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MULTIPLICATION $\underleftarrow{AB}$ AND DIVISION OF MATRICES

HASAN KELEŞ

Abstract. This study is about division and right multiplication in matrices. The discussion of the properties of multiplication and division is examined. Some results between multiplication based on the row-column relationship and division based on the same relationship are discussed. The commonalities of these results between the processes are emphasized. Examples of unrealized properties are given. The algebraic properties of the newly defined right product and division are clarified in matrices. The properties of the known multiplication operation and new situations between right multiplication and division are investigated. Some results are declared between the transpositions of matrices and the obtained rules of operations. New results are discussed belong the equations $\underleftarrow{XA} = B$ and $\underleftarrow{AX} = B$. New ideas are proposed for solving these equations. The contribution The contribution is explained the equation $AB = \underleftarrow{BA}$ to division operation. Many new properties, lemmas and theorems are presented on this subject.

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1. Introduction

Leibnitz calculated the determinant in 1693. The first use of matrices was seen in Lagrange's studies in 1700 [1]. Cayley introduced the multiplication of matrices to world of science with his first paper named "A Memoir on the Theory of Matrices" in 1850 [1, 3]. Matrix multiplication has been used as a skeleton in the application areas of many sciences [2]. Banachiewicz used the Cracovian Product to solve linear equations in the 1930s [4].

Let $\mathbb{F}$ be a field. The following notations are given:

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- The set of all $n-$ by $-n$ matrices over a field $\mathbb{F}$ is denoted by $\mathbb{K}_n(\mathbb{F}) = \{[a_{ij}]_n | a_{ij} \in \mathbb{F}\}$.
- The set of all regular matrices order n over a field $\mathbb{F}$ is denoted by $\mathbb{M}_n(\mathbb{F}) = \{[a_{ij}]_n | a_{ij} \in \mathbb{F}\}$.
- The transpose of a matrix $A \in \mathbb{K}_n(\mathbb{F})$ is denoted by A^T.

A matrix A is said to be similar to matrix B if there exists a regular matrix P such that $A = P^{-1}BP$. There is not only one matrix B that is similar to the matrix A. Any regular matrix has infinitely many similar matrices. Generally, $AB \neq BA$. This property is made operation of right product necessary. The definitions operations of left(known) and right product are given below.

Definition 1.1. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times r}$ be any two matrices and let matrix C be the product of matrix A with matrix B. The product matrix C is

$$C = AB = [c_{ik}]_{m\times r}, \text{where } c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

The row-column method is used in the right multiplication as in the left multiplication.

Definition 1.2 (Keleş [9])**.** Let $A = [a_{ij}]_{r\times m}$ and $B = [b_{jk}]_{n\times r}$ be any two matrices. The matrix $C = \left[\sum_{i=1}^{r} b_{ki}a_{ji}\right]_{n\times m}$ is called the right product of matrix A and matrix B. It is denoted by

$$\underleftarrow{AB} = C = [c_{kj}]_{n\times m}, \text{where } c_{kj} = \sum_{i=1}^{r} b_{ki}a_{ji}.$$

Example 1.3. For two matrices $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$, $\underleftarrow{AB}$ is

$$\underleftarrow{AB} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 2 \\ 1 & -2 & -1 \end{bmatrix}. \quad (1)$$

There are not $\underleftarrow{BA}$ and AB for these matrices.

Generally,

$$\underleftarrow{BA} \neq \underleftarrow{AB}.$$

Example 1.4. For two matrices $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$,

$$\underleftarrow{AB} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 1 & -2 \end{bmatrix},$$

$$\underleftarrow{BA} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 6 & 0 \end{bmatrix},$$

$$\underleftarrow{BA} \neq \underleftarrow{AB}.$$

The following theorem is related to the commutative property.

