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DIVISIBILITY AND ARITHMETIC PROPERTIES OF CERTAIN ℓ-REGULAR OVERPARTITION PAIRS†

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ABSTRACT. For an integer $\ell \geq 1$, let $\bar{B}_\ell(n)$ denotes the number of ℓ-regular over partition pairs of n. For certain conditions of ℓ, we study the divisibility of $\bar{B}_\ell(n)$ and arithmetic properties for $\bar{B}_\ell(n)$. We further obtain infinite family of congruences modulo 2^t satisfied by $\bar{B}_3(n)$ employing a result of Ono and Taguchi (2005) on nilpotency of Hecke operators.

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1. Introduction

A partition of n is a non-increasing sequence of positive integers whose sum is n. In [3], Corteel and Lovejoy introduced overpartitions. An overpartition of n is a partition where the first occurrence of a number may be overlined. In [8], Lovejoy investigated the ℓ-regular overpartition $\bar{A}_\ell(n)$, which counts the number of overpartitions of n with parts not divisible by ℓ. For example, $\bar{A}_2(3) = 4$ with the relevant partitions being $3,\ \bar{3},\ 1+1+1,\ \bar{1}+1+1$. The generating function of $\bar{A}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{A}_\ell(n) q^n = \frac{f_2 f_\ell^2}{f_1^2 f_{2\ell}},$$

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where, for any positive integer k,

$$f_k := (q^k; q^k)_\infty = \prod_{n=1}^{\infty}(1 - q^{kn}), \quad |q| < 1.$$

The arithmetic of ℓ-regular overpartition has received a great deal of attention (See for example [1,12]). In [17], Shen proved a number of arithmetic properties of $\bar{A}_\ell(n)$ and gave a combinatorial interpretation for $\bar{A}_3(9n+3)$ and $\bar{A}_3(9n+6)$ being divisible by 3. Chiranjt and Kalyan [15] studied the divisibility of $\bar{A}_\ell(n)$ by p_i^j's similar to the works of [2,5,13,14]. In [4], the authors explored the arithmetic properties of Fu's k dots bracelet partition where $k = p^\alpha$, p is a prime number with $p \geq 5$ and α is an integer with $\alpha \geq 0$.

The notation of ℓ-regular overpartition pairs $\bar{B}_\ell(n)$ were introduced by Mahadeva Naika and Shivashankar [10], which counts the number of overpartition pairs of n with no parts divisible by ℓ. The generating function of $\bar{B}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{B}_\ell(n) q^n = \frac{f_2^2 f_\ell^4}{f_1^4 f_{2\ell}^2}. \tag{1}$$

Shivashankar and Gireesh [18] established some Ramanujan like congruences for $\bar{B}_\ell(n)$, where $\ell \in \{3, 4, 5, 8\}$. In our main theorem, we study the divisibility properties of $\bar{B}_\ell(n)$, and extend the results on arithmetic properties of $\bar{B}_\ell(n)$ to other values of ℓ. Let p be prime divisor of ℓ. In our first result, we prove that $\bar{B}_\ell(n)$ is almost always divisible by any powers of p under certain conditions on p and ℓ. To be specific, we prove the following:

Theorem 1.1. *Let $\ell = {p_1}^{a_1}{p_2}^{a_2}...{p_m}^{a_m}$ be the prime factorization of an odd integer $\ell \geq 3$. If ${p_i}^{2a_i} \geq \ell$, then for every positive integer j,*

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : \bar{B}_\ell(n) \equiv 0 \pmod{p_i^j}\}}{X} = 1.$$

If ℓ is an odd prime, we obtain the Corollary 1.2 of Theorem 1.1.

Corollary 1.2. *If $p > 3$ is a prime number, then for every positive integer k, we have*

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : \bar{B}_p(n) \equiv 0 \pmod{p^k}\}}{X} = 1.$$

Next, we study the divisibility of $\bar{B}_p(n)$ modulo any powers of 2.

Theorem 1.3. *For every positive integer $k \geq 1$, and for all prime p satisfying $p \leq 2^{k-1}$, we have:*

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : \bar{B}_p(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

Theorem 1.4. *Let $\ell \equiv 3 \pmod 4$ be a positive integer and p_i's are prime numbers such that for $1 \le i \le k+1$, $p_i \equiv 3 \pmod 4$. Then for non-negative integers k, n and any integer $s \not\equiv 0 \pmod{p_{k+1}}$, we have*

$$\bar{B}_\ell\left(4p_1^2p_2^2\cdots p_k^2p_{k+1}^2 n + p_1^2p_2^2\cdots p_k^2 p_{k+1}(p_{k+1}+4s)\right) \equiv 0 \pmod{16}.$$

Different infinite families of congruence can be obtained from Theorem 1.4. Let $\ell \equiv 3 \pmod 4$ be a positive integer and p be prime such that for $p \equiv 3 \pmod 4$. Suppose $p_1 = p_2 = \cdots = p_{k+1} = p$. then for any integer $s \not\equiv 0 \pmod p$, we have:

$$\bar{B}_\ell\left(4p^{2k+2}n + p^{2k+1}(p+4s)\right) \equiv 0 \pmod{16}.$$

In particular for all non-negative integer n and $j \not\equiv 0 \pmod{11}$,

$$\bar{B}_7\left(484n + 121 + 44s\right) \equiv 0 \pmod{16}.$$

In the next theorem, we prove similar result using result of Serre [16] on the action of Hecke operators on cusp forms.

