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PAIR DIFFERENCE CORDIAL NUMBER OF A GRAPH

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Abstract. Let G be a (p, q) graph. Pair difference cordial number of a graph G is the least positive integer m such that $G \cup mK_2$ is pair difference cordial. It is denoted by $PDC_\eta(G)$. In this paper we find the pair difference cordial number of bistar, complete, helm, star, wheel.

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1. Introduction

We consider only finite, undirected and simple graphs.Cordial labeling was introduced in [1] and more cordial related labeling can refer [11,12]. The notion of pair diference cordial labeling of a graph was introduced in [4]. Also we have investigated pair difference cordial labeling behavior of several graphs like path, cycle, star, wheel etc have been investigated in [4,5,6,7,8,9,10]. Graph parameter is the interesting concept in graph theory. In this paper we have introduce a new labeling parameter called pair difference cordial number of a graph and we have obtained the pair difference cordial number of bistar, complete, helm, star, wheel .

2. Preliminaries

Definition 2.1. [4]. Let $G = (V, E)$ be a (p, q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different

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elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

Definition 2.2. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup E(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Theorem 2.3. [4]. The path P_n is pair difference cordial for all values of $n($ $n \neq 3$).

Theorem 2.4. [4]. The cycle C_n is pair difference cordial if and only if $n > 3$.

Theorem 2.5. [4]. The star $K_{1,n}$ is pair difference cordial if and only if $3 \leq n \leq 6$.

Theorem 2.6. [4]. The complete graph K_p is pair difference cordial if and only if $p \leq 2$.

Theorem 2.7. [4]. The bistar $B_{m,n}, (m \geq 2, n \geq 2)$ is pair difference cordial if and only if $m + n \leq 9$.

Theorem 2.8. [4]. *The wheel W_n is pair difference cordial if and only if n is even.*

3. Pair Difference Cordial Number of a Graph

Definition 3.1. Let G be a (p, q) graph. Pair difference cordial number of a graph G is the least positive integer m such that $G \cup K_2$ is pair difference cordial. It is denoted by $PDC_\eta(G)$.

Example 3.2. The graph G_1 given in figure 1 is not pair difference cordial, since the maximum possible edges with label 1 is 3. But the graph $G_1 \cup K_2$ is pair difference cordial. Therefore $\mathrm{PDC}_\eta(G_1) = 1$.

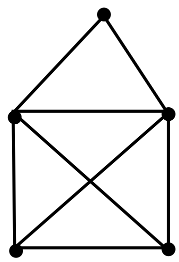


Figure 1 :G_1

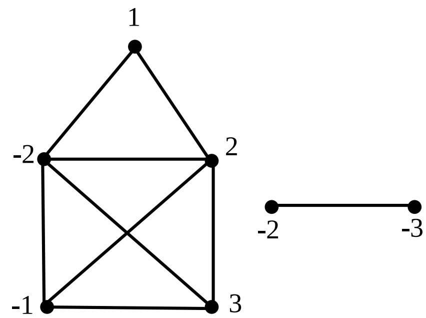


Figure 2 : $G_1 \cup K_2$

Remark 3.3. If G is pair difference cordial graph then $PDC_\eta(G) = 0$.

Theorem 3.4. *If $n \geq 2$, then*

$$PDC_\eta(K_{1,4n+r}) = \begin{cases} 4n-7 & \textit{if } r \equiv 0 \pmod 4 \\ 4n-4 & \textit{if } r \equiv 1 \pmod 4 \\ 4n-5 & \textit{if } r \equiv 2 \pmod 4 \\ 4n-6 & \textit{if } r \equiv 3 \pmod 4 \end{cases}$$

.

Proof. Let $V(K_{1,4n+r} \cup mK_2) = \{u, u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(K_{1,4n+r} \cup mK_2) = \{uu_i : 1 \leq i \leq n\} \cup \{v_i w_i : 1 \leq i \leq m\}$.
Clearly

$$K_{1,4n+r} \cup mK_2$$

has $4n + 2m + 1$ vertices and $4n + m$ edges.
There are four cases arises.