Theorem 1.5 (Keleş [9]). *If any two matrices $A, B \in \mathbb{K}_n(\mathbb{F})$, then the following equality holds.*

$$\underleftarrow{AB} = BA.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, we have $A = [a_{ij}]_n$, $B = [b_{jk}]_n$, $A^T = [a_{ji}]_n$ and $B^T = [b_{kj}]_n$ then

$$\left((BA)^T\right)^T = BA$$

by Definition 1.2 it is deduced that:

$$\underleftarrow{AB} = BA.$$

□

The Cracovian Product is done by the column by column multiplication method. The product of the reciprocal columns of two matrices with the same columns is defined as the Cracovian Product. If A, B are any two square matrices of order n then the Cracovian Product is

$$A \wedge B = B^T A, \text{ where } B^T \text{ is transpose of matrix } B \text{ in [5].}$$

Lemma 1.6 (Zehfuss [4]). *Let $A, B \in \mathbb{K}_n(\mathbb{F})$ be any two matrices. The following equality holds.*

$$\left(A \wedge B^T\right)^T = B^T \wedge A.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$(A \wedge B^T)^T = (BA)^T = A^T B^T = B^T \wedge A.$$

□

Lemma 1.7 (Zehfuss [4]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$(A^T \wedge B)^T = B \wedge A^T.$$

Proof. If $A, B \in \mathbb{M}_n(\mathbb{F})$, then

$$(A^T \wedge B)^T = (B^T A^T)^T = AB = B \wedge A^T.$$

□

Lemma 1.8 (Zehfuss [4]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$(A \wedge B)^T = B \wedge A.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$(A \wedge B)^T = (B^T A)^T = A^T B = B \wedge A.$$

□

Corollary 1.9 (Zehfuss [4]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$(B \wedge A)^T = A \wedge B.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$(B \wedge A)^T = \left(A^T B\right)^T = B^T A = A \wedge B.$$

□

Corollary 1.10 (Zehfuss [4])**.** *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\left(A^T \wedge B^T\right)^T = B^T \wedge A^T.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$(A^T \wedge B^T)^T = (BA^T)^T = AB^T = B^T \wedge A^T.$$

□

Definition 1.11 (Keleş [6, 9])**.** Let $A, B \in \mathbb{M}_n(\mathbb{F})$.

(i) The determinant of the new matrix obtained by writing the i^{th} column of the matrix B on the j^{th} column of the matrix A is called the *co-divisor by column* of the matrix B by the column on the matrix A. It is denoted by $\binom{B_{i_j}}{A}$. The matrix co-divisor by columns is $\left[\binom{B_{i_j}}{A}_{ij}\right]$.
For the two matrices satisfying the above conditions, the matrix division is also given by

$$\frac{B}{A} := \frac{1}{|A|}\left[\left(\tbinom{B}{A} i_j\right)_{ji}\right]$$

and the solution of the equation $AX = B$ is $X = \dfrac{B}{A}$.

(ii) The determinant of the new matrix obtained by writing the i^{th} row of the matrix B on the j^{th} row of the matrix A is called the *co-divisor by row* of the matrix B by the row on the matrix A. It is denoted by $\binom{BA}{ij}$. The matrix co-divisor by rows is $\left[\binom{BA}{ij}_{ij}\right]$.

Their numbers are n^2.

Let us the following theorem.

Theorem 1.12 (Keleş [8])**.** *Let $A, B \in \mathbb{M}_n(\mathbb{F})$, then the solution of the linear matrix equation $XA = B$ is*

$$X = \left(\frac{B^T}{A^T}\right)^T.$$

Proof. The solution of the equation $XA = B$ is $X = \dfrac{B}{A}$ for all $A, B, X \in \mathbb{M}_n(\mathbb{F})$.

$$XA = B \iff (XA)^T = B^T \iff A^T X^T = B^T. \tag{2}$$

$$X^T = \frac{1}{|A^T|}\left[\binom{B^T A^T}{ij}_{ij}\right] \Longrightarrow X = \frac{1}{|A^T|}\left[\binom{B^T A^T}{ij}_{ji}\right]^T \tag{3}$$

$$X = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T . \tag{4}$$

□

Corollary 1.13. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then The following equation holds.*

$$AB = \underleftarrow{BA}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then by Theorem 1.5

$$AB = \underleftarrow{BA}.$$

Because the product of the rows of matrix A and the columns of matrix B is AB. The product of $(B^T A^T)^T$ is explained the same description too. This product is equal to the right product of $\underleftarrow{BA}$. □

Example 1.14. For two matrices $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $\underleftarrow{BA}$ is

$$\underleftarrow{BA} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}.$$

The following equality is achieved.

$$\underleftarrow{BA} = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} = AB.$$

Similarity,

$$\underleftarrow{AB} = \begin{bmatrix} 3 & 8 \\ 0 & 2 \end{bmatrix} = BA.$$

The property similar to the associative property in matrix multiplication is given below.