Theorem 1.5. *Let ℓ be a positive integer and p_i's are prime numbers such that for $1 \le i \le k+1$, $p_i \equiv -1 \pmod{18}$. And let $k, n \ge 0$, then for any integer $s \not\equiv 0 \pmod{p_{k+1}}$, we have*

$$\bar{B}_{3\ell}\left(3p_1^2p_2^2\cdots p_k^2p_{k+1}^2 n + p_1^2p_2^2\cdots p_k^2 p_{k+1}(p_{k+1}+3s)\right) \equiv 0 \pmod 8.$$

In the following theorem, using a result of Ono and Taguchi [11] on nilpotency on Hecke operators, we derive infinite family of congruences modulo 2^t satisfied by $\bar{B}_3(n)$.

Theorem 1.6. *Let n be a non-negative integer. Then there is an integer $s \ge 0$ such that for every $t \ge 1$ and distinct primes $p_1, \cdots p_{s+t}$ coprime to 6, we have for n coprime to $p_1, \cdots p_{s+t}$,*

$$\bar{B}_3\left(\frac{p_1\cdots p_{s+t}\cdot n}{24}\right) \equiv 0 \pmod{2^t}.$$

2. Preliminaries

In this section we discuss some definitions and results related to Modular Forms. Let $\mathbb{H}$ denote the upper half plane. The complex vector space of weight k (positive integer) with respect to a congruence subgroup Γ will be denoted by $M_k(\Gamma)$.

Definition 2.1. Let χ be a Dirichlet character modulo N (a positive integer). Then a modular form $f \in M_k(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z), \tag{2}$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular form is denoted by $M_k(\Gamma_0(N), \chi)$. Here $\Gamma_0(N)$ will be the principal congruence subgroup of level N.

The Dedekind's eta-function ($\eta(z)$) is defined by

$$\eta(z) := q^{\frac{1}{24}} f_1 = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \tag{3}$$

where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}, \tag{4}$$

where N is a positive integer and r_δ is an integer.

Theorem 2.2. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that*

$$k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z} \tag{5}$$

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \text{ and} \tag{6}$$

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}. \tag{7}$$

Then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z), \tag{8}$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *Here*

$$\chi(d) := \left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{d}\right). \tag{9}$$

Let f be an eta-quotient satisfying the conditions of Theorem 2.2 and if f is also holomorphic at all the cusps of $\Gamma_0(N)$, then $f \in M_k(\Gamma_0(N), \chi)$. To verify the holomorphicity at cusps of $f(z)$ it suffices to check that the orders at the cusps are non-negative. The necessary criterion for determining orders of an eta-quotient at cusps is the following:

Theorem 2.3. *Let $c, d,$ and N are positive integers with $d \mid N$ and $gcd(c,d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.2 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N} \frac{gcd(d,\delta)^2 r_\delta}{gcd(d,\frac{N}{d}) d\delta}. \tag{10}$$

The following definitions of Hecke operators play important role in proving the main results.

Definition 2.4. Let m be a positive integer and

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi).$$

The Hecke operator T_m acts on $f(z)$ by

$$f(z) \mid T_m := \sum_{n=0}^{\infty} \left(\sum_{d \mid gcd(n,m)} \chi(d) d^{k-1} a\left(\frac{nm}{d^2}\right) \right) q^n. \tag{11}$$

In particular, if $m = p$ is a prime, then

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{k-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{12}$$

Definition 2.5. A modular form $f(z) \in M_k(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exist a complex number $\lambda(m)$ for which

$$f(z) \mid T_m = \lambda(m) f(z). \tag{13}$$

Theorem 2.6. *Let A denote the subset of integer weight modular forms in $M_k(\Gamma_0(N), \chi)$ whose Fourier coefficients are in O_k, the ring of algebraic integers in a number field K. Suppose $M \subset \mathcal{O}_k$ is an ideal. If $f(z) \in A$ has a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n, \tag{14}$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{M}\} = \mathcal{O}\left(\frac{X}{(logX)^{\alpha}}\right). \tag{15}$$

Which yields

$$\lim_{x \to \infty} \frac{\#\{0 < n \leq X : a(n) \equiv 0 \pmod{M}\}}{X} = 1. \tag{16}$$

Proposition 2.7. *Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$ has coefficients in $\mathcal{O}_k$, M is a positive integer, and $k > 1$. Then a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that*

$$f(z) \mid T_p \equiv 0 \pmod{M\mathcal{O}_k}.$$

Theorem 2.8. *Let n be a non-negative integer and k be a positive integer. Let χ be a quadratic Dirichlet character of conductor $9 \cdot 2^n$. There is an integer $c \geq 0$ such that for every $f(z) \in M(\Gamma_0(9 \cdot 2^n), \chi) \cap \mathbb{Z}[[q]]$ and every $t \geq 0$*

$$f(z) \mid T_{p_1} \mid T_{p_2} \cdots \mid T_{p_{c+t}} \equiv 0 \pmod{2^t}.$$

Lemma 2.9.

