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THE DYNAMICS OF EUROPEAN-STYLE OPTION PRICING IN THE FINANCIAL MARKET UTILIZING THE BLACK-SCHOLES MODEL WITH TWO ASSETS, SUPPORTED BY VARIATIONAL ITERATION TECHNIQUE

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Abstract. This article offers a thorough exploration of a modified Black-Scholes model featuring two assets. The determination of option prices is accomplished through the Black-Scholes partial differential equation, leveraging the variational iteration method. This approach represents a semi-analytical technique that incorporates the use of Lagrange multipliers. The Lagrange multiplier emerges as a beacon of efficiency, adeptly streamlining the computational intricacies, and elevating the model's efficacy to unprecedented heights. For better understanding of the presented system, a graphical and tabular interpretation is presented with the help of Maple software.

AMS Mathematics Subject Classification : 34A35, 65B10, 35Qxx.
Key words and phrases : Option price, black scholes model with two assets, variational iteration method, Lagrange multiplier.

1. Introduction

The Variational Iteration Method (VIM) stands as a potent analytical tool, adept at tackling differential equations, particularly those entrenched in non-linearity, defying resolution through conventional means. Originating in 1979 by Nayfeh and Mook, its initial concepts have since undergone refinement and expansion under the scrutiny of numerous researchers. Here's a breakdown of its operation: Commencing with an initial approximation of the solution, VIM

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hinges on a guess that aligns with the boundary or initial conditions of the problem at hand. Subsequent to this, an iterative process ensues, wherein successive corrections refine the initial approximation, aiming for a more precise solution. These corrections pivot around a correction functional, often dubbed the Lagrange multiplier, adjusted iteratively to diminish the residual error between the differential equation and its approximated counterpart. Central to VIM is the adoption of a variational principle akin to those seen in the calculus of variations. This principle orchestrates the minimization of a functional, symbolizing the deviation between the approximate and true solutions of the differential equation. The iterative journey persists until the desired accuracy threshold is met or convergence criteria are satisfied. Techniques such as the homotopy perturbation method or Adomian decomposition method may be enlisted to expedite convergence. Upon convergence, the obtained solution undergoes scrutiny through substitution into the original differential equation, ensuring compliance within the specified tolerance. VIM's forte lies in its adaptability across a broad spectrum of differential equations [25], encompassing nonlinear, fractional, and partial varieties. However, its efficacy heavily relies on the judicious selection of the initial guess and the apt formulation of the correction functional. Despite its prowess, VIM may falter in certain scenarios, particularly when grappling with highly nonlinear or singular equations. In such instances, alternative methodologies or modifications to the VIM approach may be imperative. Moreover, the computational overhead can be substantial, particularly for intricate problems necessitating numerous iterations for convergence. Effective applications of VIM can be seen in [20, 33, 34]

The Black Scholes model was suggested by Fischer Black and Myron Scholes [4] in 1973 to examine the nature of option pricing in a market. Several mathematical models related to Black-Scholes equation have been developed [3, 5, 6]. Mostly contain five important components: underlying security stock price, strike price, risk-free rate, maturity period and volatility. A derivative is a financial instrument that guarantees payment at a future date and whose payout amount is determined by the movement of an underlying asset. It's value can be generated from a variety of underlying assets, including stocks, bonds, interest rates, commodities, currencies, and so on. Options are, without any doubt, the most important component of a financial derivative. The primary based on the evidence of the Netherlands' active participation in option trading, the initial research was proposed by Corzo [8]. Implying the drift of an arithmetic Brownian motion, In [6] presented an option pricing formula for implementation. Black Scholes model in financial market has gained significant intension in present time. In this regard, numerous analytic and numeric path have been examined and suggested. In the sense of analytic purposes He [11] employs the homotopy perturbation technique. The same is true for numerical solutions while Kim [13] , Koleva [16], Lesmana [17] and Marcozzi [18] employ the finite element technique, whereas Phaochoo [24] employs the finite difference method for finding numerical solution. Furthermore, the radial basis function partition of unity

technique (RBF-PUM) which is extensively used to approximate the partial differential equation issue, is one of the approaches for solving the Black-Scholes equations in also applied.
The two assets Black Scholes model for option pricing may be stated as follows in general