Proposition 1.15 (Keleş [6])**.** *If $A, B, C \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A(BC)} = \underleftarrow{(AB)C}.$$

Proof. Suppose $A, B, C \in \mathbb{K}_n(\mathbb{F})$,

$$\underleftarrow{A(BC)} = \underleftarrow{(BC)}A = (CB)A = C(BA) = C\underleftarrow{(AB)} = \underleftarrow{(AB)C}. \tag{5}$$

□

The distributive property of right multiplication on addition is given below.

Proposition 1.16 (Keleş [6])**.** *If $A, B, C \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A\,(B + C)} = \underleftarrow{AB} + \underleftarrow{AC}.$$

Proof. Suppose $A, B, C \in \mathbb{K}_n(\mathbb{F})$,

$$\underleftarrow{A\,(B+C)} = (B+C)\,A = BA + CA = \underleftarrow{AB} + \underleftarrow{AC}.$$

□

Lemma 1.17 (Keleş [6])**.** *Let $A, B \in \mathbb{M}_n(\mathbb{F})$. The following hold.*

(i) $A = \left(\dfrac{(\underleftarrow{BA})^T}{B^T}\right)^T.$

(ii) $B = \dfrac{\underleftarrow{BA}}{A}.$

Proof. The proof of the lemma is clear by Theorem 1.12 and Corollary 1.13. □

Proposition 1.18 (Keleş [6])**.** *If $A \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{\overbrace{A \ldots A}^{s\text{-times}}} = A^s, \text{ where } s \in \mathbb{Z}^+.$$

Proposition 1.19 (Keleş [6])**.** *Suppose $A, B \in \mathbb{M}_n(\mathbb{F})$, If $\underleftarrow{BA} = I_n$ and $\underleftarrow{AB} = I_n$ then $B = A^{-1}$.*

Proof. If $\underleftarrow{BA} = I_n$ and $\underleftarrow{AB} = I_n$. By Corollary 1.13

$$\underleftarrow{BA} = AB = I_n, \ \underleftarrow{AB} = BA = I_n$$

then $B = A^{-1}$. □

The inverse of the right product is similar to that of the matrix product. It is as that.

$$\underleftarrow{(AB)}^{-1} = (BA)^{-1} = A^{-1}B^{-1} = \underleftarrow{B^{-1}A^{-1}}.$$

Similar situation is also ensured in transpose. That means:

$$\underleftarrow{(AB)}^T = (BA)^T = A^T B^T = \underleftarrow{B^T A^T}.$$

The relation between the right matrix product and the Cracovian Product is given by the following lemma.

Lemma 1.20 (Keleş [6])**.** *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$A \wedge B = \underleftarrow{AB^T}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by Theorem 1.13 then

$$A \wedge B = (B)^T A = \underleftarrow{AB^T}.$$

□

Lemma 1.21 (Keleş [6])**.** *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$B^T \wedge A^T = \underleftarrow{B^T A}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by Corollary 1.13 then

$$B^T \wedge A^T = AB^T = \underleftarrow{B^T A}.$$

□

Lemma 1.22 (Keleş [6]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$A \wedge B^T = \underleftarrow{AB}$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by Corollary 1.13, then

$$A \wedge B^T = AB = \underleftarrow{AB}.$$

□

Corollary 1.23 (Keleş [6]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{AB^T} = A \wedge B.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ then

$$\underleftarrow{AB^T} = B^T A = A \wedge B.$$

□

Lemma 1.24 (Keleş [6]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\left(A^T \wedge B\right)^T = \underleftarrow{BA}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$\left(A^T \wedge B\right)^T = \left(B^T A^T\right)^T = AB = \underleftarrow{BA}.$$

□

Theorem 1.25 (Keleş [6]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\left(A^T \wedge B^T\right)^T = \left(\underleftarrow{A^T B}\right)^T.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$\left(A^T \wedge B^T\right)^T = \left(BA^T\right)^T = AB^T \tag{6}$$

$$\left(\underleftarrow{A^T B}\right)^T = \left(BA^T\right)^T = AB^T. \tag{7}$$

By Equation (6) and Equation (7) the following equality is obtained:

$$\left(A^T \wedge B^T\right)^T = \left(\underleftarrow{A^T B}\right)^T.$$

□

Corollary 1.26 (Keleş [6]). *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A^T B} = A^T \wedge B^T.$$

2. Multiplication $\underleftarrow{AB}$ and Division of Matrices

In this section, the properties between the equations $AX = B, XA = B$ with $\underleftarrow{AX} = B, \underleftarrow{XA} = B$ are analyzed. New solutions of these equations are investigated. These solutions are compared. Different results are obtained for regular matrices. Let us start with the following theorem.