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \tag{17}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \tag{18}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}. \tag{19}$$

The identity (17) *is 2-dissection of* $\phi(q)^2$ *[7, (1.10.1)]* . (19) *is obtained from [6].*

3. Proof of Theorems

Proof of Theorem 1.1. Suppose $\ell = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where the primes p_i's are greater than 3.
Let

$$B_i(z) = \frac{\eta(24z)^{p_i^{a_i}}}{\eta(24p_i^{a_i} z)} \equiv 1 \pmod{p_i}.$$

Using Binomial theorem for any positive integers, we obtain

$$(q^k; q^k)_\infty^{p^j} \equiv (q^{pk}; q^{pk})_\infty^{p^{j-1}} \pmod{p^j}.$$

Therefore

$$B_i^{p_i^j}(z) = \frac{\eta(24z)^{p_i^{a_i+j}}}{\eta(24p_i^{a_i} z)^{p_i^j}} \equiv 1 \pmod{p_i^{j+1}}.$$

Define

$$C_{i,j,\ell}(z) = \left(\frac{\eta(48z)^2 \eta(24\ell z)^4}{\eta(24z)^4 \eta(48\ell z)^2} \right) B_i^{p_i^j}(z).$$

Then

$$C_{i,j,\ell}(z) \equiv \frac{\eta(48z)^2 \eta(24\ell z)^4}{\eta(24z)^4 \eta(48\ell z)^2} \pmod{p_i^{j+1}}. \tag{20}$$

From identities (1) and (20), we get

$$C_{i,j,\ell}(z) \equiv \sum_{n=0}^{\infty} \bar{B}_\ell(z) q^{24n} \pmod{p_i^{j+1}}. \tag{21}$$

Again let

$$C_{i,j,\ell}(z) = \frac{\eta(24z)^{p_i^{a_i+j}-4} \eta(48z)^2 \eta(24\ell z)^4}{\eta(24p_i^{a_i} z)^{p_i^j} \eta(48\ell z)^2}.$$

From Theorem 2.2, $C_{i,j,\ell(z)}$ is an eta-quotient with a positive integer weight $\frac{p_i^j(p_i^{a_i}-1)}{2}$. Now we calculate the level of $C_{i,j,\ell(z)}$. The level of $C_{i,j,\ell(z)}$ is equal to $48\ell u$, where m is the smallest positive integer satisfying

$$48\ell u\left[\frac{p_i^{a_i+j}-4}{24}+\frac{2}{48}+\frac{4}{24\ell}-\frac{p_i^j}{24p_i^{a_i}}-\frac{2}{48\ell}\right]\equiv 0 \pmod{24}.$$

Equivalently

$$u\times 2\ell\left[p_i^{a_i+j}-p_i^{j-a_i}-2\right]\equiv 0 \pmod{24}.$$

Then $u=12$ and hence the level of eta-quotient $N=576\ell$.

To check the holomorphic nature at the cusp $\frac{c}{d}$, where $d\mid 576\ell$ and $\gcd(c,d)=1$ we use Theorem 2.3. Clearly $C_{i,j,\ell}(z)$ is holomorphic at cusp $\frac{c}{d}$ if and only if

$$(p_i^{a_i+j}-4)\frac{\gcd(d,24)^2}{24}+\frac{\gcd(d,48)^2}{24}+\frac{\gcd(d,24\ell)^2}{6\ell} -\frac{\gcd(d,48\ell)^2}{24\ell}-p_i^{j-a_i}\frac{\gcd(d,24p_i^{a_i})^2}{24}\geq 0. \tag{22}$$

That is

$$(p_i^{a_i+j}-4)\frac{\gcd(d,24)^2}{\gcd(d,48\ell)}+\ell\frac{\gcd(d,48)^2}{\gcd(d,48\ell)^2}+4\frac{\gcd(d,24\ell)^2}{\gcd(d,48\ell)^2} -\ell p_i^{j-a_i}\frac{\gcd(d,24p_i^{a_i})^2}{\gcd(d,48\ell)^2}-1\geq 0. \tag{23}$$

Case(i). When $d=\{2^{r_1}3^{r_2}\cdot t\cdot p_i^k: 0\leq r_1\leq 3,\ 0\leq r_2\leq 2,$ $t\mid \ell \textit{ but } p_i\nmid t \textit{ and } 0\leq k\leq a_i\}$.

