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ON GENERALIZED FRACTIONAL INTEGRAL INEQUALITIES AND APPLICATIONS TO GLOBAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

FATMA SAID, BRAHIM KILANI AND KHALED BOUKERRIOUA*

Abstract. We obtain new fractional integral inequalities which generalize certain inequalities given in [16]. Generalized inequalities can be used to study global existence results for fractional differential equations.

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1. Introduction and Preliminaries

In [6], Henry obtained the following result about weakly singular Gronwall type inequality.

Theorem 1.1. *Let a, b, α, β be nonnegative constants with $\alpha < 1$, $\beta < 1$. Suppose that $u \in L^1[0, T]$ satisfies*

$$u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta} u(s)ds, \quad a.e. t \in (0, T]. \tag{1.1}$$

Then there is a constant $C(b, \beta, T)$ such that

$$u(t) \leq \frac{at^{-\alpha}}{1-\alpha} C(b, \beta, T), \quad a.e. t \in (0, T]. \tag{1.2}$$

Another version of a weakly singular result of Henry is given by the following Theorem.

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Theorem 1.2 (6). *Suppose $\beta > 0, \gamma > 0$, $\beta + \gamma > 1$ and $a \geq 0$, $b \geq 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \leq t < T$, and u satisfies*

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds, \quad a.e. t \in [0,T). \tag{1.3}$$

Then

$$u(t) \leq aE_{\beta,\gamma}(b\Gamma(\beta)^{\frac{1}{\beta+\gamma-1}}t), \tag{1.4}$$

where $E_{\beta,\gamma}(z)$ is given by an infinite series related to the two-parameter Mittag-Leffler function.

Since weakly singular integral inequalities are well-known tools for proving the existence, uniqueness and stability of integral equations and fractional differential equations, many researchers have embarked on the study of these inequalities and derived various versions (for example, see $[1-3, 5, 8-10, 12-13, 15, 17]$ and the references therein). In [14],Willett studied the following inequality by using the Minkowski inequality.

Lemma 1.3. *Let $1 \leq p < \infty, a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $l(t) \in L^1_{Loc}[0,+\infty)$.Suppose $u(t)$ is a nonnegative continuous function on $[0,+\infty)$ with*

$$u(t) \leq a(t) + b(t)(\int_0^t l(s)u^p(s)ds)^{\frac{1}{p}}, \quad t \in [0,\infty). \tag{1.5}$$

then

$$u(t) \leq a(t) + b(t)\frac{(\int_0^t l(s)e(s)a^p(s)ds)^{\frac{1}{p}}}{1-[1-e(t)]^{\frac{1}{p}}},$$

where

$$e(t) = exp(-\int_0^t l(s)b^p(s)ds).$$

By a new method, Tao.Zhu [16] investigated the inequality (1.5) and presented the following results.

Theorem 1.4. *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $t^{-\gamma}l(t) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)u(s)ds, \tag{1.6}$$

then

$$u(t) \leq (A(t) + B(t)\int_0^t L(s)A(s)\exp(\int_s^t L(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t \in [0,+\infty), \tag{1.7}$$

where $A(t) = 2^{q-1}a^q(t)$, $B(t) = \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{p}{q}}}{(p\beta--p+1)^{\frac{q}{p}}}$, $L(t) = t^{-q\gamma}l^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.5. *Let* $\beta \in (0,1)$, $a(t)$ *be a nonnegative and continuous function on* $[0,+\infty)$, $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $l(t) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$ *with*

$$u(t) \le a(t) + t^{1-\beta}\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)u(s)ds, \tag{1.8}$$

then

$$u(t) \le a(t) + b(t)(A(t)\exp(\int_0^t L(s)ds))^{\frac{1}{q}}, \quad t \in [0,\infty). \tag{1.9}$$