$$\frac{\partial C}{\partial \tau} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 C}{\partial S_i \partial S_j} + \sum_{i=1}^{2}(r-q_i)S_i\frac{\partial C}{\partial S_i} - rC = 0, \qquad S_1, S_2 \in [0,\infty),\ \tau \in [0,T] \tag{1}$$

with the terminal condition

$$C(S_1,S_2,T) = max\{\sum_{i=1}^{2}\beta_i S_i - K, 0\}$$

where as $K = max\{K_1, K_2\}$, and the boundary conditions:

$$C(S_1,S_2,\tau) = 0\ for\ (S_1,S_2) \to 0,$$
$$C(S_1,S_2,\tau) = \sum_{i=1}^{2}\beta_i S_i - Ke^{-r(T-\tau)}\ for\ S_1 \to \infty\ or\ S_2 \to \infty$$

where
β_i is a coefficient that ensures that all hazardous asset prices are equal,
C is a call option that is based on the underlying stock values S_1, S_2 at a given time.
ρ_{ij} is the connection between the prices of the ith and jth underlying stocks,
q_i is the dividend yield on the ith underlying stock,
r is the risk-free interest rate until it reaches the end of the period,
σ_i is the ith underlying stock's volatility,
T is the date of expiry,
K_i is the strike price of the underlying stock.

The majority of existing models include tight assumptions, such as ideal markets, volatility and risk-free rate with constant values, no dividends, continuous delta hedging and log-normal distribution of share price dynamics to name a few. The representation of a divisible number of shares is insufficient a market reality. The purpose of this work is to investigate the afore mentioned Black Scholes model using variational iteration method(VIM) [14, 22, 39, 42]. This technique is semi analytical and give outcome in series form. The suggestion of Lagrange Multiplier play very important role in the implementation of this technique. At the end tabular results are given to show the efficiency of these techniques. Similarly the graphical results are plotted for different values of parameters involve in Black Scholes model.

Many specialists and scholars employ stochastic numerical approaches because of their usability and worth. Falkner–skan model [14] is also an example of contemporary stochastic solvers that demonstrate their use.

2. Mathematical Formulation

For an European-style option, consider the two assets Black Scholes equation with efficient perfect liquidity, markets, and no dividends during the option's life. We make the assumption that σ_1, σ_2, ρ and r are constants.

$$\begin{aligned}&\frac{\partial c}{\partial \tau}+\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 c}{\partial S_1^2}+\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 c}{\partial S_2^2}+\rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 c}{\partial S_1 \partial S_2}\\ &+r[S_1\frac{\partial c}{\partial S_1}+S_2\frac{\partial c}{\partial S_2}]-rc=0,\ S_1,S_2\in[0,\infty),\ \tau\in[0,T]\end{aligned} \tag{2}$$

the terminal conditions are

$$c(S_1,S_2,\tau)=max\{\beta_1 S_1+\beta_2 S_2-K,0\}$$

boundary conditions are

$$\begin{aligned}&c(S_1,S_2,\tau)=0\ for\ (S_1,S_2)\to 0,\\ &c(S_1,S_2,\tau)=\beta_1 S_1+\beta_2 S_2-Ke^{-r(T-\tau)}\ for\ S_1\to\infty\ or\ S_2\to\infty\end{aligned}$$

By alternation in variables[7]

$$x=ln(S_1)-(r-\frac{1}{2}\sigma_1^2)\tau\ and\ y=ln(S_2)-(r-\frac{1}{2}\sigma_2^2)\tau$$

the equation (1) is defines as,

$$\frac{\partial c}{\partial \tau}+\frac{1}{2}\sigma_1^2\frac{\partial^2 c}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 c}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 c}{\partial x\partial y}-rc=0,\ (x,y,\tau)\in\mathbb{R}\times\mathbb{R}\times[0,T] \tag{3}$$

By changing again of variables, we define

$$c(x,y,\tau)=e^{-r(T-\tau)}v(x,y,\tau)$$

uaing $c(x,y,\tau)$ in equation (2),then obtain

$$\frac{\partial v}{\partial \tau}+\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}=0, \tag{4}$$

Now we use

$$t=T-\tau$$

By using in above equation we have

$$\frac{\partial v}{\partial t}=\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y},\ (x,y,t)\in\mathbb{R}\times\mathbb{R}\times[0,T] \tag{5}$$

3. Analysis of Methods

3.1. Variational iteration method.

The Variational iteration method is very useful and efficiently technique for either linear or nonlinear ordinary differential equation (ODEs) and partial differential equation (PDEs). Unlike adomian decomposition method (ADM) [2], these methods deals with nonlinear terms in the same manner as it deals with linear terms in the same steps. So, it is considered more capable method than ADM to handle nonlinear problems.