Therefore

$$\frac{\gcd(d,24)^2}{\gcd(d,48\ell)}=1/t^2p_i^{2k},\quad \frac{\gcd(d,48)^2}{\gcd(d,48\ell)^2}=1/t^2p_i^{2k},\quad \frac{\gcd(d,24\ell)^2}{\gcd(d,48\ell)^2}=1,$$
$$\frac{\gcd(d,24p_i^{a_i})^2}{\gcd(d,48\ell)^2}=1/t^2.$$

Therefore the left side of equation (23) will become

$$\frac{\ell}{t^2}\left(p_i^j\left(\frac{p_i^{a_i}}{p_i^{2k}}-\frac{1}{p_i^{a_i}}\right)-\frac{3}{p_i2k}\right)+3. \tag{24}$$

For $k=a_i$, the identity (24) ≥ 0 since $3-\frac{3\ell}{t^2p_i^{2k}}\geq 0$ as $p^{2a_i}\geq \ell$.

For $0\leq k\leq a_i$, it is clear that $\frac{p_i^{a_i}}{p_i^{2k}}-\frac{1}{p_i^{a_i}}>0$,

$$\frac{p_i^{a_i}}{p_i^{2k}}-\frac{1}{p_i^{a_i}}-\frac{1}{p_i^{2k}}\geq\frac{p^{2a_i}-p^{2(a_i-1)}-p^{a_i}}{p^{a_i+2k}}=\frac{p^{a_i}\left(p^{a_i}\left(1-\frac{1}{p^2}\right)-1\right)}{p^{a_i+2k}}\geq 0,$$

since $p_i^j\left(1-\frac{1}{p^2}\right) > 1 \ \ \forall p$.
Therefore (24) is non-negative when $p^{2a_i} \geq \ell$.
Case(ii). When $d = \{2^{r_1}3^{r_2} \cdot t \cdot p_i^k : 4 \leq r_1 \leq 6,\ 0 \leq r_2 \leq 2,\ t \mid \ell \ but$ $p_i \nmid t \ and \ 0 \leq k \leq a_i\}$.

Therefore

$$\frac{\gcd(d,24)^2}{\gcd(d,48\ell)} = 1/4t^2p_i^{2k}, \quad \frac{\gcd(d,48)^2}{\gcd(d,48\ell)^2} = 1/t^2p_i^{2k}, \quad \frac{\gcd(d,24\ell)^2}{\gcd(d,48\ell)^2} = 1/4,$$
$$\frac{\gcd(d,24p_i^{a_i})^2}{\gcd(d,48\ell)^2} = 1/4t^2.$$

Therefore left side of (23) can be written as

$$\frac{\ell p_i^j}{4t^2}\left[\frac{p_i^{a_i}}{p_i^{2k}} - \frac{1}{p_i^{a_i}}\right]. \tag{25}$$

Since $a \geq 0$, the above identity (25) is non-negative.

Therefore $C_{i,j,\ell}(z)$ is holomorphic at every cusp $\frac{c}{d}$. The character associated with $C_{i,j,\ell}(z)$ is $\chi(\bullet) = \left(\frac{(-1)^{\frac{p_i^j(p_i^{a_i}-1)}{2}}(24)^{p_i^{a_i+j}}(48)^2(24\ell)^4(48\ell)^{-2}(24p_i^{a_i})^{-p_i^j}}{\bullet}\right)$. Hence $C_{i,j,\ell}(z) \in M_{\frac{p_i^j(p_i^{a_i}-1)}{2}}(\Gamma_0(576\ell),\chi)$.

Applying Theorem 2.6, the Fourier coefficients of $C_{i,j,\ell}(z)$ are almost always divisible by $M = p_i^j$. Therefore using the identity (21), we complete the proof of Theorem 1.1.

For $p > 3$ and $\ell = p$, Corollary 1.2 directly follows from Theorem 1.1.

□

Proof of Theorem 1.3. From (1), generating function of $\bar{B}_p(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{B}_p(n)q^n = \frac{f_2^2 f_p^4}{f_1^4 f_{2p}^2}. \tag{26}$$

Let

$$E_p(z) = \frac{\eta(24pz)^2}{\eta(48pz)}.$$

Using binomial theorem we have

$$E_p^{2^k}(z) = \frac{\eta(24pz)^{2^{k+1}}}{\eta(48pz)^{2^k}} \equiv 1 \pmod{2^{k+1}}.$$

Define $F_{p,k}(z)$ by

$$F_{p,k}(z) := \frac{\eta(48z)^2\eta(24pz)^4}{\eta(24z)^4\eta(48pz)^2} E_p^{2^k}(z). \tag{27}$$