where $b(t) = \frac{2^{\frac{1}{p}}t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $A(t) = \int_0^t 2^{q-1}l^q(s)a^q(s)ds$, $L(t) = 2^{q-1}l^q(t)b^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 1.6. *Let* $1 \le p < \infty$, $a(t)$ *and* $b(t)$ *be continuous and nonnegative functions on* $[0,\infty)$, *nonnegative function* $l(t) \in L^p_{Loc}[0,+\infty)$,*and* $u(t)$ *be a continuous and nonnegative function with*

$$u(t) \le a(t) + b(t)(\int_0^t l^p(s)u^p(s)ds)^{\frac{1}{p}}). \tag{1.10}$$

Then

$$u(t) \le a(t) + b(t)(A(t)\exp\int_0^t L(s)ds)^{\frac{1}{p}}, \quad t \in [0,\infty), \tag{1.11}$$

where

$$\begin{aligned} A(t) &= \int_0^t 2^{p-1}l^p(s)a^p(s)ds; \\ L(t) &= 2^{p-1}l^p(t)b^p(t). \end{aligned} \tag{1.12}$$

In this paper, we study the following fractional integral inequality

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds, \tag{1.13}$$

where $\gamma > 0, \beta \in (0,1)$.

We provide some generalizations concerning fractional integral inequality (1.6) , which can be used to study properties of the solutions to the following initial value problem:

$$\begin{cases} D^\beta_r x(t) = f(t,x(t)) \quad t \in (0,\infty), \quad \beta \in (0,1) \\ \lim_{t\to 0^+} t^{1-\beta}x(t) = x_0, \end{cases} \tag{1.14}$$

where D^β_r is the Riemann–Liouville fractional derivative and the function f satisfies certain inequalities. For example, f satisfies an inequality of the form

$$|f(t,x) - f(t,y)| \le b(t)g(|x-y|).$$

Now, we introduce notations, definitions, and several results that are utilized throughout this paper.

Let $\beta \in (0,1)$, denote $C_\beta(0,T] = \{x : (0,T] \to \mathbb{R}$ and $x(t) = t^{-\beta}y(t)$ for some $y \in C[0,T]\}$.Let $\| x \|_\beta = sup_{0<t\preceq T} t^\beta \mid x(t) \mid$, then $C_\beta(0,T]$ endowed with the norm $\| \,.\, \|_\beta$ is a Banach space. We denote $C_\beta(0,+\infty) = \{x : (0,+\infty) \to \mathbb{R}$ and $x(t) = t^{-\beta}y(t)$ for some $y \in C[0,+\infty)\}$. $L^p_{Loc}[0,+\infty)(p \geq 1)$ is the space of all real valued functions which are Lebesgue integrable over every bounded subinterval of $[0,+\infty)$.

Definition 1.7. [11]The Riemann–Liouville fractional integral of order $\beta \in (0,1)$ of a function $f \in L^1[0,T]$ is defined by

$$(I^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds. \tag{1.15}$$

Definition 1.8. [11]The Riemann–Liouville fractional derivative of order $\beta \in (0,1)$ of a function f where $I^{1-\beta} f$ is absolutely continuous (AC) is defined by

$$(D^\beta_r f)(t) = \frac{d}{dt}(I^{1-\beta} f)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\beta} ds. \tag{1.16}$$

For more details about fractional calculus, we refer the reader to [11, 12] .

Definition 1.9. [4]A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathfrak{F}$, if it satisfies the following conditions :

$$\begin{aligned} w(x) &> 0 \text{ is nondecreasing and continuous for } x \geq 0, \\ \frac{1}{a} w(x) &\leq w\left(\frac{x}{a}\right) \text{ for } a > 0 \,. \end{aligned}$$

For example, if $w(x) = x^p, p \geq 1$, then $w(\frac{x}{a}) = (\frac{x}{a})^p \geq \frac{x^p}{a} = \frac{w(x)}{a}$ for $a \in (0,1]$.