The general form of differential equation is

$$L(v) + N(v) = g(x)$$

In this equation L, N are linear and nonlinear operator correspondingly, where the non homogeneous function is g(x). The correction functional method for VIM is defined as

$$v_{n+1}(x) = v_n(x) + \int_0^t \lambda(Lv_n(t) + N\tilde{u}_n(t) - g(t))dt$$

Where as λ is a langrange multiplier and $\tilde{v}_n$ is restricted variation such that $(\tilde{\delta}_n) = 0$

In this method, v_0 can be selected from the given initial guess. The final solution is obtained by:

$$v(x) = \lim_{n\to\infty} v_n$$

3.2. Solution Procedure by VIM.

First we will solve the Black Scholes model by using VIM. We rewrite the Black Scholes equation as follows.

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2} - \rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y} = 0 \tag{6}$$

The initial condition is

$$v(x, y, 0) = max\{\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2} - K, 0\}$$

Now we construct correctional functional for eq (4) as given

$$v_{n+1} = v_n + \int_0^t \lambda(\frac{\partial v_n}{\partial t} - \frac{1}{2}\sigma_1^2\frac{\partial^2 v_n}{\partial x^2} - \frac{1}{2}\sigma_2^2\frac{\partial^2 v_n}{\partial y^2} - \rho\sigma_1\sigma_2\frac{\partial^2 v_n}{\partial x\partial y})ds \tag{7}$$

Where as the general lagrange multiplier is λ. Where the approximate value of $\lambda = -1$.

The correctional functional for eq(10) is

$$v_{n+1} = v_n - \int_0^t (\frac{\partial v_n}{\partial t} - \frac{1}{2}\sigma_1^2\frac{\partial^2 v_n}{\partial x^2} - \frac{1}{2}\sigma_2^2\frac{\partial^2 v_n}{\partial y^2} - \rho\sigma_1\sigma_2\frac{\partial^2 v_n}{\partial x\partial y})ds \tag{8}$$

Where the initial guess is

$$v_0(x, y, 0) = max\{(\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2})(1 + t) - K, 0\}$$

Now for n=0 we have

$$v_1 = v_0 - \int_0^t (\frac{\partial v_0}{\partial t} - \frac{1}{2}\sigma_1^2 \frac{\partial^2 v_0}{\partial x^2} - \frac{1}{2}\sigma_2^2 \frac{\partial^2 v_0}{\partial y^2} - \rho\sigma_1\sigma_2 \frac{\partial^2 v_0}{\partial x \partial y})ds \tag{9}$$

To solve this we obtain

$$\begin{aligned} v_1 &= (\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &+ \frac{1}{2}t(1+\frac{t}{2})(\sigma_1^2\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^2\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) - K \end{aligned}$$

Similarly for n=1, we get

$$\begin{aligned} v_2 = (\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) + \frac{1}{2}t(\sigma_1^2\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^2\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ \frac{1}{8}t^2(1+\frac{t}{3})(\sigma_1^4\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^4\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) - K \end{aligned}$$

.

The final form of solution can be written as

$$\begin{aligned} v &= \lim_{n\to\infty} v_n \\ &= (\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) + \frac{1}{2}t(\sigma_1^2\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^2\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &\frac{1}{8}t^2(\sigma_1^4\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^4\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &+ \frac{1}{48}t^3(\sigma_1^6\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^6\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &\frac{1}{384}t^4(\sigma_1^8\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^8\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &\frac{1}{3840}t^5(\sigma_1^{10}\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^{10}\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) \\ &\frac{1}{46080}t^6(1+\frac{t}{7})(\sigma_1^{12}\beta_1 e^{x+Tr-\frac{1}{2}\sigma_1^2} + \sigma_2^{12}\beta_2 e^{y+Tr-\frac{1}{2}\sigma_2^2}) - K + \cdots \end{aligned}$$

Table 4.1. Values of parameters of the numerical solution.

