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A FAMILY OF HOLOMORPHIC FUNCTIONS ASSOCIATED WITH MUTUALLY ADJOINT FUNCTIONS†

K.R. KARTHIKEYAN, G. MURUGUSUNDARAMOORTHY AND N.E. CHO*

ABSTRACT. In this paper, making use of symmetric differential operator, we introduce a new class of ℓ-symmetric - mutually adjoint functions. To make this study more comprehensive and versatile, we have used a differential operator involving three-parameter extension of the well-known Mittag-Leffler functions. Mainly we investigated the inclusion relation and subordination conditions which are the main results of the paper. To establish connections or relations with earlier studies, we have presented applications of main results as corollaries.

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1. Introduction

Let $\mathcal{H}$ be the class of holomorphic (analytic) functions in the open unit disc $\mathcal{U} = \{z : | z |< 1\}$. Let

$$\mathcal{H}(a, n) = \{f \in \mathcal{H},\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ \ldots\}$$

be the subclass of $\mathcal{H}$. Also, let

$$\mathcal{A} = \{f \in \mathcal{H},\ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots\} \tag{1}$$

and two functions $f,\ g \in \mathcal{A}$ are called mutually adjoint if for all $z \in \mathcal{U}$

$$\operatorname{Re} \frac{zf'(z)}{f(z) + g(z)} > 0 \quad \text{and} \quad \operatorname{Re} \frac{zg'(z)}{f(z) + g(z)} > 0,$$

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which is denoted by $\mathcal{MS}^*$. Lewandowski and Stankiewicz in [7] established that if $f(z)$ and $g(z)$ are mutually adjoint, then the function $\psi(z) = \frac{f(z)+g(z)}{2}$ is starlike and both $f(z)$ and $g(z)$ are close-to-convex. We let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$. $\mathcal{S}^*$ and $\mathcal{CC}$ will denote the respective class of starlike and close-to-convex in $\mathcal{U}$.

Example 1.1. Let $f(z) = \frac{1+z}{1-z}$ and $g(z) = z-1$, $|z| < 1$ and let $T(z) = \frac{zf'(z)}{f(z)+g(z)}$ and $R(z) = \frac{zg'(z)}{f(z)+g(z)}$. Then we can see that

$$\mathrm{Re}\,[T(z)] = \mathrm{Re}\left(\frac{2}{(1-z)(3-z)}\right) > 0 \quad (z \in \mathcal{U})$$

and

$$\mathrm{Re}\,[R(z)] = \mathrm{Re}\left(\frac{1-z}{(3-z)}\right) > 0 \quad (z \in \mathcal{U}).$$

It is well known that if $f(z)$ given by (1) is in $\mathcal{S}$, then the ℓ-symmetrical function $\left[f(z^\ell)\right]^{1/\ell}$, ($\ell$ is a positive integer) is also in $\mathcal{S}$ (see [5, pg. 18]). Let ℓ be a positive integer and $\varepsilon = \exp(2\pi i/\ell)$. For $f \in \mathcal{A}$, let

$$f_\ell(z) = \frac{1}{\ell}\sum_{\nu=0}^{\ell-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^\nu}. \tag{2}$$

The class of starlike functions with respect to ℓ-symmetric points, denoted by $\mathcal{S}_\ell^s$, which was defined by Sakaguchi [10] as below:

The function f is said to be starlike with respect to ℓ-symmetric points if it satisfies the condition

$$\mathcal{S}_\ell^s = \{f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f_\ell(z)} > 0, \quad f_\ell(z) \ as \ in (2)\}$$

and shown that $f \in \mathcal{S}_\ell^s$ are univalent. Note that $\mathcal{S}_1^s = \mathcal{S}^*$.

