J. Appl. & Pure Math. Vol. 6(2024), No. 3 - 4, pp. 177 - 189
https://doi.org/10.23091/japm.2024.177

CLASSES OF HIGHER ORDER CONVERGENT ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, we suggest and analyze new higher order classes of iterative methods for solving nonlinear equations by using variational iteration technique. We present several examples to illustrate the efficiency of the proposed methods. Comparison with other similar methods is also given. New methods can be considered as an alternative of the existing methods. This technique can be used to suggest a wide class of new iterative methods for solving nonlinear equations.

AMS Mathematics Subject Classification : 34A34, 34G20, 35Qxx.
Key words and phrases : Nonlinear equations, variational iteration technique, Newton's method, Taylor series, convergence, examples.

1. INTRODUCTION

It is well known that a wide class of problems, which arises in various branches of mathematical and engineering science can be studied by the nonlinear equation of the form $f(x) = 0$. Numerical methods for finding the approximate solutions of the nonlinear equation are being developed by using several different techniques including Taylor series, quadrature formulas, homotopy perturbation method, decomposition techniques and variational iteration technique, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein. The classical Newton's method for solving nonlinear equation is written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Received April 27, 2024. Revised June 22, 2024. Accepted July 8, 2024.

This is an important and basic method [23], which converges quadratically. To improve the order of convergence, many modified methods have been suggested in open literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Motivated and inspired by the research going on in this direction, we suggest and analyze new iterative methods for solving the nonlinear equations. In this paper, we implement the variational iteration technique by considering an auxiliary function involving an arbitrary predictor function. The predictor function is $\phi(x)$ having convergence order $p \geq 1$. The predictor function helps to obtain iterative methods of convergence order $p+1$. This is the modified form of variational iteration technique for finding simple roots of nonlinear equations. We would like to mention that the variational iteration technique was introduced by Inokuti et al. [10]. However, the technique was developed for solving a variety of diverse problems [14, 15, 16, 17, 18]. Essentially using the same idea Noor and Shah [15, 16] has suggested and analyzed some iterative methods for finding simple roots and multiple roots of the nonlinear equations. Now we have applied variational iteration technique for obtaining higher order methods. New methods are modified with less number of functional evaluations which raised the efficiency index of these methods. We also discuss the convergence criteria of these new iterative methods. Comparison with other similar methods is also given. Several examples are given to re-confirm the efficiency of the suggested methods.

2. CONSTRUCTION OF ITERATIVE METHODS

We consider the nonlinear equation

$$f(x) = 0. \tag{1}$$

This nonlinear equation can be written in the following equivalent form as:

$$x = H(x). \tag{2}$$

Let $\phi(x)$ be an iteration function of order $p \geq 1$, and $g(x)$ be an auxiliary arbitrary function. We consider the function defined as:

$$H(x) = \phi(x) + \lambda\,[f(x)g(x)]^p, \tag{3}$$

where λ is Lagrange's multiplier.

Using the optimality condition, we obtain the value of as:

$$\lambda = -\frac{\phi'(x)}{p[f(x)g(x)]^{p-1}[f'(x)g(x) + f(x)g'(x)]}. \tag{4}$$

From (3) and (4), we obtain

$$H(x) = \phi(x) - \frac{\phi'(x)f(x)g(x)}{p\,[f'(x)g(x) + f(x)g'(x)]}. \tag{5}$$

Now combining (2) and (5), we obtain

$$x = H(x) = \phi(x) - \frac{\phi'(x)f(x)g(x)}{p\,[f'(x)g(x) + f(x)g'(x)]}. \tag{6}$$

This is another fixed point problem. We use this fixed point formulation to suggest the following iterative scheme as:

Algorithm 2.1. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = \phi(x_n) - \frac{\phi'(x_n)f(x_n)g(x_n)}{p\,[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}.$$