Parameters	Value
β_1	2
β_2	1
strike price, K (dollars)	40
maturity rate, T (years)	1
risk-free intrest rate (per year), r	5 percent
the underlying first assets' volatility (per year), σ_1	10 percent
the underlying second assets' volatility (per year), σ_2	20 percent
correlation, ρ	0.5
τ	0.1

Table 4.2

t	Exact	VIM	Error
0.0	22.00900775	22.00900774	0.00000001
0.1	22.04775701	22.04775701	0.00000000
0.2	22.08654114	22.08654113	0.00000001
0.3	22.12536019	22.12536019	0.00000000
0.4	22.16421422	22.16421421	0.00000001
0.5	22.20310325	22.203103255	0.00000000
0.6	22.24202735	22.24202733	0.00000002
0.7	22.28098656	22.28098654	0.00000002
0.8	22.31998091	22.31998091	0.00000000
0.9	22.35901049	22.35901048	0.00000001
1	22.39807534	22.39807533	0.00000001

In Table 4.2, the comparison of VIM with exact solution is given for several values of time. It can easily observe that the proposed technique gives accurate results. This shows that these techniques are very useful and efficient for the application of such type of problems.
In Figure 1, the option price v with respect to asset x is plotted for different values of time. By using VIM it is clearly seen that both techniques gives similar result. It is noted that the option prices v surge after x=3.9589 over a range of x from 2 to 6. Similarly in figure 7 the result of v is plotted for the asset y. It is noted that the option prices v soar after y=4.1371 over a range of y from 2 to 6.

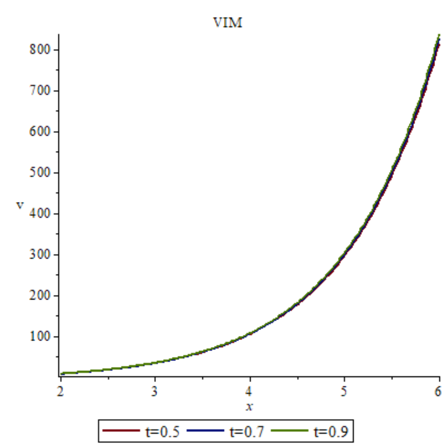

Figure 1. The effect of different values of time on option price w.r.t asset price x

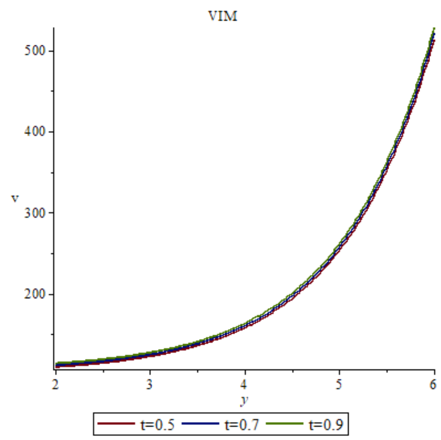

FIGURE 2. The effect of different values of time on option price w.r.t asset price y

In Figure 2, it is noted that with increase in volatility rate σ_1 the option price increases rapidly with the passage of time. Higher the volatility rate the option price increases more rapidly.

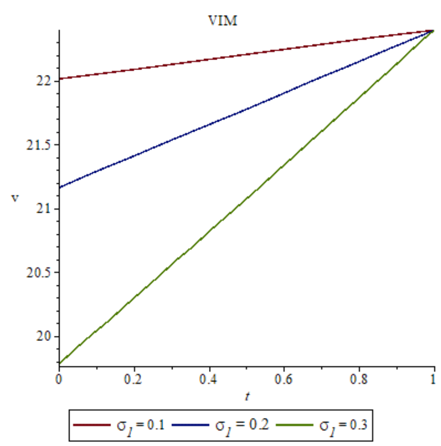

FIGURE 3. The influence of volatility rates on the option price w.r.t time

In Figure 4, the volatility rate σ_2 varies. Similarly here the option price v increases with the increase in volatility. But the increase in option price is less than as compare to Figure 3.