Taking modulo 2^{k+1} in the above identity, we get

$$F_{p,k}(z) \equiv \frac{\eta(48z)^2\eta(24pz)^4}{\eta(24z)^4\eta(48pz)^2}. \tag{28}$$

From identities (26) and (28), we obtain

$$F_{p,k}(z) \equiv \sum_{n=0}^{\infty} \bar{B}_p(n) q^{24n} \pmod{2^{k+1}}. \tag{29}$$

From (27), we have

$$F_{p,k}(z) := \frac{\eta(48z)^2 \cdot \eta(24pz)^{2^{k+1}+4}}{\eta(24z)^4 \cdot \eta(48pz)^{2^k+2}}. \tag{30}$$

From the Theorem 2.2, if $k \geq 1$, $F_{p,k}(z)$ is an eta-quotient with level $N = 192p$ and a positive integer weight 2^{k-1}. The cusps of $\Gamma_0(192p)$ are represented by $\frac{c}{d}$, where $d \mid 192p$ and $\gcd(c,d) = 1$. Using Theorem 2.3, we say that $F_{p,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,48)^2 \cdot 2}{48} + \frac{\gcd(d,24p)^2 \cdot (2^{k+1}+4)}{24p} - \frac{\gcd(d,24)^2 \cdot 4}{24} - \frac{\gcd(d,48p)^2 \cdot (2^k+2)}{48p} \geq 0.$$

If and only if

$$K = A \cdot 2p + B \cdot (2^{k+2} + 2^3) - C \cdot 8p - (2^k + 2) \geq 0, \tag{31}$$

where $A = \frac{\gcd(d,48)^2}{\gcd(d,48p)^2}$, $B = \frac{\gcd(d,24p)^2}{\gcd(d,48p)^2}$, $C = \frac{\gcd(d,24)^2}{\gcd(d,48p)^2}$, respectively.

The table given below shows all the possible values of K. Now we find that for the given condition $p \leq 2^{k-1}$, $K \geq 0$ for all $d \mid 192p$.

$d \mid 192p$	A	B	C	K
$2^\alpha 3^\beta,\ 0 \leq \alpha \leq 3,\ \beta = 0,1$	1	1	1	$-6p + 3 \cdot 2^k + 6$
$2^\alpha 3^\beta p,\ 0 \leq \alpha \leq 3,\ \beta = 0,1$	$\frac{1}{p^2}$	1	$\frac{1}{p^2}$	$-\frac{6}{p} + 3 \cdot 2^k + 6$
$2^\alpha 3^\beta,\ 4 \leq \alpha \leq 6,\ \beta = 0,1$	1	$\frac{1}{4}$	$\frac{1}{4}$	0
$2^\alpha 3^\beta p,\ 4 \leq \alpha \leq 6,\ \beta = 0,1$	$\frac{1}{p^2}$	$\frac{1}{4}$	$\frac{1}{4p^2}$	0

Hence $F_{p,k}(z)$ is holomorphic at a every cusp $\frac{c}{d}$. The character associated with $F_{p,k}(z)$ is $\chi(\bullet) = \left(\frac{(-1)^{2^{k-1}}(24)^{2^{k+1}}(48)^{2^k+4}p^{3\cdot 2^k+6}}{\bullet}\right)$. Theorem 2.2 gives that $F_{p,k}(z) \in M_{2^{k-1}}(\Gamma_0(192p), \chi)$ for all $p \leq 2^{k-1}$ where $k \geq 1$. And the Fourier coefficient of $F_{p,k}(z)$ are all integers. By Theorem 2.6, the Fourier coefficient of $F_{p,k}(z)$ are almost always divisible by 2^k. From (26), $\bar{B}_p(n)$ is almost always divisible by 2^k, thus completes the proof of Theorem 1.3.

□

Proof of Theorem 1.4. Using identity (18) and (17) in (1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{B}_\ell(n)q^n\\ &= \frac{f_2^2}{f_{2\ell}^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\left(\frac{f_{4\ell}^{10}}{f_{2\ell}^2f_{8\ell}^4}-4q^\ell\frac{f_{2\ell}^2f_{8\ell}^4}{f_{4\ell}^2}\right) \qquad (32)\\ &= \frac{f_2^2}{f_{2\ell}^2}\left(\frac{f_4^{14}f_{4\ell}^{10}}{f_2^{14}f_8^4f_{2\ell}^2f_{8\ell}^4}+4q\frac{f_4^2f_8^4f_{4\ell}^{10}}{f_2^{10}f_{2\ell}^2f_{8\ell}^4}-4q^\ell\frac{f_4^{14}f_{2\ell}^2f_{8\ell}^4}{f_2^{14}f_8^4f_{4\ell}^2}-16q^{\ell+1}\frac{f_4^2f_8^4f_{2\ell}^2f_{8\ell}^4}{f_2^{10}f_{4\ell}^2}\right).\end{aligned}$$