Case 1. $r \equiv 0 \pmod 4$

Define $f : V(K_{1,4n+r} \cup (4n-7)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \left\lfloor \frac{4n+2m+1}{2} \right\rfloor\}$ as follows:
When $n = 2$, Assign the labels $1, 2$ to the vertices v_1, w_1 and assign the labels $-2, -1, -3, -3, 4, 5, 6, -4, -5, -6$ to the vertices $u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$.

When $n \geq 3$, Now assign the labels $1, 3, 5, \cdots, 4n-7$ to the vertices $v_1, v_3, v_5, \cdots, v_m$ and assign the labels $2, 4, 6 \cdots, 4n-6$ to the vertices $w_1, w_3, w_5, \cdots, w_m$. Next assign the labels $-1, -3, -5, \cdots, -(4n-8)$ to the vertices $v_2, v_4, v_6, \cdots, v_{m-1}$ and assign the labels $-2, -4, -6 \cdots, -(4n-9)$ to the vertices $w_2, w_4, w_6, \cdots, w_{m-1}$. Assign the label $-(4n-6)$ to the vertex u. Finally assign the distinct labels to the pendent vertices of the star from $\pm(4n-3), \pm(4n-2), \cdots, \pm(m+2n)$ to the vertices $u_4, u_5, u_6, \cdots, u_n$ and assign the labels $-(4n-7), -(4n-5), -(4n-5)$

to the vertices u_1, u_2, u_3.

Therefore f is pair difference cordial labeling of $K_{1,4n+r} \cup (4n-7)K_2$. The maximum number of edges with labels 1 from the star is, $\Delta_{f_1} = 3$, if n is even.But the size of the star is n. Hence $4n-7$ is the least integer such that $K_{1,4n+r} \cup (4n-7)K_2$ is pair difference cordial.

Case 2. $r \equiv 1 \pmod 4$

Define $f : V(K_{1,4n+r} \cup (4n-4)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{4n+2m+1}{2} \rfloor\}$ as follows:
Now assign the labels $1, 3, 5, \cdots, 4n-5$ to the vertices $v_1, v_3, v_5, \cdots, v_m$ and assign the labels $2, 4, 6 \cdots, 4n-4$ to the vertices $w_1, w_3, w_5, \cdots, w_m$. Next assign the labels $-1, -3, -5, \cdots, -(4n-5)$ to the vertices $v_2, v_4, v_6, \cdots, v_{m-1}$ and assign the labels $-2, -4, -6 \cdots, -(4n-4)$ to the vertices $w_2, w_4, w_6, \cdots, w_{m-1}$. Assign the labels $-(4n-2), 4n-2, -(4n-3), 4n-3$ to the vertex u, u_1, u_2, u_3. Finally assign the distinct labels to the pendent vertices of the star from $\pm(4n-1), \pm(4n), \cdots, \pm(m+2n)$ to the vertices $u_4, u_5, u_6, \cdots, u_n$.

Therefore f is pair difference cordial labeling of $K_{1,4n+r} \cup (4n-4)K_2$. The maximum number of edges with labels 1 from the star is, $\Delta_{f_1} = 2$, if n is odd.But the size of the star is n. Hence $4n-4$ is the least integer such that $K_{1,4n+r} \cup (4n-4)K_2$ is pair difference cordial.