This is the main recurrence relation involving the iteration function $\phi(x_n)$, which generates the iterative methods of order $p+1$. This scheme is also introduced in [10]. We select $\phi(x_n)$ as predictor having the order of convergence $p \geq 1$. We note that, if we take $\phi(x_n) = x_n$, and $p = 1$, then Algorithm 2.1 collapses to the main recurrence relation [11]. It is important to mention that Noor [11] has introduced some efficient iterative methods which can be considered as the alternate of Newton method and Halley method and its variants are also the specialty of the work. For simplicity, we first consider the well known Newton method as an auxiliary iterative function and consider the associated function as

$$\phi(x) = x - \frac{f(x)}{f'(x)}. \tag{7}$$

Then

$$\phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}. \tag{8}$$

Using (7) and (8) with $p = 1$, in Algorithm 2.1, we have

$$x = x - \frac{f(x)}{f'(x)} - \frac{f(x)g(x)f''(x)}{2[f'(x)]^2\,[f'(x)g(x) + f(x)g'(x)]}, \tag{9}$$

which is another fixed point formulation. This fixed point formula enables us to suggest the iterative method for solving nonlinear equations as:

Algorithm 2.2. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)g(x_n)f''(x_n)}{2\,[f'(x_n)]^2\,[f'(x_n)g(x_n) + f(x_n)g'(x_n)]},$$

For different values of the auxiliary function $g(x)$ we can obtain several Householder type iterative methods for solving nonlinear equations. Now we again

implement the Algorithm 2.1 to obtain some iterative methods of fifth-order convergence. For this purpose we will select the well known Traub's method[23] as predictor. Different values of the auxiliary function will diversify the main scheme and new methods will be suggested. Second derivative will be removed by suitable approximation and methods will be modified with better efficiency index by decreasing the functional evaluation by appropriate substitution.
Let us consider the auxiliary function

$$\phi(x) = y - \frac{f(y)}{f'(y)}, \tag{10}$$

where

$$y = x - \frac{f(x)}{f'(x)}. \tag{11}$$

$$\phi'(x) = \frac{f(y)f''(y)}{[f'(y)]^2} y', \tag{12}$$

$$\phi'(x) = \frac{f(y)f''(y)}{[f'(y)]^2} y', \tag{13}$$

$$y' = \frac{f(x)f''(x)}{[f'(x)]^2}. \tag{14}$$

Using (10) to (13) with $p = 4$, in relation (6), we obtain the modified form as:

$$x = y - \frac{f(y)}{f'(y)} - \frac{f(y)f''(y)}{[f'(y)]^2} \frac{f(x)f''(x)}{[f'(x)]^2} \left[\frac{f(x)g(x)}{4\,[f'(x)g(x) + f(x)g'(x)]} \right]. \tag{15}$$

Algorithm 2.3. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f(y_n)f''(y_n)}{[f'(y_n)]^2} \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2} \left[\frac{f(x_n)g(x_n)}{4\,[f'(x_n)g(x_n) + f(x_n)g'(x_n)]} \right].$$

We replace second derivative by a suitable substitution involving only the first derivative. Using the Taylor series, we have

$$f(y) \approx f(x) + (y - x)f'(x) + \frac{(y - x)^2}{2} f''(x). \tag{16}$$

Taking

$$y = x - \frac{f(x)}{f'(x)}. \tag{17}$$

From (15) and (16), we have

$$f(y) = \frac{[f(x)]^2 f''(x)}{2[f'(x)]^2}. \tag{18}$$

From which, we have

$$f''(x) = \frac{2f(y)[f'(x)]^2}{[f(x)]^2}. \tag{19}$$

Similarly using the Taylor series, we have also

$$f(z) \approx f(y) + (z-y)f'(y) + \frac{(z-y)^2}{2} f''(y). \tag{20}$$

Taking

$$z = y - \frac{f(y)}{f'(y)}. \tag{21}$$

From (19) and (20), we obtain

$$f(z) = \frac{[f(y)]^2 f''(y)}{2[f'(y)]^2}. \tag{22}$$

Simplifying, we obtain

$$f''(y) = \frac{2f(z)[f'(y)]^2}{[f(y)]^2}. \tag{23}$$

Using (18) and (22), in Algorithm 2.3, we obtain the following fixed point formulation

$$x = z - \frac{f(z)g(x)}{[f'(x)g(x) + f(x)g'(x)]}, \tag{24}$$

which allows the following iterative method for solving nonlinear equations as:

Algorithm 2.4. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}.$$

Algorithm 2.4 is the main recurrence relation for generating the iterative methods. To elaborate and convey the main idea, we now consider some special cases of Algorithm 2.4 for particular choice of the auxiliary $g(x_n)$. and $n = 0, 1, 2, \cdots$. Now we want to improve the efficiency index of the derived methods. We will obtain the new methods with less number of functional evaluations with the same convergence order. We will eliminate $f'(y_n)$ and will replace it by some

approximation.
We consider the following finite difference technique:

$$f''(x) \approx \frac{f'(y) - f'(x)}{y - x}. \tag{25}$$

With the help of (18) and (24), we obtain

$$f'(y) \approx \frac{f'(x)}{f(x)} \left[f(x) - 2f(y) \right]. \tag{26}$$

Now using (25) in Algorithm 2.4, for all values of n. We get the modified relation as:

Algorithm 2.5. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n)}{[f(x_n) - 2f(y_n)]},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n) - \alpha f(x_n)}, n = 0, 1, 2, \cdots .$$

Algorithm 2.6. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n)}{[f(x_n) - 2f(y_n)]},$$

$$x_{n+1} = z_n - \frac{f(z_n)f(x_n)}{f'(x_n)f(x_n) - 2\alpha f(y_n)},$$

Algorithm 2.7. For a given x_0 calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n)}{[f(x_n) - 2f(y_n)]},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n) + \alpha \left[f(x_n) - 2f(y_n) \right]},$$

3. CONVERGENCE ANALYSIS

In this section, we consider the convergence criteria of Algorithm 2.1 which is the main and general scheme.

Theorem 3.1. Let $\phi(x_n)$ be the iteration function of order $p \geq 1$ at r, where r is a root of $f(x)$. Suppose that $\phi(x)$ is continuously differentiable at r, then the Algorithm 2.1 has convergence order at least $p+1$.

Proof. Let us consider the function (6) and assume that r is the root of $f(x)$. and $\phi(x)$ is of pth-order convergent iterative function, so we have the following results

$$f(r) = 0, \tag{27}$$

$$\begin{aligned} &\phi'(r) = 0, \\ &\vdots \end{aligned} \tag{28}$$

$$\phi^{(p-1)}(r) = 0, \tag{29}$$

$$\phi^{p}(r) \neq 0 \tag{30}$$

Let us consider

$$\eta(x) = \frac{f(x)g(x)}{f'(x)g(x) + f(x)g'(x)}. \tag{31}$$

Where

$$\eta(r) = 0. \tag{32}$$

because $(f(r) = 0)$

Using

$$f(r) = 0 \tag{33}$$

we have

$$\eta(r) = 0 \tag{34}$$

and

$$\eta'(r) = 1. \tag{35}$$

From (6), we have

$$H(x) = \phi(x) - \frac{1}{p}\phi'(x)\,\eta(x). \tag{36}$$

From which it follows that

$$H(r) = r. \tag{37}$$

Differentiating (34), we obtain

$$H^{(p)}(x) = \phi^{(p)}(x) - \frac{1}{p}\sum_{n=0}^{p} \begin{pmatrix} p \\ n \end{pmatrix} \phi^{(p-n+1)}(x)\eta^{(n)}(x), \tag{38}$$

and

$$H^{(p+1)}(x) = \phi^{(p+1)}(x) - \frac{1}{p}\sum_{n=0}^{p+1} \binom{p+1}{n} \phi^{(p-n+2)}(x)\eta^{(n)}(x). \tag{39}$$