Theorem 1.10. ([2]). *Let $f(t,x)$ be a function that is continuous on the set*

$$B = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\},$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x : (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (1.17) *on $(0,T]$ if and only if it satisfies the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s)) ds. \tag{1.17}$$

on $(0,T]$

Lemma 1.11. [16]*Suppose* $f : (0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, and there exist nonnegative functions* $l(t), k(t)$ *with* $t^{\beta-1} l(t) \in C(0,T] \cap L^q[0,T]$ *and* $k(t) \in C(0,T] \cap L^q[0,T] (q > \frac{1}{\beta}, \beta \in (0,1))$ *such that*

$$\mid f(t,x) \mid \leq l(t) \mid x \mid + k(t),$$

for all $(t,x) \in (0,T] \times \mathbb{R}$. *Then the Volterra integral equation* (1.17) *has at least one solution in* $C_{1-\beta}(0,T]$.

Lemma 1.12. [4] *Suppose that* $a \geq 0$, $p \geq q \geq 0$ *and* $p \neq 0$, *then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} \varepsilon^{\frac{q-p}{p}} a + \frac{p-q}{p} \varepsilon^{\frac{q}{p}}, \tag{1.18}$$

for any $\varepsilon > 0$.

Lemma 1.13. [7] *Let* $\alpha, \beta, \lambda,$ *and* p *be positive constants. Then*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\lambda-1)} ds = \frac{t^\theta}{\alpha} \beta \left[\frac{p(\lambda-1)+1}{\alpha},\ p(\beta-1)+1 \right] \quad , \qquad t \in \mathbb{R}_+ \tag{1.19}$$

where

$$B\left[\zeta, \eta\right] = \int_0^1 s^{\zeta-1} (1-s)^{\eta-1} ds \quad (\Re e\ \zeta > 0, \Re e\ \eta > 0), \tag{1.20}$$

is the well-known beta function and

$$\theta = p\left[\alpha(\beta-1) + \lambda - 1\right] + 1 \geq 0. \tag{1.21}$$

2. A generalized fractionnal integral inequalities

In this section, we will now prove several results regarding the generalization of fractional integral inequalities (1.5) and (1.6), which can be employed to investigate the global existence of solutions of fractional differential equation (1.14).

Theorem 2.1. *Let* $\beta \in (0,1)$ *and* $\gamma \geq 0$, $a(t)$ *and* $b(t)$ *be nonnegative and continuous functions on* $[0,+\infty)$ *with* $b(t)$ *is nondecreasing;* $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$. *Let* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be a continuous function such that*

$$0 \leq f(t,x) - f(t,y) \leq L(t,y)(x-y), \quad x \geq y \geq 0, \tag{2.1}$$

for $t \in \mathbb{R}_+$, *where* $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is a continuous function with* $t^{-\gamma}\ l(t) L(t, a(t)) \in L^q_{Loc}[0,+\infty) (q > \frac{1}{\beta})$. *If*

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) f(s,u(s)) ds \tag{2.2}$$

then

$$u(t) \leq a(t)+b(t)(A(t)+B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t\in[0,+\infty), \tag{2.3}$$

where $A(t)=2^{q-1}\tilde{a}^q(t)$, $B(t)=\frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta-p+1)^{\frac{q}{p}}}$, $R(t)=t^{-q\gamma}r^q(t), p\in(1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.
And

$$\tilde{a}(t)=\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s))ds \ , \ r(t)=l(t)L(t,a(t)) \ . \tag{2.4}$$

Proof. Let

$$z(t)=\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds, \tag{2.5}$$

then, we have $z(0)=0$ and

$$u(t)\leq a(t)+b(t)z(t). \tag{2.6}$$

So it follows that

$$z(t)\leq \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s)+b(s)z(s))ds, \tag{2.7}$$

from (2.1), we have

$$f(t,a(t)+b(t)z(t))\leq L(t,a(t))b(t)z(t)+f(t,a(t)), \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$z(t)\leq \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)[L(s,a(s))b(s)z(s)+f(s,a(s))]ds. \tag{2.9}$$