Case 3. $r \equiv 2 \pmod 4$

Define $f : V(K_{1,4n+r} \cup (4n-5)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{4n+2m+1}{2} \rfloor\}$ as follows:
Now assign the labels $1, 3, 5, \cdots, 4n-6$ to the vertices $v_1, v_3, v_5, \cdots, v_m$ and assign the labels $2, 4, 6 \cdots, 4n-6$ to the vertices $w_1, w_3, w_5, \cdots, w_m$. Next assign the labels $-1, -3, -5, \cdots, -(4n-7)$ to the vertices $v_2, v_4, v_6, \cdots, v_{m-1}$ and assign the labels $-2, -4, -6 \cdots, -(4n-6)$ to the vertices $w_2, w_4, w_6, \cdots, w_{m-1}$. Assign the label $-(4n-4)$ to the vertex u. Finally assign the distinct labels to the pendent vertices of the star from $\pm(4n-2), \pm(4n-1), \cdots, \pm(m+2n)$ to the vertices $u_4, u_5, u_6, \cdots, u_n$ and assign the labels $-(4n-5), -(4n-3), -(4n-3)$ to the vertices u_1, u_2, u_3.

Therefore f is pair difference cordial labeling of $K_{1,4n+r} \cup (4n-5)K_2$. The maximum number of edges with labels 1 from the star is $\Delta_{f_1} = 3$, if n is even. But the size of the star is n.
Hence $4n-5$ is the least integer such that $K_{1,4n+r} \cup (4n-5)K_2$ is pair difference cordial.

Case 4. $r \equiv 3 \pmod 4$

Define $f : V(K_{1,4n+r} \cup (4n-6)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{4n+2m+1}{2} \rfloor\}$ as follows :

Now assign the labels $1, 3, 5, \cdots, 4n-7$ to the vertices $v_1, v_3, v_5, \cdots, v_m$ and assign the labels $2, 4, 6 \cdots, 4n-6$ to the vertices $w_1, w_3, w_5, \cdots, w_m$. Next assign the labels $-1, -3, -5, \cdots, -(4n-7)$ to the vertices $v_2, v_4, v_6, \cdots, v_{m-1}$ and assign the labels $-2, -4, -6 \cdots, -(4n-6)$ to the vertices $w_2, w_4, w_6, \cdots, w_{m-1}$. Assign the labels $-(4n-4), 4n-4, -(4n-5), 4n-5$ to the vertex u, u_1, u_2, u_3. Finally assign the distinct labels to the pendent vertices of the star from $\pm(4n-3), \pm(4n-2), \cdots, \pm(m+2n)$ to the vertices $u_4, u_5, u_6, \cdots, u_n$.

Therefore f is pair difference cordial labeling of $K_{1,4n+r} \cup (4n-6)K_2$. The maximum number of edges with labels 1 from the star is $\Delta_{f_1} = 2$, if n is odd.But the size of the star is n. Hence $4n-6$ is the least integer such that $K_{1,4n+r} \cup (4n-6)K_2$ is pair difference cordial.

□

Theorem 3.5.

$$PDC_\eta(P_n) = \begin{cases} 1, & \textit{if } n = 3 \\ 0, & \textit{if } n \neq 3 \end{cases}$$

Proof. When $n \neq 3$, Assign the labels $1, 2, 2$ to the vertices of P_3 and assign the labels $-1, -2$ to the vertices of K_2.

When $n = 3$, Follows from theorem 2.3.

□

Corollary 3.6.

$$PDC_\eta(K_{1,n}) = \begin{cases} 0, & \textit{if } 1 \leq n \leq 6 \\ 2, & \textit{if } n = 7 \end{cases}$$

Proof. **Case 1.** $n = 1, 2$.

Since $K_{1,1} \cong P_2$,$K_{1,2} \cong P_3$, By theorem 3.5, $PDC_\eta(K_{1,1}) = 0$ and $PDC_\eta(K_{1,2}) = 0$.

Case 2. $3 \leq n \leq 6$.

Follows from theorem 2.5, $PDC_\eta(K_{1,n}) = 0$ where $3 \leq n \leq 6$.

Case 3. $n = 7$.

Assign the labels $1, 2, 3, -1, -2, -3, -4$ to the vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ and assign the labels $2, 5, 6, -5, -6$ to the vertices u, v_1, w_1, v_2, w_2.
Therefore $PDC_\eta(K_{1,7}) = 2$.