Simple computations yield that

$$H^{(p)}(r) = 0, \tag{40}$$

because

$$\phi'(r) = \phi''(r) = ... \ \phi^{(p-1)}(r) = 0, \ \ \eta(r) = 0, \ \ \eta'(r) = 1,$$

and

$$H^{(p+1)}(r) = -\frac{1}{p}\phi^{(p+1)}(r) \neq 0, \tag{41}$$

because

$$\phi^{(p+1)}(r) \neq 0. \tag{42}$$

Hence proved that Algorithm 2.1 generates the iterative methods of order at least $p + 1$.

4. NUMERICAL RESULTS

In this section, we present some numerical examples to illustrate the efficiency and performance of the new developed methods. We compare the new developed fifth-order convergent methods described as Algorithm 2.5 (FAS1), Algorithm 2.6 (FAS2) and Algorithm 2.7 (FAS3) with some known methods i.e Ham et al.'s method (HCM)[7], Kou's method (KM) [16], and Javidi's method JM [14], The following stopping criteria are used for computer programs:
(i) $|x_{n+1} - x_n| < \varepsilon$, (ii) $|f(x_{n+1})| < \varepsilon$.
Computational order of convergence is calculated by the formula [**?**, 22]

$$\rho \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}.$$

We note that, all new developed methods do not require the calculation of second derivative to carry out the iterations. All calculations are done using the Maple using 60 digits floating point arithmetics (Digits :=300). We use $\varepsilon = 10^{-64}$. We test the following some examples for numerical results and comparison.

Example 4.1. We consider the nonlinear equation $f_1(x) = \sin^2 x - x^2 + 1$. We consider the different values of parameter α for all the methods to compare the numerical results in the following Table.

Table 4.1(Numerical comparison for example 4.1)

Method	IT	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	COC
$\alpha = 0$					
FAS1	4	1.4044916482153412260350868177 9	1.20e-299	2.04e-276	4.99878
FAS2	4	1.40449164821534122603508681779	1.20e-299	2.04e-276	4.99878
FAS3	4	1.40449164821534122603508681779	1.20e-299	2.04e-276	4.99878
KM	4	1.40449164821534122603508681779	1.20e-299	4.47e-700	4.89027
HCM	5	1.40449164821534122603508681779	1.20e-299	1.90e-297	4.99913
JM	5	1.40449164821534122603508681779	1.20e-299	3.46e-283	4.99882
$\alpha = 0.5$					
FAS1	5	1.40449164821534122603508681779	1.20e-299	1.61e-262	4.99829
FAS2	5	1.40449164821534122603508681779	1.20e-299	9.46e-282	4.99914
FAS3	5	1.40449164821534122603508681779	1.20e-299	3.22e-293	4.99931
KM	4	1.40449164821534122603508681779	1.20e-299	4.47e-700	4.89027
HCM	5	1.40449164821534122603508681779	1.20e-299	1.90e-297	4.99913
JM	5	1.40449164821534122603508681779	1.20e-299	3.46e-283	4.99882
$\alpha = 1$					
FAS1	4	1.40449164821534122603508681779	1.20e-299	4.30e-890	4.48558
FAS2	4	1.40449164821534122603508681779	1.20e-299	2.92e-288	4.99957
FAS3	4	1.40449164821534122603508681779	1.20e-299	1.30e-257	4.99773
KM	4	1.40449164821534122603508681779	1.20e-299	4.47e-700	4.89027
HCM	5	1.40449164821534122603508681779	1.20e-299	1.90e-297	4.99913
JM	5	1.40449164821534122603508681779	1.20e-299	3.46e-283	4.99882

Table 4.1 showcases the comprehensive numerical results derived from Example 4.1. The meticulous computations commence with an initial guess of $x_0 = 2$. Notably, the iteration count for the newly introduced methods aligns precisely with that of established fifth-order methods, illustrating the remarkable efficiency and precision of our approach. Last column shows the computational order of convergence of all methods regarding example 4.1.