The inequality (2.9) can be reformulated as

$$z(t)\leq \widetilde{a}(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}r(s)z(s)ds, \tag{2.10}$$

where $\widetilde{a}$ and r are defined as in (2.4).
By Theorem 1.4, and using (2.6) we obtain the inequality (2.3) and complete the proof. □

Remark 2.1. Assume $f(t,u(t))=u(t), inequality (2.2) in$ Theorem 2.1 implies inequality (3.1)*in* Theorem 3.1 in [16].

Corollary 2.2. *Let* $\beta\in(0,1)$ *and* $\gamma\geq 0$, $a(t)$ *and* $b(t)$ *be nonnegative and continuous functions on* $[0,+\infty)$*with* $b(t)$ *is nondecreasing,* $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$.*Suppose* $g\colon \mathbb{R}_+\to\mathbb{R}_+$ *is a differentiable increasing function* *on* $]0,+\infty[$ *with continuous non-increasing first derivative* $g^{'}$ *on* $]0,+\infty[$ *with* $t^{-\gamma}l(t)g'(a(t)) \in L^q_{Loc}[0,+\infty)(q>\frac{1}{\beta})$.*If*

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)g(u(s))ds,$$

then

$$u(t) \le a(t)+b(t)(A(t)+B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t\in[0,+\infty),$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta-p+1)^{\frac{q}{p}}}$, $R(t) = t^{-q\gamma}r^q(t)$ and $p\in(1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$.
With

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)g(a(s))ds\ ,\ r(t) = \ l(t)g'(a(t))\ .$$

Proof. Applying the mean value Theorem for the function g, then for every $x \ge y > 0$, there exists $c \in]\ y,x[$such that

$$g(x)-g(y) = g^{'}(c)(x-y) \le g^{'}(0)(x-y),$$

The rest of proof is essentially identical to the proof of Theorem 2.1. □

Corollary 2.3. *Let* $\beta\in(0,1)$ *and* $\gamma\ge 0$*,* $a(t)$ *and* $b(t)$ *be nonnegative and continuous functions on* $[0,+\infty)$ *with* $b(t)$ *is nondecreasing,*$l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$*. Suppose* $\frac{t^{-\gamma}l(t)}{1+a(t)} \in L^q_{Loc}[0,+\infty)(q>\frac{1}{\beta})$.*If*

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)\ln(u(s)+1)ds, \quad t\in[0,\infty),$$

then

$$u(t) \le a(t)+b(t)(A(t)+B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t\in[0,+\infty),$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta--p+1)^{\frac{q}{p}}}$, $R(t) = t^{-q\gamma}r^q(t)$ *and* $p\in(1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.
With

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)\ln(1+a(t))ds\ ,\ r(t) = \frac{b(t)\ l(t)}{\ln(1+a(t))}\ .$$

Theorem 2.4. *Let* $\beta\in(0,1)$ *and* $\gamma\ge 0$*,* $a(t)$ *and* $b(t)$ *be nonnegative and continuous functions on* $[0,+\infty)$*,* $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$*. Let*

$f:\mathbb{R}_+^2\to\mathbb{R}_+$ be a continuous function and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0)=0$ such that

$$0\le f(t,x)-f(t,y)\le L(t,y)\phi^{-1}(x-y), \tag{2.11}$$

for $t\in\mathbb{R}_+$ and $x\ge y\ge 0$, where $L:\mathbb{R}_+^2\to\mathbb{R}_+$ is a continuous function and ϕ^{-1} is the inverse function of ϕ and

$$\phi^{-1}(x.y)\le\phi^{-1}(x)\phi^{-1}(y), \tag{2.12}$$

for $x,y\in\mathbb{R}_+$.If $t^{-\gamma}l(t)\ L(t,a(t))\in L^q_{Loc}[0,+\infty)(q>\frac{1}{\beta})$ and