For $\ell \equiv 3 \pmod 4$, we have

$$\sum_{n=0}^{\infty}\bar{B}_\ell(2n+1)q^n = 4\frac{f_2^2f_4^4f_{2\ell}^{10}}{f_1^8f_\ell^4f_{4\ell}^4}-4q^{\frac{\ell-1}{2}}\frac{f_2^{14}f_{4\ell}^4}{f_1^{12}f_4^4f_{2\ell}^2}. \qquad (33)$$

On employing binomial theorem, we get

$$\sum_{n=0}^{\infty}\bar{B}_\ell(2n+1)q^n \equiv 4f_2^6-4q^{\frac{\ell-1}{2}}f_{2\ell}^6 \pmod{16}. \qquad (34)$$

Extracting coefficient of q^{2n}, we get

$$\sum_{n=0}^{\infty}\bar{B}_\ell(4n+1)q^n \equiv 4f_1^6 \pmod{16}. \qquad (35)$$

Which implies,

$$\sum_{n=0}^{\infty}\bar{B}_\ell(4n+1)q^{4n+1} \equiv 4\,\eta(4z)^6 \pmod{16}. \qquad (36)$$

Using Theorem 2.2, we get $\eta(4z)^6 \in M_3\left(\Gamma_0(16),\left(\frac{-1}{d}\right)\right)$. Therefore $\eta(4z)^6$ has the Fourier series expansion

$$\eta(4z)^6 = q-6q^5+9q^9+10q^{13}-30q^{17}+\cdots=\sum_{n=1}^{\infty}a(n)q^n.$$

$a(n)=0$ for $n \not\equiv 1 \pmod 4$, and for all $n \geq 0$, we have

$$\bar{B}_\ell(4n+1) \equiv 4a(4n+1) \pmod{16}. \qquad (37)$$

From [9] it is clear that $\eta(4z)^6$ is a Hecke eigenform. Also note that $a(n)=0$ if $n \not\equiv 1 \pmod 4$ and for all $n \geq 0$. From the definitions 2.4 and 2.5, we have

$$\eta(4z)^6 \mid T_p := \sum_{n=1}^{\infty}\left(a(pn)+\left(\frac{-1}{p}\right)\,p^2a\left(\frac{n}{p}\right)\right)q^n = \lambda(p)a(n). \qquad (38)$$

Setting $n=1$ and noting that $a(1)=1$, we readily obtain $a(p)=\lambda(p)$. Since $a(p)=0$ for all $p\not\equiv 1 \pmod 4$, we have $\lambda(p)=0$. Then, we obtain

$$a(pn)+\left(\frac{-1}{p}\right)\, p^2 a\left(\frac{n}{p}\right)=0. \tag{39}$$

Now, consider $p\nmid n$. Then from the identity (39), we get

$$a\left(p^2n+pr\right)=0. \tag{40}$$

For $p\mid n$, from identity (39), we obtain

$$a\left(p^2n\right)=-\left(\frac{-1}{p}\right)\, p^2\, a(n). \tag{41}$$

On replacing n by $4n-pr+1$ in (40), we obtain

$$a\left(4p^2n+p^2+pr\left(1-p^2\right)\right)=0. \tag{42}$$

Using (37) in (42), we obtain

$$\bar{B}_\ell\left(4p^2n+p^2+pr\left(1-p^2\right)\right)\equiv 0 \pmod{16}. \tag{43}$$

Again applying (37) in (41) with n replaced by $4n+1$, we obtain

$$\bar{B}_\ell\left(4p^2n+p^2\right)\equiv -\left(\frac{-1}{p}\right)\, p^2 \bar{B}_\ell\left(4n+1\right) \pmod{16}. \tag{44}$$

Since $\gcd\left(\frac{1-p^2}{4},p\right)=1$, if r runs over a residue system excluding the multiples of p, then so does $\frac{(1-p^2)r}{4}$. Thus for $s\not\equiv 0 \pmod p$, we can rewrite (43) as

$$\bar{B}_\ell\left(4p^2n+p^2+4ps\right)\equiv 0 \pmod{16}. \tag{45}$$

Suppose $p_i\geq 5$ and $p_i\not\equiv 1 \pmod 4$,then

$$\begin{aligned}
&\bar{B}_\ell\left(4p_1^2p_2^2\cdots p_k^2n+p_1^2p_2^2\cdots p_k^2\right) && (46)\\
&=\bar{B}_\ell\left(4p_1^2\left(p_2^2\cdots p_k^2n+\frac{p_2^2\cdots p_k^2-1}{4}\right)+p_1^2\right).\\
&\equiv -\left(\frac{-1}{p_1}\right)\, p_1^2\, \bar{B}_\ell\left(4\left(p_2^2\cdots p_k^2n+\frac{p_2^2\cdots p_k^2-1}{4}\right)+1\right) \pmod{16}.\\
&=-\left(\frac{-1}{p_1}\right)\, p_1^2\, \bar{B}_\ell\left(4p_2^2\cdots p_k^2n+p_2^2\cdots p_k^2\right)\\
&\vdots\\
&\equiv (-1)^k\left(\frac{-1}{p_1}\right)\cdots\left(\frac{-1}{p_k}\right)p_1^2\cdots p_k^2\, \bar{B}_\ell\left(4n+1\right) \pmod{16} && (47)
\end{aligned}$$