With primary aim of unifying the class of mutually adjoint close-to-convex functions and the class of functions starlike with respect to ℓ-symmetric points, Aouf et al. [2] defined the class of functions $\Omega(\ell, U, V)$ satisfying the subordination condition

$$\frac{z\, g_i'(z)}{\frac{1}{\ell}\sum_{\nu=1}^{\ell} g_\nu(z)} \prec \frac{1+Uz}{1+Vz} \quad (z \in \mathcal{U}; i = 1, 2, \ldots, \ell;\ -1 \le V < U \le 1),$$

where $g_1, \ldots, g_\ell \in \mathcal{A}$. Note that Aouf et al. [2] defined the class $\Omega(\ell, U, V)$ involving multiplier transformation. Notice that $\ell = 2$, $U = 1$ and $V = -1$, the class $\Omega(\ell, U, V)$ reduces to the class $\mathcal{MS}^*$. Also, letting $g_i(z) = f(z)$, $g_\nu(z) = \omega^{-\nu} f(\omega^\nu z)$ $(f \in \mathcal{A};\ \nu = 1, \ldots, \ell;\ \omega = e^{2\pi i/n})$, $U = 1$ and $V = -1$ in $\Omega(\ell, U, V)$, then the class $\Omega(\ell, U, V)$ reduces to the class $\mathcal{S}_\ell^s$.

Recently, Breaz et al. [3] defined the function

$$\Gamma(U, V;\, p;\, \sigma; \Psi) = \frac{[(1+U)p + \sigma(V-U)]\,\Psi(z) + [(1-U)p - \sigma(V-U)]}{[(V+1)\Psi(z) + (1-V)]}, \tag{3}$$

where $\Psi(z) \in \mathcal{P}$, a well-known class of functions with positive real part and is of the form

$$\Psi(z) = 1 + R_1 z + R_2 z^2 + \cdots . \tag{4}$$

Lemma 1.2. [4] *If $p \in \mathcal{P}$, and $\Psi(z)$ is given by (4), then*

$$|R_n| \leq 2 \ for\ n \geq 1, \tag{5}$$

where $\mathcal{P}$ is the family of all functions Ψ analytic in $\mathcal{U}$ for which

$$\mathrm{Re}\,(\Psi(z)) > 0 \quad (z \in \Delta).$$

Using Hadamard product, we let following operator $\Xi(\theta,\, \vartheta,\, \rho)f : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\Xi(\theta,\, \vartheta,\, \rho)f(z) = \left[f(z) * \mathcal{R}^{\rho}_{\theta,\, \vartheta}(z)\right] = z + \sum_{n=2}^{\infty} \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma\left(\vartheta + \theta(n-1)\right)(n-1)!} a_n z^n, \tag{6}$$

where $\mathcal{R}^{\rho}_{\theta,\, \vartheta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta+\theta(n-1))(n-1)!} z^n \, (z,\, \theta,\, \vartheta,\, \rho \in \mathbb{C}, Re(\theta) > 0)$. The function $\mathcal{R}^{\rho}_{\theta,\, \vartheta}(z)$ is the normalized form of the Mittag-Leffler three parameter function (popularly known as Prabhakar function [9]).

Throughout this paper, we assume that $-1 \leq V < U \leq 1$, $\ell \in \mathbb{N}$, $\varepsilon = \exp(2\pi i/\ell)$ and

$$\Xi(\theta,\, \vartheta,\, \rho)f_{j,\, \ell}(z) = \frac{1}{\ell} \sum_{\nu=0}^{\ell-1} \varepsilon^{-\nu j} \left[\Xi(\theta,\, \vartheta,\, \rho)f(\varepsilon^{\nu} z)\right] = z + \cdots \tag{7}$$

$$(f \in \mathcal{A};\, \ell = 1, 2,\, 3,\, \ldots).$$

And let $\Xi(\theta,\, \vartheta,\, \rho)f_{1,\, \ell}(z) = \Xi(\theta,\, \vartheta,\, \rho)f_{\ell}(z)$

Motivated by Aouf et al. [2], we now define the following.

Definition 1.3. For $\Gamma(U, V;\, p;\, \sigma; \Psi)$ defined as in (3), a function $f \in \mathcal{A}$ is said to be in $\mathcal{A}^{m}_{k,\, \sigma}(\theta,\, \vartheta,\, \rho;\, U, V;\, \Psi)$ if and only if

$$\frac{z\left[\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)\right]}{\frac{1}{\ell}\sum_{\nu=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_{\nu}(z)} \prec \Gamma(U, V;\, 1;\, \sigma; \Psi), \quad (z \in \mathcal{U}; i = 1, 2, \ldots, \ell), \tag{8}$$

where $\frac{1}{\ell}\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_{\nu}(z) \neq 0$.