$$u(t)\le a(t)+b(t)\phi(\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds), \tag{2.13}$$

then

$$u(t)$$
$$\le a(t)+b(t)\phi(A(t)+B(t)\int_0^t K(s)A(s)\exp(\int_s^t K(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}},\quad t\in[0,+\infty), \tag{2.14}$$

where $A(t)=2^{q-1}\tilde{a}^q(t)$, $B(t)=\frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}(\phi^{-1}(b(t)))^q}{(p\beta-p+1)^{\frac{q}{p}}}$, $K(t)=t^{-q\gamma}m^q(t)$ and $p\in(1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$.

With

$$\tilde{a}(t)=\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s))ds\ ,\ m(t)=\ l(t)L(t,a(t))\ . \tag{2.15}$$

Proof. Let

$$z(t)=\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds. \tag{2.16}$$

Then $z(0)=0$, and (2.13) can be written as

$$u(t)\le a(t)+b(t)\phi(z(t)). \tag{2.17}$$

It follows that

$$z(t)\le\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s)+b(s)\phi(z(s))ds, \tag{2.18}$$

from (2.11) and (2.12), we observe that

$$\begin{aligned} f(t,a(t)+b(t)\phi(z(t)))\quad &\le L(t,a(t))\phi^{-1}(b(t)\phi(z(t)))+f(t,a(t))\\ &\le L(t,a(t))\phi^{-1}(b(t))z(t)+f(t,a(t)).\end{aligned} \tag{2.19}$$

Using (2.18) and (2.19), we obtain

$$z(t)\le\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)[L(s,a(s))\phi^{-1}(b(s))z(s)+f(s,a(s))]ds\ . \tag{2.20}$$

The inequality (2.20) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + \phi^{-1}(b(t)) \int_0^t (t-s)^{\beta-1} s^{-\gamma} m(s) z(s) ds, \quad t \in [0, \infty), \tag{2.21}$$

where $\widetilde{a}$ and m are defined as in (2.15).

Applying Theorem 1.4 to (2.21) and using (2.17), we can get the desired inequality (2.14). □

Remark 2.2. Assume $f(t, u(t)) = u(t)$ and $\phi(x) = x$, inequality (2.13) in Theorem 2.4 implies inequality 3.1 in Theorem 3.1 in [16].

Corollary 2.5. *Let $\beta \in (0, 1)$, $a(t)$ be a nonnegative and continuous function on $[0, +\infty)$, $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$. Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0$ satisfy (2.11) − (2.12) and $u(t)$ be a continuous, nonnegative function on $[0, +\infty)$.Suppose $t^{\beta-1} l(t) L(t, a(t)) \in L^q_{Loc}[0, +\infty)(q > \frac{1}{\beta})$.If*

$$u(t) \leq a(t) + t^{1-\beta} \phi\Big(\int_0^t (t-s)^{\beta-1} s^{\beta-1} l(s) f(s, u(s)) ds\Big). \tag{2.22}$$

Then

$$u(t) \leq a(t) + t^{1-\beta} \phi\Big(A(t) + B(t) \int_0^t K(s) A(s) \exp\Big(\int_s^t K(\tau) B(\tau) d\tau\Big) ds\Big)^{\frac{1}{q}}, t \in [0, \infty), \tag{2.23}$$

where $A(t) = 2^{q-1} \tilde{a}^q(t)$, $B(t) = \frac{2^{q-1} t^{q\beta - q + \frac{p}{q}} (\phi^{-1}(t^{1-\beta}))^q}{(p\beta - p + 1)^{\frac{q}{p}}}$, and $p \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

With

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1} s^{\beta-1} l(s) f(s, a(s)) ds \ ,$$
$$K(t) = t^{-q(\beta-1)} m^q(t), m(t) = l(t) L(t, a(t)) \ . \tag{2.24}$$