□

Theorem 3.7. *If $n \geq 3$, then*

$$PDC_\eta(W_n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. **Case A.** n is even.
Follows from theorem 2.8. $PDC_\eta(W_n) = 0$.

Case B. n is odd.
Let $V(W_n \cup mK_2) = \{u, u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(W_n \cup mK_2) = \{uu_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1u_n\} \cup \{v_iw_i : 1 \leq i \leq m\}$.
Clearly $W_n \cup mK_2$ has $n + 2m + 1$ vertices and $2n + m$ edges.
There are three cases arises.
Case 1. $n \equiv 0 \pmod 3$

When $n = 3$, assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u and assign the labels $-1, -3, -2, -4$ to the vertices v_1, v_2, w_1, w_2.

When $n > 3$,

Define $f : V(W_n \cup (\frac{n+1}{2})K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+1}{2} \rfloor\}$ as follows :
Assign the labels $1, 2, 3, \cdots, n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$ and assign the label $n + 1$ to the vertex u.

Assign the labels $-1, -3, -5, \cdots, \frac{2n}{3} - 1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{3}}$ and assign the labels $-2, -4, -6, \cdots, \frac{2n}{3}$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n}{3}}$.

Next assign the labels $-(\frac{2n}{3} + 1)$,$-(\frac{2n}{3} + 3)$ to the vertices $v_{\frac{n}{3}+1}$,$w_{\frac{n}{3}+1}$ and assign the labels $-(\frac{2n}{3} + 5)$,$-(\frac{2n}{3} + 7)$ to the vertices $v_{\frac{n}{3}+2}$,$w_{\frac{n}{3}+2}$.We assign the labels $-(\frac{2n}{3} + 9)$,$-(\frac{2n}{3} + 11)$ to the vertices $v_{\frac{n}{3}+3}$,$w_{\frac{n}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n}{3}+\frac{n-6}{3}}$,$w_{\frac{n}{3}+\frac{n-6}{3}}$. Note that the vertices $v_{\frac{n}{3}+\frac{n-6}{3}}$,$w_{\frac{n}{3}+\frac{n-6}{3}}$ get the labels $-(n-2), -(n)$.

Now we assign the labels $-(\frac{2n}{3} + 2)$,$-(\frac{2n}{3} + 4)$ to the vertices $v_{\frac{n}{3}+\frac{n-6}{3}+1}$,$w_{\frac{n}{3}+\frac{n-6}{3}+1}$ and assign the labels $-(\frac{2n}{3} + 6)$,$-(\frac{2n}{3} + 8)$ to the vertices $v_{\frac{n}{3}+\frac{n-6}{3}+2}$,$w_{\frac{n}{3}+\frac{n-6}{3}+2}$.We assign the labels $-(\frac{2n}{3} + 10)$,$-(\frac{2n}{3} + 12)$ to the vertices $v_{\frac{n}{3}+3}$

$,w_{\frac{n}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n+1}{2}}$ $,w_{\frac{n+1}{2}}$. Note that the vertices $v_{\frac{n+1}{2}}$ $,w_{\frac{n+1}{2}}$ get the labels $-(n-1), -(n+1)$.

Case 2. $n \equiv 1 \pmod 3$

Define $f : V(W_n \cup (\frac{n+1}{2})K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+1}{2} \rfloor\}$ as follows :
Assign the labels $1, 2, 3, \cdots, n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$ and assign the label $n+1$ to the vertex u.

Assign the labels $-1, -3, -5, \cdots, \frac{2n-2}{3} - 1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{3}}$ and assign the labels $-2, -4, -6, \cdots, \frac{2n-2}{3}$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n-1}{3}}$.