Remark 1.1. For appropriate choice of the parameters, we can see that $\mathcal{MS}^*$ and $\Omega(\ell,\, U, V)$ can be obtained as a special case of $\mathcal{A}_{\sigma}(\theta,\, \vartheta,\, \rho;\, U, V;\, \Psi)$. Now we list a few special cases to illustrate that $\mathcal{A}_{\sigma}(\theta,\, \vartheta,\, \rho;\, U, V;\, \Psi)$ is a complete generalization of various subclasses of starlike functions.

(1) If we let $\theta = \sigma = 0$, $\rho = 1$, $f_i(z) = f(z)$, $f_{\nu}(z) = \omega^{-\nu j} f(\omega^{\nu} z)$ ($f \in \mathcal{A};\, \nu = 1, \ldots, \ell;\, \omega = e^{2\pi i/n}$), $U = 1$ and $V = -1$, then the class $\mathcal{A}_{\sigma}(\theta,\, \vartheta,\, \rho;\, U, V;\, \Psi)$ reduces to the class

$$\mathcal{S}^{s}_{j,\, \ell} = \left\{ f \in \mathcal{A} :\, \frac{zf'(z)}{f_{j,\, \ell}(z)} \prec \Psi(z) \right\}.$$

The class $\mathcal{S}^s_{j,\,\ell}(\Psi)$ was introduced and studied by Karthikeyan in [6].

(2) If we let $\theta = \sigma = 0$, $\rho = 1$, $f_i(z) = f(z)$, $f_\nu(z) = \omega^{-\nu j} f(\omega^\nu z) + \omega^{\nu j}\overline{f(\omega^\nu \overline{z})}$ ($f \in \mathcal{A}$; $\nu = 1, \ldots, \ell$; $\omega = e^{2\pi i/n}$), $U = 1$ and $V = -1$, then the class $\mathcal{A}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V;\, \Psi)$ reduces to the class

$$\mathcal{S}^c_{2j,\,\ell} = \left\{ f \in \mathcal{A} :\ \frac{zf'(z)}{f_{2j,\,\ell}(z)} \prec \Psi(z) \right\}.$$

The class $\mathcal{S}^c_{2j,\,\ell}(\Psi)$ was introduced and studied by Selvaraj et al. in [12] (also see [14]).

Further, for convenience we now define the following.

Definition 1.4. For $\Gamma(U, V;\, p;\, \sigma; \Psi)$ defined as in (3), a function $f \in \mathcal{A}$ is said to be in $\mathcal{M}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V, X, Y)$ if and only if

$$\frac{z\Xi(\theta,\, \vartheta,\, \rho)f'(z)}{\Xi(\theta,\, \vartheta,\, \rho)f_\ell(z) + \Xi(\theta,\, \vartheta,\, \rho)g_\ell(z)} \prec \Gamma(U, V;\, 1;\, \sigma; \Phi), \tag{9}$$

and

$$\frac{z\,\Xi(\theta,\, \vartheta,\, \rho)g'(z)}{\Xi(\theta,\, \vartheta,\, \rho)f_\ell(z) + \Xi(\theta,\, \vartheta,\, \rho)g_\ell(z)} \prec \Gamma(X, Y;\, 1;\, \sigma; \Psi) \tag{10}$$

where $\Xi(\theta,\, \vartheta,\, \rho)f_\ell(z) \neq 0$ and $\Xi(\theta,\, \vartheta,\, \rho)g_\ell(z) \neq 0$ are defined as in (7).

Remark 1.2. If we let $\theta = \sigma = 0$, $\rho = 1$, $\ell = 1$, $U = X = 1$, $V = Y = -1$ and $\Psi(z) = \frac{1+z}{1-z}$, then the class $\mathcal{M}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V, X, Y)$ reduces to the class $\mathcal{MS}^*$ introduced and studied by Lewandowski and Stankiewicz in [7].

2. Integral Representation and Subordination Results.

We begin this section by obtaining the integral representation for functions in the class $\mathcal{M}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V, X, Y)$.