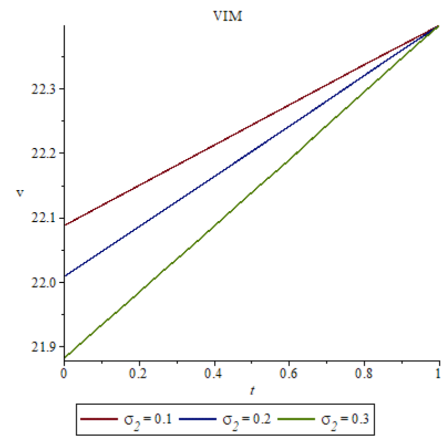

FIGURE 4. The influence of volatility rates on the option price w.r.t time

In Figure 5, the option price v is figurized over a range of $0 \leq x \leq 6$ and $0 \leq y \leq 6$ surrounding at the strike price. The outcomes depict rise when the stock prices growth. By setting y = 3.4589, the solution v is plotted in Figure 5. With the increasing x the option price reached to zero with the increase of x from 0 to 3. Similarily for $x > 3$ the option price rise exponentially.

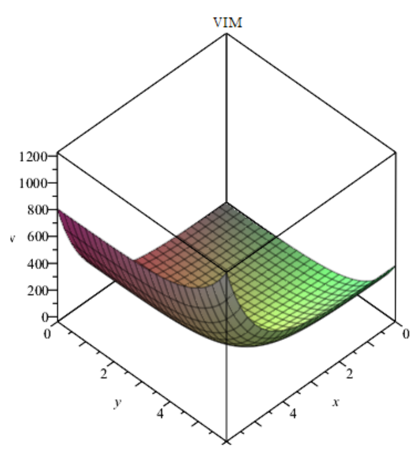

FIGURE 5. Transformed explicit solution v at a day before an expiration date.

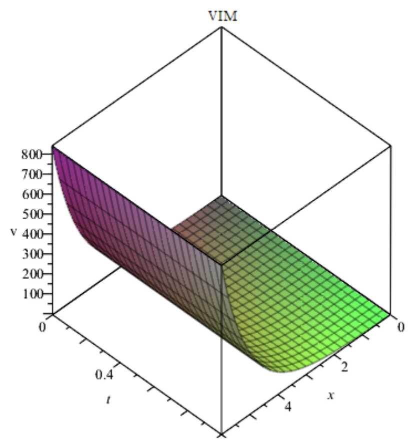

FIGURE 6. Transformed explicit solution v at a day before an expiration date.

Figure 6, depict the surface plot of call option with x = 4.2452 over a range of stock price $0 \leq y \leq 6$ and time $0 \leq t \leq 1$. As y increases from 0 to 1.8 the option price is still touches to zero. Then the option price suddenly rise for $y > 3$. Figure 7, depict the surface plot of call option with x = 4.2452 over a

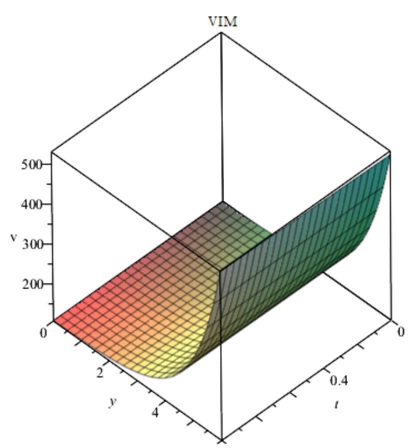

FIGURE 7. Transformed explicit solution v at a day before an expiration date.

range of stock price $0 \leq y \leq 6$ and time $0 \leq t \leq 1$. As y increases from 0 to 1.8 the option price is still touches to zero. Then the option price suddenly rise for $y > 3$.

4. Conclusion

The Black Scholes model with two assets is modified version of classical Black Scholes model using for determining the option prices. In this present study analytical technique variational iteration method(VIM) is used to find the series solution by using Maple. The Black-Scholes model stands as a cornerstone in finance and investment for several compelling reasons. It furnishes a mathematical framework for appraising options, financial instruments conferring the right, yet not the obligation, to buy or sell an underlying asset at a predetermined price within a specified timeframe. Through precise option pricing, investors gain the necessary insights to navigate their investment decisions adeptly, thereby enhancing risk management and optimizing returns within their portfolios. Options, pivotal in risk mitigation strategies for both investors and financial institutions, find accurate valuation through the Black-Scholes model, enabling hedging against adverse price fluctuations in the underlying assets. Moreover, the model's influence extends to fostering the creation of diverse financial products and investment tactics. Undoubtedly, the Black-Scholes model retains its pivotal status in financial theory and practice, offering invaluable perspectives on option pricing, risk management, and the dynamics of financial markets. Despite its inherent limitations and assumptions, it continues to shape investors' perceptions and strategies in the intricate realm of derivatives and securities trading.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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