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ZEROS OF THE EULER-FIBONACCI POLYNOMIALS

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Abstract. In this paper, we investigate the distribution of the zeros of the Euler-Fibonacci polynomials by using computer.

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1. Introduction

In this paper, we investigate the distribution of zeros of the Euler-Fibonacci polynomials by using computer. Throughout this paper, we always make use of the following notations: $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

The authors [1, 2, 4] introduced generating functions for Euleri numbers E_n and Euler polynomials $E_n(x)$ as follow

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2}{e^t+1}, \quad \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right) e^{xt}.$$

Now, we give some definitions (for these definitions see [11, 12]) that we will use throughout the article. The F-factorial is defined as

$$F_n! = F_n \cdot F_{n-1} \cdot F_{n-2} \cdots F_1, \quad F_0! = 1.$$

where F_n is n-th Fibonacci numbers. The Fibonomial coefficients are defined as $(0 \le k \le n)$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!}$$

with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$.

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The binomial theorem for the F-analogues (or-Golden binomial theorem) are given by

$$(x+y)_F^n = \sum_{k=0}^{n} (-1)^{\binom{n}{2}} \binom{n}{k}_F x^{n-k} y^k$$

The F-exponential functions $e_F(x)$ and $E_F(x)$ are defined as:

$$e_F(x) = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}, \quad E_F(x) = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{x^n}{F_n!}.$$

The following identity holds

$$e_F^x E_F^x = e_F^{(x+y)_F}$$

The author [6] defined generating functions for Euler-Fibonacci numbers $E_{n,F}$ and Euler-Fibonacci polynomials $E_{n,F}(x)$ as follow

$$\sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!} = \frac{2}{e_F(t)+1},$$
$$\sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^n}{F_n!} = \left(\frac{2}{e_F(t)+1}\right) e_F(xt).$$

Theorem 1.1. *For $n \geq 1$, we have*

$$(1) \quad E_{n,F}(x) = \sum_{l=0}^{n} \binom{n}{l}_F E_{l,F} x^{n-l}.$$

$$(2) \quad \sum_{l=0}^{n} \binom{n}{l}_F E_{l,F}(x) + E_{n,F}(x) = 2x^n.$$

For the first few Euler-Fibonacci numbers we have,

$$E_{0,F} = 1, \quad E_{1,F} = -\frac{1}{2}, \quad E_{2,F} = -\frac{1}{4}, \quad E_{3,F} = \frac{1}{4},$$
$$E_{4,F} = \frac{5}{8}, \quad E_{5,F} = -\frac{13}{16}, \quad E_{6,F} = -\frac{41}{4}, \quad E_{7,F} = -\frac{87}{8},$$
$$E_{8,F} = \frac{16995}{16}, \quad E_{9,F} = \frac{40367}{16}, \quad E_{10,F} = -\frac{22615103}{32},$$
$$E_{11,F} = -\frac{889776019}{64}, \quad E_{12,F} = \frac{24141921365}{8}, \quad E_{13,F} = \frac{4412564523437}{16},$$
$$E_{14,F} = -\frac{2609751415277683}{32}, \quad E_{15,F} = -\frac{980874706013690667}{32}.$$

2. Zeros of the Euler-Fibonacci polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Euler-Fibonacci polynomials $E_{n,F}(x)$. The Euler-Fibonacci polynomials $E_{n,F}(x)$. can be determined explicitly. A few of them are

$$E_{0,F}(x)=1,$$

$$E_{1,F}(x)=-\frac{1}{2}+x,$$

$$E_{2,F}(x)=-\frac{1}{4}-\frac{x}{2}+x^2,$$

$$E_{3,F}(x)=\frac{1}{4}-\frac{x}{2}-x^2+x^3,$$

$$E_{4,F}(x)=\frac{5}{8}+\frac{3x}{4}-\frac{3x^2}{2}-\frac{3x^3}{2}+x^4,$$

$$E_{5,F}(x)=-\frac{13}{16}+\frac{25x}{8}+\frac{15x^2}{4}-\frac{15x^3}{4}-\frac{5x^4}{2}+x^5,$$

$$E_{6,F}(x)=-\frac{41}{4}-\frac{13x}{2}+25x^2+15x^3-10x^4-4x^5+x^6.$$

$$E_{7,F}(x)=-\frac{87}{8}-\frac{533x}{4}-\frac{169x^2}{2}+\frac{325x^3}{2}+65x^4-26x^5-\frac{13x^6}{2}+x^7,$$

$$E_{8,F}(x)=\frac{16995}{16}+\frac{1827x}{8}-\frac{11193x^2}{4}-\frac{3549x^3}{4}+\frac{2275x^4}{2}+273x^5 -\frac{273x^6}{4}-\frac{21x^7}{2}+x^8,$$