Example 4.2. We consider the nonlinear equation $f_2(x) = x^2 - e^x - 3x + 2$. We consider the different values of parameter α for all the methods to compare the numerical results in the following Table.

Table 4.2(Numerical comparison for example 4.2)

Method	IT	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	COC
$\alpha = 0$					
FAS1	5	0.25753028543986076045536730494	0.00e-001	1.33e-152	4.88991
FAS2	5	0.25753028543986076045536730494	0.00e-001	1.33e-152	4.88991
FAS3	5	0.25753028543986076045536730494	0.00e-001	1.33e-152	4.88991
KM	6	0.25753028543986076045536730494	0.00e-001	4.60e-123	5.16943
HCM	5	0.25753028543986076045536730494	0.00e-001	3.27e-173	4.83259
JM	5	0.25753028543986076045536730494	1.00e-299	1.15e-152	4.78567
$\alpha = 0.5$					
FAS1	5	0.25753028543986076045536730494	0.00e-001	8.01e-137	4.93270
FAS2	5	0.25753028543986076045536730494	0.00e-001	3.92e-154	4.89475
FAS3	5	0.25753028543986076045536730494	0.00e-001	4.52e-127	4.95395
KM	6	0.25753028543986076045536730494	0.00e-001	4.60e-123	5.16943
HCM	5	0.25753028543986076045536730494	0.00e-001	3.27e-173	4.83259
JM	5	0.25753028543986076045536730494	1.00e-299	1.15e-152	4.78567
$\alpha = 1$					
FAS1	5	0.25753028543986076045536730494	0.00e-001	5.18e-127	4.88705
FAS2	5	0.25753028543986076045536730494	0.00e-001	7.48e-156	4.90065
FAS3	5	0.25753028543986076045536730494	0.00e-001	3.60e-116	5.00178
KM	6	0.25753028543986076045536730494	0.00e-001	4.60e-123	5.16943
HCM	5	0.25753028543986076045536730494	0.00e-001	3.27e-173	4.83259
JM	5	0.25753028543986076045536730494	1.00e-299	1.15e-152	4.78567

Table 4.2 shows the efficiency of the methods for example 4.2. We use the initial guess $x_0 = 3.5$ for the computer program. Number of iterations and computational order of convergence gives us an idea about the better performance of the new methods.

Example 4.3. We consider the nonlinear equation $f_3(x) = \cos x - x$. We consider the different values of parameter α for all the methods to compare the numerical results in the following Table.

Table 4.3(Numerical comparison for example 4.3)

Method	IT	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1} - x_n\rvert$	COC
$\alpha = 0$					
FAS1	4	0.73908513321516064165531208767	1.00e-300	7.61e-098	4.93459
FAS2	4	0.73908513321516064165531208767	1.00e-300	7.61e-098	4.93459
FAS3	4	0.73908513321516064165531208767	1.00e-300	7.61e-098	4.93459
KM	4	0.73908513321516064165531208767	1.00e-300	1.28e-089	4.91236
HCM	4	0.73908513321516064165531208767	1.00e-300	6.37e-110	4.70798
JM	4	0.73908513321516064165531208767	1.00e-300	1.05e-117	4.69894
$\alpha = 0.5$					
FAS1	4	0.73908513321516064165531208767	1.00e-300	3.19e-096	5.09159
FAS2	4	0.73908513321516064165531208767	1.00e-300	8.77e-096	4.91217
FAS3	4	0.73908513321516064165531208767	1.00e-300	4.77e-087	4.91266
KM	4	0.73908513321516064165531208767	1.00e-300	1.28e-089	4.91236
HCM	4	0.73908513321516064165531208767	1.00e-300	6.37e-110	4.70798
JM	4	0.73908513321516064165531208767	1.00e-300	1.05e-117	4.69894
$\alpha = 1$					
FAS1	4	0.73908513321516064165531208767	1.00e-300	3.24e-070	5.03982
FAS2	4	0.73908513321516064165531208767	1.00e-300	4.44e-094	4.99451
FAS3	4	0.73908513321516064165531208767	1.00e-300	9.12e-083	4.87857
KM	4	0.73908513321516064165531208767	1.00e-300	1.28e-089	4.91236
HCM	4	0.73908513321516064165531208767	1.00e-300	6.37e-110	4.70798
JM	4	0.73908513321516064165531208767	1.00e-300	1.05e-117	4.69894