Theorem 2.1. *$f \in \mathcal{M}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V, X, Y)$ if and only if there exists $R(z) = \Gamma\,(U, V;\, 1;\, \sigma; \Psi[w(z)])$ and $T(z) = \Gamma\,(X, Y;\, 1;\, \sigma; \Psi[w(z)])$ in $\mathcal{P}$ such that*

$$\Xi(\theta,\, \vartheta,\, \rho)f_\ell(z) = \int_0^z R(\zeta) \left[\exp \int_0^\zeta \frac{R(\eta) + T(\eta) - 2}{2\eta} d\eta \right] d\zeta$$

and

$$\Xi(\theta,\, \vartheta,\, \rho)g_\ell(z) = \int_0^z T(\zeta) \left[\exp \int_0^\zeta \frac{R(\eta) + T(\eta) - 2}{2\eta} d\eta \right] d\zeta,$$

where $w(z)$ is the Schwartz function.

Proof. Let $f \in \mathcal{M}_\sigma(\theta,\, \vartheta,\, \rho;\, U, V, X, Y)$. Replacing z by $\varepsilon^\nu z$ in (9), (10) and using the fact that $\Xi(\theta,\, \vartheta,\, \rho)f'(\varepsilon^\nu z) = \Xi(\theta,\, \vartheta,\, \rho)f'_\ell(z)$ we can establish that

$$\frac{z\Xi(\theta,\, \vartheta,\, \rho)f'_\ell(z)}{\Xi(\theta,\, \vartheta,\, \rho)f_\ell(z) + \Xi(\theta,\, \vartheta,\, \rho)g_\ell(z)} \prec \Gamma(U, V;\, 1;\, \sigma; \Phi)$$

and

$$\frac{z\Xi(\theta,\ \vartheta,\ \rho)g_\ell'(z)}{\Xi(\theta,\ \vartheta,\ \rho)f_\ell(z)+\Xi(\theta,\ \vartheta,\ \rho)g_\ell(z)} \prec \Gamma(X,Y;\ 1;\ \sigma;\Psi).$$

Therefore $f\in\mathcal{M}_\sigma(\theta,\ \vartheta,\ \rho;\ U,V,X,Y)$ implies $f_\ell\in\mathcal{M}_\sigma(\theta,\ \vartheta,\ \rho;\ U,V,X,Y)$.

By definition 1.4, (9) and (10) can be equivalently written in the form

$$\frac{z\Xi(\theta,\ \vartheta,\ \rho)f_\ell'(z)}{\Xi(\theta,\ \vartheta,\ \rho)f_\ell(z)+\Xi(\theta,\ \vartheta,\ \rho)g_\ell(z)} = R(z) \tag{11}$$

and

$$\frac{z\Xi(\theta,\ \vartheta,\ \rho)g_\ell'(z)}{\Xi(\theta,\ \vartheta,\ \rho)f_\ell(z)+\Xi(\theta,\ \vartheta,\ \rho)g_\ell(z)} = T(z), \tag{12}$$

where $R(z)=\Gamma\left(U,V;\ 1;\ \sigma;\Psi[w(z)]\right)$ and $T(z)=\Gamma\left(X,Y;\ 1;\ \sigma;\Psi[w(z)]\right)$. Using Logarithmic differentiation on (11), we get

$$\frac{\Xi(\theta,\ \vartheta,\ \rho)f_\ell''(z)}{\Xi(\theta,\ \vartheta,\ \rho)f_\ell'(z)} = \frac{R'(z)}{R(z)} + \frac{R(z)+T(z)-2}{2z}.$$

On integrating the above expression, we get

$$\Xi(\theta,\ \vartheta,\ \rho)f_\ell(z) = \int_0^z R(\zeta)\left[\exp\int_0^\zeta \frac{R(\eta)+T(\eta)-2}{2\eta}d\eta\right]d\zeta.$$

Similarly, from (12) we can establish

$$\Xi(\theta,\ \vartheta,\ \rho)g_\ell(z) = \int_0^z T(\zeta)\left[\exp\int_0^\zeta \frac{R(\eta)+T(\eta)-2}{2\eta}d\eta\right]d\zeta.$$

Adding (11) and (12), retracing the steps as in [7, p. 49] we can establish the sufficiency part. Hence, the proof of the theorem is completed. □

Letting $\theta=\sigma=0$, $\mu=\rho=\ell=1$, $U=X=1$ and $V=Y=-1$ in Theorem 2.1, we have the following result.

