delta(1-\kappa)}{2}|\langle\mathfrak{B}^*\mathfrak{Q}x,x\rangle|^k.$$

Taking sup, we obtain the first part.

To obtain the second part, notice the following

$$w^k(\mathfrak{B}^*\mathfrak{Q})$$

$$\begin{aligned}
&\leq \frac{\Delta(\kappa)+1}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|+\frac{\Delta(1-\kappa)}{2}w^k(\mathfrak{B}^*\mathfrak{Q})\\
&\leq \frac{\Delta(\kappa)+1}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|+\frac{\Delta(1-\kappa)}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\| \quad (9)\\
&= \frac{\Delta(\kappa)+1}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|+\frac{\Delta(1-\kappa)}{4}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|\\
&= \frac{1}{2}\left\||\mathfrak{Q}|^{2k}+|\mathfrak{B}|^{2k}\right\|.
\end{aligned}$$

□

Corollary 4.7. *Clearly, a refinement of the inequality (9) is given for arbitrary function Δ that fulfills the conditions given in (12).*

5. Conclusion

Refinement of the Buzano's inequality has been given which as a consequence allowed us to obtain various generalizations of both vector and numerical radius type inequalities. Refinement and generalization of the results reported in the literature is shown as to showcase the validity of the obtained results. Further questions can be asked whether further refinements of the obtained inequalities are possible. Application of the numerical radius can be seen in the estimation of zeros of a polynomial as shown in the work of Bhunia et al. [6].

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors would like to thank the Editor-in-Chief and the anonymous reviewers for their insightful suggestions and careful reading of the manuscript.

References

1. A. Abu-Omar and F. Kittaneh, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mountain J. Math. **45** (2015), 1055-1065.
2. J. Aujla and F. Silva, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. **369** (2003), 217-233.
3. M. Al-Dolat and I. Jaradat, *A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds*, Filomat **37** (2023), 971-977.
4. M.W. Alomari, *On Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Ricerche mat (2022). https://doi.org/10.1007/s11587-022-00689-2
5. M.W. Alomari, M. Bakherad, M. Hajmohamadi, C. Chesneau, V. Leiva and C. Martin-Barreiro, *Improvement of Furuta's inequality with applications to numerical radius*, Mathematics **11** (2023), 36.
6. P. Bhunia, S. Bag and K. Paul, *Numerical radius inequalities and its applications in estimation of zeros of polynomials*, Linear Algebra Appl. **573** (2019), 166-177.

7. P. Bhunia, M. Gürdal, K. Paul, A. Sen and R. Tapdigoglu, *On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities*, Numer. Funct. Anal. Optim. **44** (2023), 970-986.
8. P. Bhunia, S.S. Dragomir, M.S. Moslehian and K. Paul, *Lectures on Numerical Radius Inequalities*, Springer Cham., 2022.
9. I. Chalendar, E. Fricain, M. Gürdal and M. Karaev, *Compactness and Berezin symbols*, Acta Sci. Math. (Szeged) **78** (2012), 315-329.
10. S.S. Dragomir, *Some inequalities generalizing Kato's and Furuta's results*, Filomat **28** (2014), 179-195.
11. S.S. Dragomir, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math. **5** (2009), 269-278.
12. M. El-Haddad and F. Kittaneh, *Numerical radius inequalities for Hilbert space operators II*, Studia Math. **182** (2007), 133-140.
13. T. Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc. **120** (1994), 785-787.
14. M.T. Garayev, H. Guediri, M. Gürdal and G.M. Alsahli, *On some problems for operators on the reproducing kernel Hilbert space*, Linear Multilinear Algebra **69** (2021), 2059-2077.
15. B. Güntürk and M. Gürdal, *On some refining inequalities via Berezin symbols*, Honam Math. J. to appear, 2024.
16. M. Gürdal and M. Alomari, *Improvements of some Berezin radius inequalities*, Constr. Math. Anal. **5** (2022), 141-153.
17. M. Gürdal and H. Başaran, *Advanced refinements of Berezin number inequalities*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **72** (2023), 386-396.
18. M. Gürdal and R. Tapdigoglu, *New Berezin radius upper bounds*, Proc. Inst. Math. Mech. **49** (2023), 210-218.
19. M.B. Huban, H. Başaran and M. Gürdal, *New upper bounds related to the Berezin number inequalities*, J. Inequal. Spec. Funct. **12** (2021), 1-12.
20. M.B. Huban, H. Başaran and M. Gürdal, *Berezin number inequalities via convex functions*, Filomat **36** (2022), 2333-2344.
21. T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208-212.
22. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), 11-17.
23. F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), 73-80.
24. F. Kittaneh and H.R. Moradi, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **23** (2020), 1117-1125.
25. C.A. McCarthy, c_p, Israel J. Math. **5** (1967), 249-271.
26. R.K. Nayak, *Advancement of numerical radius inequalities of operator and product of operators*, Iran J. Sci. (2024). https://doi.org/10.1007/s40995-024-01603-1
27. V. Stojiljković, R. Ramaswamy, O.A.A. Abdelnaby and S. Radenovic, *Some refinements of the tensorial inequalities in Hilbert spaces*, Symmetry **15** (2023), 925.
28. V. Stojiljkovic, *Twice differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces*, Eur. J. Pure Appl. Math. **16** (2023), 1421-1433.
29. V. Stojiljkovic, *Twice differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert Spaces*, Electron. J. Math. Anal. Appl. **11** (2023), 1-15.
30. V. Stojiljkovic and S.S. Dragomir, *Refinement of the Cauchy-Schwartz inequality with refinements and generalizations of the numerical radius type inequalities for operators*, Ann. Math. Comp. Sci. **21** (2024), 33-43.
31. V. Stojiljković and M. Gürdal, *Berezin radius type inequalities for functional Hilbert space operators*, Electron. J. Math. **7** (2024), 35-44.

Vuk Stojiljković Some of his research interests are operator theory, mathematical inequalities, special functions, fractional calculus.

Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 3, 21000 Novi Sad, Serbia.
Mathematical Grammar School, Kraljice Natalije 37, Belgrade, Serbia.
e-mail: vuk.stojiljkovic999@gmail.com

Mehmet Gürdal received Ph.D. from Suleyman Demirel University. He is currently a professor at Suleyman Demirel University. His research interests are operator theory, functional analysis and complex analysis.

Department of Mathematics, Süleyman Demirel University, 32260, Isparta, Turkey.
e-mail: gurdalmehmet@sdu.edu.tr