Next assign the labels $-(\frac{2n-2}{3}+1)$,$-(\frac{2n-2}{3}+3)$ to the vertices $v_{\frac{n-1}{3}+1}$ $,w_{\frac{n-1}{3}+1}$ and assign the labels $-(\frac{2n-2}{3}+5)$,$-(\frac{2n-2}{3}+7)$ to the vertices $v_{\frac{n-1}{3}+2}$ $,w_{\frac{n-1}{3}+2}$.We assign the labels $-(\frac{2n-2}{3}+9)$,$-(\frac{2n-2}{3}+11)$ to the vertices $v_{\frac{n-1}{3}+3}$ $,w_{\frac{n-1}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}}$ $,w_{\frac{n-1}{3}+\frac{n-4}{3}}$. Note that the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}}$ $,w_{\frac{n-1}{3}+\frac{n-4}{3}}$ get the labels $-(n-1), -(n+1)$.

Now we assign the labels $-(\frac{2n-2}{3}+2)$,$-(\frac{2n-2}{3}+4)$ to the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}+1}$ $,w_{\frac{n-1}{3}+\frac{n-4}{3}+1}$ and assign the labels $-(\frac{2n}{3}+6)$,$-(\frac{2n}{3}+8)$ to the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}+2}$ $,w_{\frac{n-1}{3}+\frac{n-4}{3}+2}$.We assign the labels $-(\frac{2n}{3}+10)$,$-(\frac{2n}{3}+12)$ to the vertices $v_{\frac{n-1}{3}+3}$ $,w_{\frac{n-4}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n+1}{2}}$ $,w_{\frac{n+1}{2}}$. Note that the vertices $v_{\frac{n+1}{2}}$ $,w_{\frac{n+1}{2}}$ get the labels $-(n-2), -(n)$.

Case 3. $n \equiv 2 \pmod 3$

Define $f : V(W_n \cup (\frac{n+1}{2})K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+1}{2} \rfloor\}$ as follows :
Assign the labels $1, 2, 3, \cdots, n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$ and assign the label $n+1$ to the vertex u.

Assign the labels $-1, -3, -5, \cdots, \frac{2n-4}{3} - 1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-2}{3}}$ and assign the labels $-2, -4, -6, \cdots, \frac{2n-4}{3}$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n-2}{3}}$.

Next assign the labels $-(\frac{2n-4}{3}+1)$,$-(\frac{2n-4}{3}+3)$ to the vertices $v_{\frac{n-2}{3}+1}$ $,w_{\frac{n-2}{3}+1}$ and assign the labels $-(\frac{2n-4}{3}+5)$,$-(\frac{2n-4}{3}+7)$ to the vertices $v_{\frac{n-2}{3}+2}$ $,w_{\frac{n-2}{3}+2}$.We assign the labels $-(\frac{2n-4}{3}+9)$,$-(\frac{2n-4}{3}+11)$ to the vertices $v_{\frac{n-2}{3}+3}$ $,w_{\frac{n-2}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n-2}{3}+\frac{n-2}{3}}$ $,w_{\frac{n-2}{3}+\frac{n-2}{3}}$. Note that the vertices $v_{\frac{n-2}{3}+\frac{n-2}{3}}$ $,w_{\frac{n-2}{3}+\frac{n-2}{3}}$ get the labels $-(n-1), -(n+1)$.

Now we assign the labels $-(\frac{2n-2}{3}+2)$,$-(\frac{2n-2}{3}+4)$ to the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}+1}$,$w_{\frac{n-1}{3}+\frac{n-4}{3}+1}$ and assign the labels $-(\frac{2n}{3}+6)$,$-(\frac{2n}{3}+8)$ to the vertices $v_{\frac{n-1}{3}+\frac{n-4}{3}+2}$,$w_{\frac{n-1}{3}+\frac{n-4}{3}+2}$.We assign the labels $-(\frac{2n}{3}+10)$,$-(\frac{2n}{3}+12)$ to the vertices $v_{\frac{n-1}{3}+3}$,$w_{\frac{n-4}{3}+3}$. Proceeding like this until the vertices $v_{\frac{n+1}{2}}$,$w_{\frac{n+1}{2}}$. Note that the vertices $v_{\frac{n+1}{2}}$,$w_{\frac{n+1}{2}}$ get the labels $-(n-2), -(n)$.