Corollary 2.2. [7] *$f\in\mathcal{MS}^*$ if and only if there exists $\Phi(z)$, $\Psi(z)\in\mathcal{P}$ such that*

$$f(z) = \int_0^z \Phi(\zeta)\left[\exp\int_0^\zeta \frac{\Phi(\eta)+\Psi(\eta)-2}{2\eta}d\eta\right]d\zeta$$

and

$$g(z) = \int_0^z \Psi(\zeta)\left[\exp\int_0^\zeta \frac{\Phi(\eta)+\Psi(\eta)-2}{2\eta}d\eta\right]d\zeta.$$

Remark 2.1. In general, we note that $\Gamma(U,V;\ 1;\ \sigma;\Psi)$ need not be convex univalent in $\mathcal{A}$. However, the function $\Gamma(U,V;\ 1;\ \sigma;\Psi)$ is convex depending on the choice of $\Psi(z)$ (see [3]).

Lemma 2.3. *Let ℓ be convex in $\mathcal{U}$, with $\ell(0) = d$, $\nu \neq 0$ and* $\operatorname{Re}\nu \geq 0$. *If $r \in \mathcal{H}(d, n)$ and*

$$r(z) + \frac{zr'(z)}{\nu} \prec \ell(z),$$

then

$$r(z) \prec q(z) \prec \ell(z),$$

where

$$q(z) = \frac{\nu}{n\, z^{\nu/n}} \int_0^z t^{(\nu/n)-1} \ell(t) dt.$$

The function q is convex and is the best (d, n)-dominant.

Theorem 2.4. *Let $\Xi(\theta, \vartheta, \rho)f \in \mathcal{A}$ for all $z \in \mathcal{U} \setminus \{0\}$. Also let $\Gamma(U, V;\, 1;\, \sigma; \Psi)$ be convex univalent in $\mathcal{U}$ with $[\Gamma(U, V;\, 1;\, \sigma; \Psi)]_{z=0} = 1$ and* $\operatorname{Re}\, \Gamma(U, V;\, 1;\, \sigma; \Psi) > 0$. *Further, suppose that*

$$\left(\frac{z\,[\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)]}{\frac{1}{\ell}\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_j(z)}\right)^2 \left[3 + 2\left\{\frac{z(\Xi(\theta,\, \vartheta,\, \rho)f_i''(z)}{\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)} - \frac{\sum_{j=1}^{\ell} z\Xi(\theta,\, \vartheta,\, \rho)f_j'(z)}{\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_j(z)}\right\}\right] \prec \Gamma(U, V;\, 1;\, \sigma; \Psi). \tag{13}$$

Then

$$\frac{z\,[\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)]}{\frac{1}{\ell}\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_j(z)} \prec K(z) = \sqrt{\Omega(z)}, \tag{14}$$

where

$$\Omega(z) = \frac{1}{z}\int_0^z \Gamma(U, V;\, 1;\, \sigma; \Psi)\, dt$$

and K is convex and is the best dominant.

Proof. Let

$$r(z) = \frac{z\,[\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)]}{\frac{1}{\ell}\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_j(z)} \quad (z \in \mathcal{U};\ \mu \geq 0).$$

Then $r(z) \in \mathcal{H}(1, 1)$ with $r(z) \neq 0$. By assumption, $\Gamma(U, V;\, 1;\, \sigma; \Psi)$ is convex univalent in $\mathcal{U}$ which in turn implies $\sqrt{\Gamma(U, V;\, 1;\, \sigma; \Psi)}$ is convex and univalent in $\mathcal{U}$. Suppose that $T(z) = r^2(z)$. Then $T(z) \in \mathcal{H}$ with $T(z) \neq 0$ in $\mathcal{U}$.