$$E_{9,F}(x)=\frac{40367}{16}+\frac{288915x}{8}+\frac{31059x^2}{4}-\frac{190281x^3}{4}-\frac{20111x^4}{2}+7735x^5 +\frac{4641x^6}{4}-\frac{357x^7}{2}-17x^8+x^9,$$

$$E_{10,F}(x)=-\frac{22615103}{32}+\frac{2220185x}{16}+\frac{15890325x^2}{8}+\frac{1708245x^3}{8}-\frac{3488485x^4}{4} -\frac{221221x^5}{2}+\frac{425425x^6}{8}+\frac{19635x^7}{4}-\frac{935x^8}{2}-\frac{55x^9}{2}+x^{10},$$

$$E_{11,F}(x)=-\frac{889776019}{64}-\frac{2012744167x}{32}+\frac{197596465x^2}{16}+\frac{1414238925x^3}{16} +\frac{50677935x^4}{8}-\frac{62095033x^5}{4}-\frac{19688669x^6}{16}+\frac{2912525x^7}{8} +\frac{83215x^8}{4}-\frac{4895x^9}{4}-\frac{89x^{10}}{2}+x^{11}.$$

We investigate the zeros of the Euler-Fibonacci polynomials $E_{n,F}(x) = 0$. by using a computer. We plot the zeros of the Euler-Fibonacci polynomials $E_{n,F}(x) = 0$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 15$.

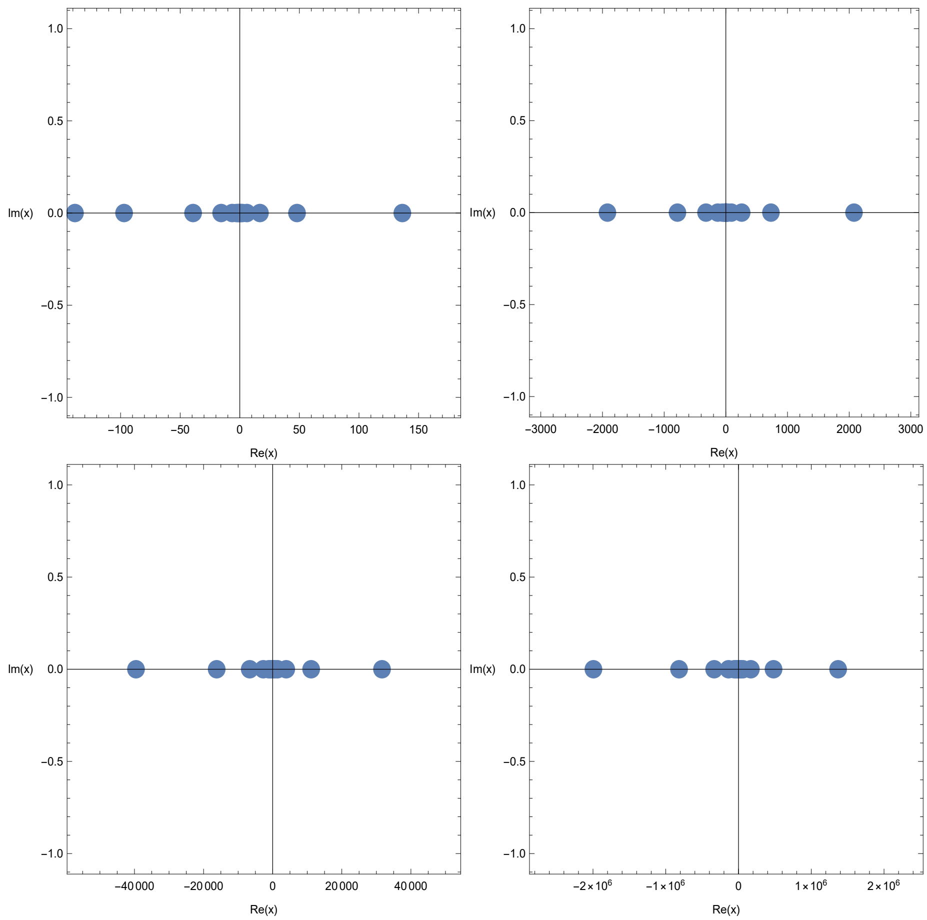

FIGURE 1. Zeros of $E_{n,F}(x) = 0$

In Figure 1(top-right), we choose $n = 25$. In Figure 1(bottom-left), we choose $n = 35$. In Figure 1(bottom-right), we choose $n = 45$.

Stacks of zeros of the Euler polynomials $E_n(x) = 0$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 2).