Table 4.3 shows the efficiency of the methods for example 4.3. We use the initial guess $x_0 = 1.5$ for the computer program. Number of iterations and computational order of convergence gives us an idea about the better performance of the new methods.

Example 4.4. We consider the nonlinear equation
$f_4(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$.
We consider the different values of α parameter α for all the methods to compare the numerical results in the following Table.

Table 4.4 shows the efficiency of the methods for example 4.4. We use the initial guess $x_0 = -2.0$ for the computer program. Number of iterations and computational order of convergence gives us an idea about the better performance of the new methods.

Table 4.4(Numerical comparison for example 4.4)

Method	IT	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	COC
$\alpha=0$					
FAS1	5	-1.20764782713091892700941675836	0.00e-001	5.02e-073	5.05618
FAS2	5	-1.20764782713091892700941675836	0.00e-001	5.02e-073	5.05618
FAS3	5	-1.20764782713091892700941675836	0.00e-001	5.02e-073	5.05618
KM	8	-1.20764782713091892700941675836	0.00e-001	5.26e-185	5.00463
HCM	6	-1.20764782713091892700941675836	0.00e-001	0.00e-001	4.99987
JM	6	-1.20764782713091892700941675836	0.00e-001	2.65e-200	4.99770
$\alpha=0.5$					
FAS1	5	-1.20764782713091892700941675836	0.00e-001	8.90e-075	5.07422
FAS2	5	-1.20764782713091892700941675836	0.00e-001	6.49e-073	5.05246
FAS3	5	-1.20764782713091892700941675836	0.00e-001	3.53e-072	5.03374
KM	8	-1.20764782713091892700941675836	0.00e-001	5.26e-185	5.00463
HCM	6	-1.20764782713091892700941675836	0.00e-001	0.00e-001	4.99987
JM	6	-1.20764782713091892700941675836	0.00e-001	2.65e-200	4.99770
$\alpha=1$					
FAS1	5	-1.20764782713091892700941675836	0.00e-001	1.86e-070	5.02371
FAS2	5	-1.20764782713091892700941675836	0.00e-001	2.95e-073	5.06389
FAS3	5	-1.20764782713091892700941675836	0.00e-001	3.87e-075	5.11515
KM	8	-1.20764782713091892700941675836	0.00e-001	5.26e-185	5.00463
HCM	6	-1.20764782713091892700941675836	0.00e-001	0.00e-001	4.99987
JM	6	-1.20764782713091892700941675836	0.00e-001	2.65e-200	4.99770

5. CONCLUSION

In this paper, we have presented iterative methods for solving nonlinear equations by using the variational iteration technique. This technique also generates the Halley-like and Householder type iterative methods. Using appropriate substitutions, we modified the methods and obtained the second derivative-free predictor-corrector type of iterative methods. Per iteration methods require three computations of the given function and its derivative. These fifth-order methods are compared with some existing methods and the proposed methods have been observed to have at least better performance. The sixth-ordered convergent methods are also free from second derivative and have better performance. If we consider the definition of efficiency index [23] as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method, Algorithms 2.13-2.15 all obtained fifth-order convergent methods have the efficiency index equal to $5^{\frac{1}{4}} \approx 1.495348781$. The presented approach can also be applied further to obtain higher order convergent methods.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions, which improved the quality of this paper.

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