Using logarithmic differentiation, we have

$$\frac{zT'(z)}{T(z)} = 2\left[1 + \frac{z(\Xi(\theta,\, \vartheta,\, \rho)f_i''(z)}{\Xi(\theta,\, \vartheta,\, \rho)f_i'(z)} - \frac{\sum_{j=1}^{\ell} z\Xi(\theta,\, \vartheta,\, \rho)f_j'(z)}{\sum_{j=1}^{\ell} \Xi(\theta,\, \vartheta,\, \rho)f_j(z)}\right].$$

Thus by (2.4), we have

$$T(z) + zT'(z) \prec \Gamma(U, V;\, 1;\, \sigma; \Psi) \quad (z \in \mathcal{U}). \tag{15}$$

Now by Lemma 2.3, we deduce that

$$T(z) \prec \Omega(z) \prec \Gamma(U, V;\, 1;\, \sigma; \Psi).$$

Since $\operatorname{Re}\Gamma(U,V;\,1;\,\sigma;\Psi)>0$ and $\Omega(z)\prec\Gamma(U,V;\,1;\,\sigma;\Psi)$, we have $\operatorname{Re}\Omega(z)>0$. $\sqrt{\Omega(z)}$ is univalent by the virtue of Ω being univalent and $r^2(z)\prec\Omega(z)$ implies that $r(z)\prec\sqrt{\Omega(z)}$ which establishes the assertion. □

Corollary 2.5. *Let $\Xi(\theta,\,\vartheta,\,\rho)f\in\mathcal{A}$ for all $z\in\mathcal{U}\setminus\{0\}$. If*

$$\operatorname{Re}\left\{\left(\frac{z\left[\Xi(\theta,\,\vartheta,\,\rho)f_i'(z)\right]}{\frac{1}{\ell}\sum_{j=1}^{\ell}\Xi(\theta,\,\vartheta,\,\rho)f_j(z)}\right)^2\left[3+2\left\{\frac{z(\Xi(\theta,\,\vartheta,\,\rho)f_i''(z)}{\Xi(\theta,\,\vartheta,\,\rho)f_i'(z)}-\frac{\sum_{j=1}^{\ell}z\Xi(\theta,\,\vartheta,\,\rho)f_j'(z)}{\sum_{j=1}^{\ell}\Xi(\theta,\,\vartheta,\,\rho)f_j(z)}\right\}\right]\right\}>0,$$

then

$$\operatorname{Re}\left[\frac{z\left[\Xi(\theta,\,\vartheta,\,\rho)f_i'(z)\right]}{\frac{1}{\ell}\sum_{j=1}^{\ell}\Xi(\theta,\,\vartheta,\,\rho)f_j(z)}\right]>\omega(\varsigma),$$

where $\omega(\varsigma)=\sqrt{[2(1-\varsigma)\cdot\log 2+(2\varsigma-1)]}$. The inequality is sharp

Proof. Let $\sigma=0$, $U=1$, $V=-1$ and $\Psi(z)=\frac{1+(2\varsigma-1)z}{1+z}$, $0\le\varsigma<1$ in Theorem 2.4, we can easily get the desired result. □

If we let $\theta=0$ and $\rho=1$ in the Corollary 2.5, then we have the following

Corollary 2.6. *Let $f\in\mathcal{A}$ with $f^{'}(z)$ and $f(z)\neq 0$ for all $z\in\mathcal{U}\setminus\{0\}$. If*

$$Re\left\{\left(\frac{zf^{'}(z)}{f(z)}\right)^2\left[3+\frac{2\,zf^{''}(z)}{f^{'}(z)}-\frac{2\,zf^{'}(z)}{f(z)}\right]\right\}>\varsigma,$$

then

$$\operatorname{Re}\frac{zf^{'}(z)}{f(z)}>\omega(\varsigma),$$

where $\omega(\varsigma)=\sqrt{[2(1-\varsigma)\cdot\log 2+(2\varsigma-1)]}$. This inequality is sharp

3. Coefficient Inequalities

Theorem 3.1. *Let $f_\nu(z)=z+\sum_{n=2}^{\infty}a_{\nu,\,n}z^n$ be defined in $\mathcal{A}$ and let $f\in\mathcal{A}_\sigma(\theta,\,\vartheta,\,\rho;\,U,V;\,\Psi)$. Then for $n\ge 2$,*

$$|n\,a_n-b_n|\le\frac{(U-V)(1-\sigma)|R_1|\sum_{t=1}^{n-1}|\Delta_n(\vartheta,\rho,\theta)\,b_t|}{2\,|\Delta_n(\vartheta,\rho,\theta)|},\tag{16}$$

where $b_n=\frac{1}{\ell}\left[a_{1,\,n}+\cdots+a_{\ell,\,n}\right]$ and $\Delta_n(\vartheta,\rho,\theta)=\frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta+\theta(n-1))(n-1)!}$.