Therefore f is pair difference cordial labeling of $W_n \cup (\frac{n+1}{2})K_2$. In above all the cases, the maximum number of edges with labels 1 from the wheel is $\Delta_{f_1} = n$, if n is even.But the size of the wheel is $2n$.
Hence $\frac{n+1}{2}$ is the least integer such that $W_n \cup (\frac{n+1}{2})K_2$ is pair difference cordial.

□

Theorem 3.8. *If $n \geq 3$, then*

$$PDC_\eta(H_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 2 \\ n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Proof. Let $V(H_n \cup mK_2) = \{x, x_i, y_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(W_n \cup mK_2) = \{xx_i, x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1x_n\} \cup \{v_iw_i : 1 \leq i \leq m\}$.
Clearly $H_n \cup mK_2$ has $2n + 2m + 1$ vertices and $3n + m$ edges.
There are three cases arises.

Case 1. $n \equiv 0 \pmod 2$

Define $f : V(H_n \cup (n)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{2n+2m+1}{2} \rfloor\}$ as follows :
Assign the labels $1, 2, 3, \cdots, n$ to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $y_1, y_2, y_3, \cdots, y_n$. Assign the label $n+1$ to the vertex x.

Assign the labels $n+1, n+3, n+5, \cdots, n+m-1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$ and assign the labels $n+2, n+4, n+6, \cdots, n+m$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n}{2}}$. Next assign the labels $-(n+1), -(n+3), -(n+5), \cdots, -(n+m-1)$ to the vertices $v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \cdots, v_n$ and assign the labels $-(n+2), -(n+4), -(n+6), \cdots, -(n+m)$ to the vertices $w_{\frac{n+2}{2}}, w_{\frac{n+4}{2}}, w_{\frac{n+6}{2}}, \cdots, w_n$.

Therefore f is pair difference cordial labeling of $H_n \cup (n)K_2$. The maximum number of edges with labels 1 from the helm is, $\Delta_{f_1} = 2n-1$. But the size of the star is $3n$.
Hence n is the least integer such that $H_n \cup (n)K_2$ is pair difference cordial.

Case 2. $n \equiv 1 \pmod 2$

Define $f : V(H_n \cup (n+1)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{2n+2m+1}{2} \rfloor\}$ as follows :
Assign the labels $1, 2, 3, \cdots, n$ to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $y_1, y_2, y_3, \cdots, y_n$. Assign the label $n+1$ to the vertex x.

Assign the labels $n+1, n+3, n+5, \cdots, n+m-1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}}$ and assign the labels $n+2, n+4, n+6, \cdots, n+m$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n+1}{2}}$. Next assign the labels $-(n+1), -(n+3), -(n+5), \cdots, -(n+m-1)$ to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \cdots, v_{n+1}$ and assign the labels $-(n+2), -(n+4), -(n+6), \cdots, -(n+m)$ to the vertices $w_{\frac{n+3}{2}}, w_{\frac{n+5}{2}}, w_{\frac{n+7}{2}}, \cdots, w_{n+1}$.

Therefore f is pair difference cordial labeling of $H_n \cup (n+1)K_2$. The maximum number of edges with labels 1 from the helm is, $\Delta_{f_1} = 2n-1$. But the size of the helm is $3n$.
Hence $n+1$ is the least integer such that $H_n \cup (n+1)K_2$ is pair difference cordial. □

Theorem 3.9. *If $n \geq 5$, then*

$$PDC_\eta(B_{n,n}) = \begin{cases} 0 & 1 \leq n \leq 4 \\ 2n-8 & n \geq 5. \end{cases}$$

Proof. **Case A.** $1 \leq n \leq 4$

Follows from theorem 2.7. $PDC_\eta(B_{n,n})$ =0.

Case B. $n \geq 5$.