Consider $s\not\equiv 0 \pmod{p_{k+1}}$, then identities (45) and (46) implies

$$\bar{B}_\ell\left(4p_1^2p_2^2\cdots p_k^2p_{k+1}^2n+p_1^2p_2^2\cdots p_k^2p_{k+1}(p_{k+1}+4s)\right)\equiv 0 \pmod{16}. \tag{48}$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. From (1), we obtain

$$\sum_{n=0}^{\infty} \bar{B}_{3\ell}(n)q^n = \frac{f_2^2 f_{3\ell}^4}{f_1^4 f_{6\ell}^2}. \tag{49}$$

Using (19) in (49), we obtain

$$\sum_{n=0}^{\infty} \bar{B}_{3\ell}(n)q^n = \frac{f_{3\ell}^4}{f_{6\ell}^2}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2. \tag{50}$$

Extracting coefficient of q^{3n+1}, we obtain

$$\sum_{n=0}^{\infty} \bar{B}_{3\ell}(n)q^n = 4\frac{f_\ell^4}{f_{2\ell}^2}\left(\frac{f_2^7 f_3^9}{f_1^{15} f_6^3}\right). \tag{51}$$

Applying Binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \bar{B}_{3\ell}(3n+1)q^n \equiv 4\frac{f_3^3}{f_1} \pmod 8. \tag{52}$$

Therefore

$$\sum_{n=0}^{\infty} \bar{B}_{3\ell}(3n+1)q^{3n+1} \equiv 4\frac{\eta(9z)^3}{\eta(3z)}. \tag{53}$$

Define

$$A(z) = \frac{\eta(9z)^3}{\eta(3z)}. \tag{54}$$

From Theorem 2.2 and Theorem 2.3, $A(z)$ is a modular form with weight $k = 1$, level $N = 9$ and character $\chi(d) = \left(\frac{-243}{d}\right)$. Hence we have the Fourier series expansion

$$\frac{\eta(9z)^3}{\eta(3z)} = q + q^4 + 2q^7 + 2q^{13} + q^{16} + ... = \sum_{n=1}^{\infty} a(n)q^n. \tag{55}$$

Since there is no constant term in the expansion, $A(z)$ is a cusp form. From (53) and (55), we obtain

$$\bar{B}_{3\ell}(3n+1) \equiv 4 \cdot a(3n+1) \pmod 8. \tag{56}$$

From Proposition 2.7, for $p \equiv -1 \pmod{18}$ $A(z)$ satisfies

$$A(z) \mid T_p \equiv 0 \pmod 2. \tag{57}$$

Thus

$$\sum_{n=0}^{\infty}\left(a(pn) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right)\right)q^n \equiv 0 \pmod 2. \tag{58}$$

Hence

$$a(pn) \equiv a\left(\frac{n}{p}\right) \pmod{2}. \tag{59}$$

Replacing n by $pn+r$ where $p \nmid r$ in (59), we obtain

$$a(p^2n+pr) \equiv 0 \pmod{2}, \text{ since } a\left(\frac{pn+r}{p}\right)=0. \tag{60}$$

Again, replacing n by pn in (59), we obtain

$$a\left(p^2n\right) \equiv a(n) \pmod{2}. \tag{61}$$

Also replacing n by $3n-pr+1$ in (60), we obtain

$$a\left(3p^2n+p^2+pr\left(1-p^2\right)\right) \equiv 0 \pmod{2}. \tag{62}$$

Using (56) in (62), we obtain

$$\bar{B}_{3\ell}\left(3p^2n+p^2+pr\left(1-p^2\right)\right) \equiv 0 \pmod{8}. \tag{63}$$

Again using (56) in (61) with n replaced by $3n+1$, we obtain

$$\bar{B}_{3\ell}\left(3p^2n+p^2\right) \equiv \bar{B}_{3\ell}\left(3n+1\right) \pmod{8}. \tag{64}$$

Since $p \equiv -1 \pmod{18}$, $r\left(1-p^2\right)=3s$, where $s \not\equiv 0 \pmod{p}$ We can rewrite (63) as

$$\bar{B}_{3\ell}\left(3p^2n+p^2+3ps\right) \equiv 0 \pmod{8}. \tag{65}$$

For primes $p_i \geq 17$, $p_i \equiv 17 \pmod{18}$, also we have

$$3p_1^2p_2^2\cdots p_k^2n+p_1^2p_2^2\cdots p_k^2 = 3p_1^2\left(p_2^2\cdots p_k^2n+\frac{p_2^2\cdots p_k^2-1}{3}\right)+p_1^2.$$