Proof. By the definition of $\mathcal{A}_\sigma(\theta,\,\vartheta,\,\rho;\,U,V;\,\Psi)$, we have

$$\frac{z\left[\Xi(\theta,\,\vartheta,\,\rho)f_i'(z)\right]}{\frac{1}{\ell}\sum_{\nu=1}^{\ell}\Xi(\theta,\,\vartheta,\,\rho)f_\nu(z)}=p(z),\tag{17}$$

where $p(z)\in\mathcal{P}$ is subordinate to $p(z)\prec\frac{(U+1)\psi(z)-(U-1)}{(V+1)\psi(z)-(V-1)}$ and $\Psi(z)$ is defined as in (4).

Equivalently, (17) can be written as

$$\sum_{n=2}^{\infty}\Delta_n(\vartheta,\rho,\theta)na_nz^n=\left(\sum_{n=1}^{\infty}p_nz^n\right)\left(\sum_{n=1}^{\infty}\Delta_n(\vartheta,\rho,\theta)b_nz^n\right)$$

$$(b_1 = \Delta_1(\vartheta, \rho, \theta) = 1).$$

Equating the coefficient of z^n on both sides

$$\begin{aligned}|[n\,a_n - b_n]\,\Delta_n(\vartheta,\rho,\theta)| &= [\Delta_{n-1}(\vartheta,\rho,\theta)b_{n-1}p_1 + \cdots + p_{n-1}\Delta_1(\vartheta,\rho,\theta)b_1] \\ &= \sum_{t=1}^{n-1} |p_t\,\Delta_n(\vartheta,\rho,\theta)\,b_t| \le \sum_{t=1}^{n-1} |p_t\,\Delta_n(\vartheta,\rho,\theta)|\;|b_t|.\end{aligned}$$

From [3, Lemma 4], we have $|p_t| \le \frac{|R_1|(U-V)(1-\sigma)}{2}$, $t \ge 1$. On computation we have

$$|[n\,a_n - b_n]| \le \frac{(U-V)(1-\sigma)|R_1|\sum_{t=1}^{n-1}|\Delta_n(\vartheta,\rho,\theta)\,b_t|}{2\,|\Delta_n(\vartheta,\rho,\theta)|}. \tag{18}$$

□

If we let $\sigma = 0$, $f_i(z) = f(z)$, $f_\nu(z) = \omega^{-\nu j} f(\omega^\nu z)$ $(f \in \mathcal{A};\ \nu = 1,\ldots,\ell;\ \omega = e^{2\pi i/n})$ in Theorem 3.1, then we have:

Theorem 3.2. *Let* $\Xi(\theta,\,\vartheta,\,\rho)f_{j,\,\ell}(z) = \frac{1}{\ell}\sum_{\nu=1}^{\ell}\varepsilon^{-\nu j}\,[\Xi(\theta,\,\vartheta,\,\rho)f(\varepsilon^\nu z)]$*. If* $f \in \mathcal{A}$ *satisfies the condition*

$$\frac{z\Xi(\theta,\,\vartheta,\,\rho)f'(z)}{\Xi(\theta,\,\vartheta,\,\rho)f_{j,\,\ell}(z)} \prec \frac{(U+1)\Psi(z)-(U-1)}{(V+1)\Psi(z)-(V-1)},$$

then for $n \ge 2$, $-1 \le V < U \le 1$,

$$|a_n| \le \frac{1}{|\Delta_n(\vartheta,\rho,\theta)|}\prod_{t=1}^{n-1}\frac{|(U-V)R_1\Upsilon_{t,j} - 2\,[t-\Upsilon_{t,j}]\,V|}{2\,|(t+1)-\Upsilon_{t+1,j}|}, \tag{19}$$

where $\Upsilon_{n,j} = \frac{1}{\ell}\sum_{\nu=1}^{\ell}\omega^{(n-j)\nu}$.