Theorem 2.6. *Let $1 \leq p \leq q < \infty$, $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0, \infty)$, $l(t)$ be a nonnegative and continuous function on $[0, +\infty)$.Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0$ satisfies (2.11) − (2.12).Suppose $l(t) L(t, \frac{p}{q} \varepsilon^{\frac{p-q}{q}} a(t) + \frac{q-p}{q} \varepsilon^{\frac{p}{q}}) \in L_{Loc}[0, +\infty)$,and $u(t)$ be a continuous and non-negative function .If*

$$u^q(t) \leq a(t) + b(t) \phi\Big(\int_0^t l(s) f(s, u^p(s) ds)\Big). \tag{2.25}$$

Then

$$u(t) \leq (a(t)+b(t)\phi(\widetilde{a}(t)+\phi(\frac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))\frac{\int_0^t k(s)e(s)\widetilde{a}(s)ds}{1-[1-e(t)]})^{\frac{1}{q}}, \quad t\in[0,\infty). \tag{2.26}$$

where $\widetilde{a}(t)=\int_0^t l(s)f(s,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(s)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})ds$, $k(t)=l(t)L(t,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})$ and $e(t)=exp(-\int k(s)\phi(\frac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))ds)$.

Proof. Let

$$z(t)=\int_0^t l(s)f(s,u^p(s)ds. \tag{2.27}$$

Then $z(0)=0$ and from (2.25) can be written as

$$u^p(t)\leq(a(t)+b(t)\phi(z(t)))^{\frac{p}{q}}. \tag{2.28}$$

So it follows that

$$z(t)\leq\int_0^t l(s)f(s,(a(s)+b(s)\phi(z(s)))^{\frac{p}{q}})ds\ , \tag{2.29}$$

from (1.18), (2.11) and (2.12), we have

$$\begin{aligned} f(t,(a(t)+b(t)\phi(z(t)))^{\frac{p}{q}}) &\leq L(t,\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\tfrac{q-p}{q}\varepsilon^{\frac{p}{q}})\phi^{-1}(\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t)\phi(z(t)))\\ &\quad+f(t,\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\tfrac{q-p}{q}\varepsilon^{\frac{p}{q}}).\\ &\leq L(t,\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\tfrac{q-p}{q}\varepsilon^{\frac{p}{q}})\phi^{-1}(\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))z(t)\\ &\quad+f(t,\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\tfrac{q-p}{q}\varepsilon^{\frac{p}{q}}). \end{aligned} \tag{2.30}$$

Using (2.30) and (2.29), we obtain

$$\begin{aligned} z(t)\leq\int_0^t l(s)\big[L(t,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})\phi^{-1}(\frac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))z(t)\\ +f(t,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})\big]ds. \end{aligned} \tag{2.31}$$

The inequality (2.31) can be restated as

$$z(t)\leq\widetilde{a}(t)+\phi^{-1}(\frac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))\int_0^t k(s)z(s)ds, \quad t\in[0,\infty), \tag{2.32}$$

where $\widetilde{a}$ and k are defined as in Theorem 2.6.

Applying Lemma 1.3 for $p=1$ to inequality (2.32) and using (2.28) we get the required inequality in (2.26). □

Remark 2.3. If $f(t,u(t))=u(t)$ and $\phi(x)=x^{\frac{1}{p}}, q=1$, inequality (2.25) can be reduced to inequality (1.5) discussed by Willett in [14].

3. Further Results

In this section, refinements of fractional integral inequalities are presented, in which the right-hand side contains a nonlinear fractional integral term involving class $\mathfrak{F}$ functions. .