Let $V(B_{n,n} \cup mK_2) = \{x_i, y_i, x, y : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(B_{n,n} \cup mK_2) = \{xx_i, yy_i : 1 \leq i \leq n\} \cup \{xy\} \cup \{v_iw_i : 1 \leq i \leq m\}$.
Clearly $B_{n,n} \cup mK_2$ has $2n+2+2m$ vertices and $2n+m+1$ edges.

Define $f : V(B_{n,n} \cup (2n-8)K_2) \to \{\pm 1, \pm 2, \cdots, \pm(n+m+1)$ as follows :

Assign the labels $1, 3, 2$ to the vertices x_1, x_2, x and assign the labels $4, 5, 6, \cdots, n+1$ to the vertices $x_3, x_4, x_5, \cdots, x_n$. Next assign the labels $-1, -3, -2$ to the vertices y_1, y_2, y and assign the labels $-4, -5, -6, \cdots, -(n+1)$ to the vertices $y_3, y_4, y_5, \cdots, y_n$.

Now we assign the labels $n+2, n+3$ to the vertices v_1, w_1 and assign the labels $-(n+2), -(n+3)$ to the vertices v_2, w_2 and next assign the labels $n+4, n+5$ to the vertices v_3, w_3 and assign the labels $-(n+2), -(n+3)$ to the vertices v_4, w_4. Proceeding like this until we reach the vertices $v_{m-1}, w_{m-1}, v_m, w_m$. Note that the vertices v_m, w_m get the labels $(n+m), (n+m+1), -(n+m), -(n+m+1)$ respectively.

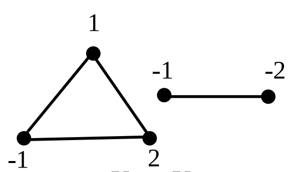
□

Theorem 3.10.

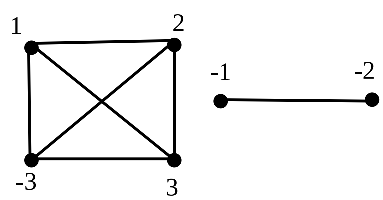
$$PDC_\eta(K_n) = \begin{cases} 0 & \textit{if } n = 1, 2 \\ 1 & \textit{if } n = 3, 4 \\ 2 & \textit{if } n = 5 \end{cases}$$

Proof. Follows from theorem 2.6. $PDC_\eta(K_n) = 0, n = 1, 2$.

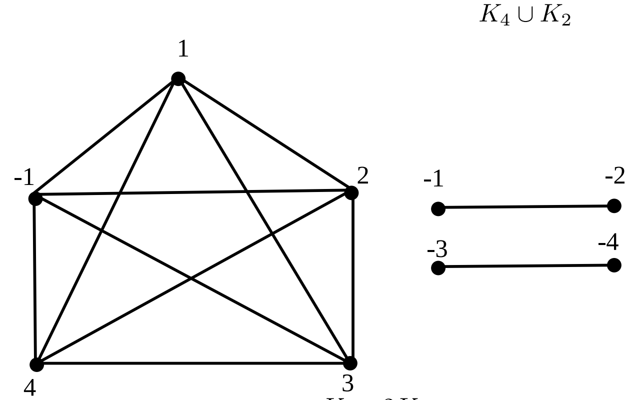
K_3, K_4, K_5 are not pair difference cordial. The figure given below shows that $K_3 \cup K_2$,$K_4 \cup K_2$,$K_5 \cup 2K_2$ are pair difference cordial.



$K_3 \cup K_2$



$K_4 \cup K_2$



$K_5 \cup 2K_2$

Therefore $PDC_\eta(K_3) = 1$,$PDC_\eta(K_4) = 1$,$PDC_\eta(K_5) = 2$.