Proof. By definition, $\Xi(\theta,\,\vartheta,\,\rho)f_{j,\,\ell}(z) = \sum_{n=1}^{\infty}\Delta_n(\vartheta,\rho,\theta)\Upsilon_{n,j}a_n z^n$, where $\Upsilon_{n,j} = \frac{1}{\ell}\sum_{\nu=1}^{\ell}\omega^{(n-j)\nu}$ $(\Upsilon_{1,j} = 1 = a_1 = \Delta_1(\vartheta,\rho,\theta))$. Replacing $b_n = \Upsilon_{n,j}a_n$ in (18), we have

$$|a_n| \le \frac{(U-V)(1-\sigma)|R_1|\sum_{t=1}^{n-1}|\Delta_n(\vartheta,\rho,\theta)\,\Upsilon_{n,j}a_n|}{2\,|\Delta_n(\vartheta,\rho,\theta)|\,|n-\Upsilon_{n,j}|}. \tag{20}$$

Let $n = 2$ in (20). Then

$$|a_2| \le \frac{(U-V)\,|R_1\,\Upsilon_{1,j}|}{2\,|\Delta_2(\vartheta,\rho,\theta)|\,|2-\Upsilon_{2,j}|}. \tag{21}$$

Letting $n = 2$ in (19), we get

$$|a_2| \le \frac{1}{|\Delta_2(\vartheta,\rho,\theta)|}\prod_{t=1}^{2-1}\frac{|(U-V)R_1\Upsilon_{t,j} - 2\,[t-\Upsilon_{t,j}]\,V|}{2\,|(t+1)-\Upsilon_{t+1,j}|}$$

$$= \frac{1}{|\Delta_2(\vartheta,\rho,\theta)|}\frac{|(U-V)R_1\Upsilon_{1,j}-2\,[1-\Upsilon_{1,j}]\,V|}{2\,|2-\Upsilon_{2,j}|} = \frac{(U-V)\,|R_1\,\Upsilon_{1,j}|}{2\,|\Delta_2(\vartheta,\rho,\theta)|\,|2-\Upsilon_{2,j}|}. \tag{22}$$

From (21) and (22), we find that the hypothesis is correct for $n = 2$. Using induction hypothesis and retracing the steps as in [13, Theorem 2.1], we can establish the assertion of the theorem. □

If we let $\theta = 0$, $\rho = 1$, $\sigma = 0$ and $\Psi(z) = \frac{1+z}{1-z}$ in Theorem 3.2, then we get the following result.

Corollary 3.3. [1, Theorem 2] *If $f \in \mathcal{A}$ satisfies the condition*

$$\frac{zf'(z)}{f_{j,\ell}(z)} \prec \frac{1+Uz}{1+Vz},$$

then for $n \geq 2$, $-1 \leq V < U \leq 1$,

$$|a_n| \leq \prod_{t=1}^{n-1} \frac{|\Upsilon_{t,j}|\,[(U-V)-1]+t}{|t+1-\Upsilon_{t+1,j}|}.$$

If we let $j = \ell = 1$, $U = 1-2\eta$ and $V = -1$ in corollary 3.3

Corollary 3.4. *If $f \in \mathcal{A}$ satisfies the condition*

$$Re\frac{zf'(z)}{f(z)} > \eta,$$

then for $n \geq 2$,

$$|a_n| \leq \prod_{t=0}^{n-1} \frac{2(1-\eta)+t}{1+t}.$$

Remark 3.1. Results obtained by Senguttuvan et al. [13] can be obtained as special case of Theorem 3.2, except for a difference in the coefficient of the defined operator.

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K.R. Karthikeyan received his Ph.D. from National University of Science & Technology. His research interests are Complex Analysis and Special Functions.

Department of Applied Mathematics and Science, National University of Science & Technology, Muscat P.O. Box 620, Oman.
e-mail: karthikeyan@nu.edu.om

G. Murugusundaramoorthy received his Ph.D. from University of Madras. His research interests are Complex Analysis, Geometric Function Theory and Special Functions.

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamilnadu, India.
e-mail: gmsmoorthy@yahoo.com

Nak Eun Cho received Ph.D. from Kyungpook National University. His research interests are Complex Analysis, Geometric Function Theory and Special Functions.

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 608-737, Korea.
e-mail: necho@pknu.ac.kr