Theorem 2.1. *For any matrices $A, B, X \in \mathbb{M}_n(\mathbb{F})$, if $\underleftarrow{XA} = B$, then the following equality holds.*

$$X = \frac{B}{A}.$$

Proof. If $\underleftarrow{XA} = B$ for $A, B \in \mathbb{M}_n(\mathbb{F})$, then by Theorem 1.5

$$\underleftarrow{XA} = AX,$$

and Definition 1.2, we get

$$X = \frac{B}{A}.$$

□

Corollary 2.2. *For any matrices $A, B, X \in \mathbb{M}_n(\mathbb{F})$, if $\underleftarrow{AX} = B$, then*

$$X = \left(\frac{B^T}{A^T}\right)^T.$$

Proof. The proof is clear by Theorem 1.12. □

The relationship between the equations and the Cracovian Product is analyzed with the following lemma.

Lemma 2.3. *For any $A, B, X \in \mathbb{M}_n(\mathbb{F})$, if $X \wedge A = B$, then*

$$X = \frac{B}{A^T}.$$

Proof. if $X \wedge A = B$, then by Lemma 1.20 we have

$$\underleftarrow{XA^T} = B.$$

and

$$A^T X = B \implies X = \frac{B}{A^T}.$$

□

Lemma 2.4. *For any $A, X, B \in \mathbb{M}_n(\mathbb{F})$, if $A \wedge X = B$ then*

$$X = \frac{B^T}{A^T}.$$

Proof. If $A \wedge X = B$ by Equation (7) and Corollary 1.26 then

$$(A \wedge X)^T = \left(X^T A\right)^T = A^T X = B^T.$$

$$X = \frac{B^T}{A^T}.$$

□

Theorem 2.5. *For any matrices $A, B, X, Y \in \mathbb{M}_n(\mathbb{F})$, if $\underleftarrow{(X+Y)A} = B$, there exist $B_1, B_2 \in \mathbb{M}_n(\mathbb{F})$ such that $\underleftarrow{XA} = B_1, \underleftarrow{YA} = B_2$ and $B = B_1 + B_2$ then the following equality holds.*

$$X + Y = \frac{B}{A}.$$

Proof. If $\underleftarrow{(X+Y)A} = B$, then we get

$$A(X+Y) = B \implies AX + AY = B \tag{8}$$

Matrix B is always written as $B = B_1 + B_2$,

$$\underleftarrow{XA} = B_1, \underleftarrow{YA} = B_2, \tag{9}$$

and

$$X + Y = \frac{B}{A}.$$

□

3. Discussion and Results

This study deals with the investigation of the approach to matrix division, the right multiplication and the Cracovian Product. The compatibility of left multiplication with division and multiplication is analyzed. The fact that the matrices in the left product have infinite factors makes left simplification possible. This property in the right product allows to the right reduction operation which is discussed with division. The contribution of division is emphasized in simplification. This situation is provided an easy approach to solution proposals in systems. In addition, this article is created new bridges for multiplication and division operations that are known in future applications. The suggested contribution opens up new research areas for further applications. The study also focused on multiplication of matrices. We have obtained results on the operations of some known matrices were obtained. Some of these results are as follows:

(i) The relations matrix product and right product.
(ii) The properties Cracovian Product, right product and division.
(iii) The connections Cracovian Product, right product and division and equations.
(iv) The transpose and linkage of operations.

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Hasan Keleş received M.Sc. from Sakarya University. Since 2000 he has been at Karadeniz Technical University. His research interests Conic section, division of matrices, fuzzy and Logic B_3.

Department of Mathematics, Karadeniz Technical University, Trabzon 61080, Türkiye.
e-mail: hasankeles@ktu.edu.tr