□

Theorem 3.11. *If $n \geq 6$, then*

$$PDC_\eta(K_n) = \begin{cases} \frac{(n-1)(n-4)}{2} & \textit{if } n \equiv 0 \textit{ mod } 2 \\ \frac{(n-1)(n-4)}{2} - 1 & \textit{if } n \equiv 1 \textit{ mod } 2 \end{cases}$$

Proof. Let $V(K_n \cup mK_2) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(K_n \cup mK_2) = E(K_n) \cup \{v_i w_i : 1 \leq i \leq m\}$.
Clearly $K_n \cup mK_2$ has $n + 2m$ vertices and $\frac{n(n-1)}{2} + 2m$ edges.
There are two cases arises.

Case 1. $n \equiv 0 \pmod 2$
Define $f : V(K_n \cup (\frac{(n-1)(n-4)}{2})K_2) \to \{\pm 1, \pm 2, \cdots, \pm \frac{n+2m}{2}\}$ as follows :

Assign the labels $1, 2, 3, \cdots, n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$. Next we assign the labels $-1, -2$ to the vertices v_1, w_1 and assign the labels $-3, -4$ to the vertices v_2, w_2. Assign the labels $-5, -6$ to the vertices v_3, w_3. Proceeding like this until we reach the vertices $v_{\frac{n}{2}}, w_{\frac{n}{2}}$. Note that the vertices $v_{\frac{n}{2}}, w_{\frac{n}{2}}$ get the labels $-(n-1), -(n)$.

Now we assign the labels $(n+1), (n+2), -(n+1), -(n+2)$ to the vertices $v_{\frac{n+2}{2}}, w_{\frac{n+2}{2}}$,
$v_{\frac{n+4}{2}}, w_{\frac{n+4}{2}}$. We assign the labels $(n+3), (n+4), -(n+3), -(n+4)$ to the vertices $v_{\frac{n+6}{2}}, w_{\frac{n+6}{2}}, v_{\frac{n+8}{2}}, w_{\frac{n+8}{2}}$. Proceeding like this until we reach the vertices v_m, w_m. Note that the vertices v_m, w_m get the labels $-\frac{n+2m}{2} - 1, -\frac{n+2m}{2}$.

Case 2. $n \equiv 1 \pmod 2$
Define $f : V(K_n \cup (\frac{(n-1)(n-4)}{2} - 1)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \frac{n+2m-1}{2}\}$ as follows :

Assign the labels $1, 2, 3, \cdots, n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$. Next we assign the labels $-1, -2$ to the vertices v_1, w_1 and assign the labels $-3, -4$ to the vertices v_2, w_2. Assign the labels $-5, -6$ to the vertices v_3, w_3. Proceeding like this until we reach the vertices $v_{\frac{n-1}{2}}, w_{\frac{n-1}{2}}$. Note that the vertices $v_{\frac{n-1}{2}}, w_{\frac{n-1}{2}}$ get the labels $-(n-2), -(n-1)$.

Next assign the labels $-n, -(n-1)$ to the vertices $v_{\frac{n+1}{2}}, w_{\frac{n+1}{2}}$. Now we assign the labels $(n+1), (n+2), -(n+1), -(n+2)$ to the vertices $v_{\frac{n+3}{2}}, w_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, w_{\frac{n+5}{2}}$. We assign the labels $(n+3), (n+4), -(n+3), -(n+4)$ to the vertices $v_{\frac{n+7}{2}}, w_{\frac{n+7}{2}}$, $v_{\frac{n+9}{2}}, w_{\frac{n+9}{2}}$. Proceeding like this until we reach the vertices v_m, w_m. Note that the vertices v_m, w_m get the labels $-\frac{n+2m-1}{2} - 1, -\frac{n+2m-1}{2}$.

□

4. conclusion

In this paper we have obtained the pair difference cordial number of some graphs like bistar,complete,star,helm,wheel. The pair difference cordial number of other non pair difference cordial graphs like product graphs like $K_m \times P_2, m > 3$, $G \times K_n, n \geq 3$, $G \times W_n, n \geq 4$, Book graph $B_m, m \geq 6$ and the shadow graphs of complete graph and star graphs are the open problems for the future research work.

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Data availability : Not applicable

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