Now applying (64) repeatedly, we obtain

$$\bar{B}_{3\ell}\left(3p_1^2p_2^2\cdots p_k^2n+p_1^2p_2^2\cdots p_k^2\right) \equiv \bar{B}_{3\ell}\left(3n+1\right) \pmod{8}. \tag{66}$$

Consider $s \not\equiv 0 \pmod{p_{k+1}}$, then identities (65) and (66) gives

$$\bar{B}_{3\ell}\left(3p_1^2p_2^2\cdots p_k^2p_{k+1}^2n+p_1^2p_2^2\cdots p_k^2p_{k+1}(p_{k+1}+3s)\right) \equiv 0 \pmod{8}. \tag{67}$$

This completes the proof of Theorem 1.5.

□

Proof of Theorem 1.6. Taking $p=3$ in (29), we have

$$F_{3,k}(z) \equiv \sum_{n=0}^{\infty} \bar{B}_3(n)q^{24n} \pmod{2^{k+1}}.$$

This implies

$$F_{3,k}(z) := \sum_{n=0}^{\infty} A(n)q^n \equiv \sum_{n=0}^{\infty} \bar{B}_3\left(\frac{n}{24}\right)q^n \pmod{2^{k+1}}. \tag{68}$$

We have $F_{3,k}(z) \in M_{2^{k-1}}\left(\Gamma_0(9 \cdot 2^6), \chi\right)$. Using Theorem 2.8, we get that there is an integer $s \geq 0$ such that for any $t \geq 1$,

$$F_{3,k}(z) \mid T_{p_1} \mid T_{p_2} \cdots \mid T_{p_{s+t}} \equiv 0 \pmod{2^t}$$

where $p_1, p_2, \cdots p_{s+t}$ are coprime to 6. From the definition of Hecke operators, if $p_1, p_2, \cdots p_{s+t}$ are distinct primes which are coprime to n. Then

$$A\left(p_1 \cdots p_{s+t} \cdot n\right) \equiv 0 \pmod{2^t}. \tag{69}$$

From identities (68) and (69), we complete the proof of Theorem 1.6. □

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References

1. A.M. Alanazi, A.O. Munagi and J.A. Sellers, *An infinite family of congruences for ℓ-regular overpartitions*, Integers **16** (2016), #A37, 8 Pages.
2. K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs*, Int. J. Number Theory **4** (2008), 303-322.
3. S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Am. Math. Soc. **356** (2004), 1623-1635.
4. S.N. Fathima, N.V. Majid, M.A. Sriraj and P. Siva Kota Reddy, *Fu's p^α dots bracelet partition*, Glob. Stoch. Anal. **11** (2024), 1-10.
5. B. Gordon and K. Ono, *Divisibility of certain partition functions by powers of primes*, Ramanujan J. **1** (1997), 25-34.
6. M.D. Hirschhorn and J.A. Sellers, *Arithmetic properties of partitions with odd parts distinct*, Ramanujan J. **22** (2010), 273-284.
7. M.D. Hirschhorn, *The power of q: A personal journey*, Developments in Mathematics, **49**, Cham: Springer, 2017.
8. J. Lovejoy, *Gordon's theorem for overpartitions*, J. Comb. Theory, Ser. A **103** (2003), 393-401.
9. Y. Martin, *Multiplicative η-quotients*, Trans. Am. Math. Soc. **348** (1996), 4825-4856.
10. M.S. Mahadeva Naika and C. Shivashankar, *Arithmetic properties of ℓ-regular overpartition pairs*, Turk. J. Math. **41** (2017), 756-774.
11. K. Ono and Y. Taguchi, *2-adic properties of certain modular forms and their applications to arithmetic functions*, Int. J. Number Theory **1** (2005), 75-101.
12. C. Ray and R. Barman, *Infinite families of congruences for k-regular overpartitions*, Int. J. Number Theory **14** (2018), 19-29.
13. C. Ray and R. Barman, *Arithmetic properties of cubic and overcubic partition pairs*, Ramanujan J. **52** (2020), 243-252.
14. C. Ray and R. Barman, *On Andrews' integer partitions with even parts below odd parts*, J. Number Theory **215** (2020), 321-338.
15. C. Ray and K. Chakraborty, *Certain eta-quotients and ℓ-regular overpartitions*, Ramanujan J. **57** (2022), 453-470.
16. J.-P. Serre, *Valeurs propres des opérateurs de Hecke modulo ℓ*, Astérisque **24-25** (1975), 109-117

17. E.Y.Y. Shen, *Arithmetic properties of l-regular overpartitions*, Int. J. Number Theory **12** (2016), 841-852.
18. C. Shivashankar and D.S. Gireesh, *New congruences for ℓ-regular overpartition pairs*, Afr. Mat. **33** (2022), Id/No 82, 12 Pages.

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