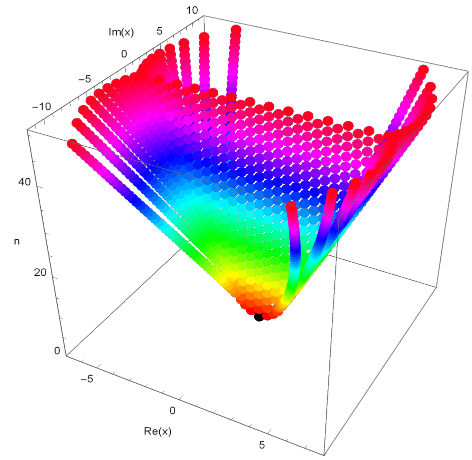

FIGURE 2. Stacks of zeros of $E_n(x) = 0$ for $1 \leq n \leq 40$

Stacks of zeros of the Euler-Fibonacci polynomials $E_{n,F}(x) = 0$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 3).

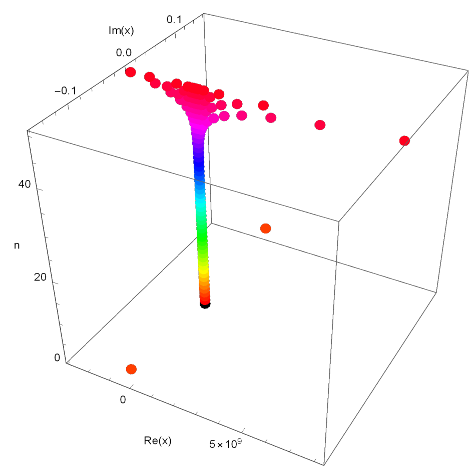

FIGURE 3. Stacks of zeros of $E_{n,F}(x) = 0$ for $1 \leq n \leq 40$

The plot of real zeros of Euler polynomials $E_n(x) = 0$ for $1 \leq n \leq 40$ structure are presented(Figure4).

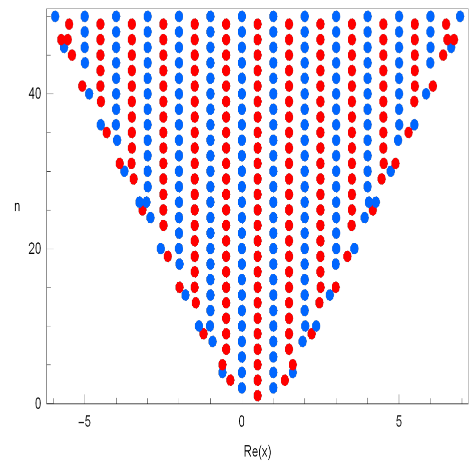

FIGURE 4. Real zeros of $E_n(x) = 0 = 0$ for $1 \leq n \leq 50$

The plot of real zeros of Euler-Fibonacci polynomials $E_{n,F}(x) = 0$ for $1 \leq n \leq 50$ structure are presented(Figure 5).

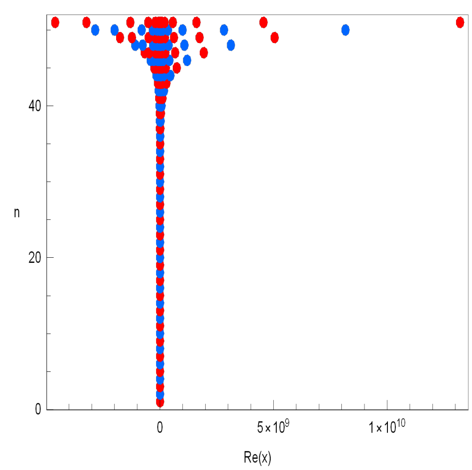

FIGURE 5. Real zeros of $E_{n,F}(x) = 0 = 0$ for $1 \leq n \leq 50$

Next, we calculated an approximate solution satisfying Euler-Fibonacci polynomials $E_{n,F}(x) = 0$ for $x \in \mathbb{C}$. The results are given in Table 1.

Table 1. Approximate solutions of $E_{n,F}(x) = 0$

degree n	x
1	0.50000
2	$-0.30902, \quad 0.80902$
3	$-0.58504, \quad 0.34445, \quad 1.2406$
4	$-0.62348 - 0.15690i, \quad -0.62348 + 0.15690i,$ $0.76158, \quad 1.9854$
5	$-1.1041, \quad -0.94468, \quad 0.21728,$ $1.1144, \quad 3.2171$
6	$-1.8098, \quad -1.1845, \quad -0.71841,$ $0.71086, \quad 1.7998, \quad 5.2020$
7	$-2.9500, \quad -2.0958, \quad -0.88759,$ $0.078327, \quad 1.0363, \quad 2.9000,$ 8.4188
8	$-4.7588, \quad -3.3229, \quad -1.3169,$ $-0.73430, \quad 0.65446, \quad 1.6582,$ $4.6995, \quad 13.621$
9	$-7.7092, \quad -5.4213, \quad -2.1919,$ $-0.88794, \quad -0.071429, \quad 0.96992,$ $2.6730, \quad 7.5993, \quad 22.040$

Conflicts of interest : The authors declare no Conflicts of interest.

Data availability : Not applicable

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