Theorem 3.1. *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$, such that $a(t)(a(t) \neq 0)$ is nondecreasing function, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ with $t^{-\gamma} l(t) L(t, a^2(t)) L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$. Let $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a continuous function such that*

$$0 \leq f(t,x) - f(t,y) \leq L(t,y)(x-y), \quad x \geq y \geq 0, \tag{3.1}$$

for $t \in \mathbb{R}_+$, where $L : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a continuous function .If

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) f(s, u(s)) ds, \quad t \in [0,\infty), \tag{3.2}$$

then

$$u(t) \leq a(t) \left\{ 1 + b(t)(A(t) + B(t) \int_0^t R(s) A(s) \exp(\int_s^t R(\tau) B(\tau) d\tau) ds)^{\frac{1}{q}} \right\}, \quad t \in [0,+\infty), \tag{3.3}$$

where
$A(t) = 2^{q-1} \tilde{a}^q(t)$, $\quad B(t) = \frac{2^{q-1} t^{q\beta - q + \frac{p}{q}} b^q(t)}{(p\beta - p + 1)^{\frac{q}{p}}}$, $\quad R(t) = t^{-q\gamma} r^q(t), p \in (1,+\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$,
with
$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) \frac{1}{a(s)} f(s, a^2(s)) ds$,$r(t) = l(t) L(t, a^2(t))$.

Proof. The inequality (3.2) can be written as :

$$\frac{u(t)}{a(t)} \leq 1 + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) \frac{f(s, a(s)\frac{u(s)}{a(s)})}{a(s)} ds. \tag{3.4}$$

Setting $w(t) = \frac{u(t)}{a(t)}$, one can reformulate (3.4) as

$$w(t) \leq 1 + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) \frac{f(s, a(s) w(s))}{a(s)} ds.$$

Let $g(t, w(t)) = \frac{1}{a(t)} f(t, a(t) w(t))$, it is easy to see that $g(t,x)$ satisfies :

$$g(t,x) - g(t,y) \leq L(t, a(t) y)(x-y), \tag{3.5}$$

then

$$w(t) \leq 1 + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(s, w(s)) ds, \tag{3.6}$$

Applying Theorem 2.1 to the inequality (3.6),we get the required inequality in (3.3). $\square$

Theorem 3.2. *Let $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,T)$ $(0<T\le+\infty)$, $l(t)\in L^q_{Loc}[0,T)\cap C[0,T[(q>1)$. Let $g\in C[0,+\infty[$ be a nondecreasing, nonnegative function. and $u(t)$ be a continuous and nonnegative function on $[0,T)$ .If*

$$u(t)\le a(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)g\left(u\left(s\right)\right)ds. \tag{3.7}$$

Then

$$u(t)\le\left\{\Omega_q^{-1}\left[\Omega_q(\widetilde{a}(t))+\widetilde{b}(t)\int_0^t l^q(s)ds\right]\right\}^{\frac{1}{q}}\quad t\in[0,T_1]\,. \tag{3.8}$$

Where

$$\begin{aligned}\widetilde{a}(t)&=2^{q-1}a(t),\\ \widetilde{b}(t)&=2^{q-1}(t^\theta\beta\left[-p\gamma+1,\ p(\beta-1)+1\right])^{\frac{q}{p}}b^q(t),\\ \Omega_q\left(\upsilon\right)&=\int_{\upsilon_0}^{\upsilon}\frac{ds}{g^q\left(s^{\frac{1}{q}}\right)},\end{aligned} \tag{3.9}$$

and $T_1\in(0,T)$ is such that $\Omega_q(\widetilde{a}(t))+\widetilde{b}(t)\int_0^t l^q(s))ds\in Dom(\Omega_q^{-1})$.

Proof. Let

$$z(t)=a(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)g(u(s))ds,$$

then

$$\begin{aligned}u(t)&\le z(t),\\ z(t)&\le a(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)g(z(s))ds.\end{aligned} \tag{3.10}$$

Let $1\le p<+\infty$, such that $\frac{1}{p}+\frac{1}{q}=1$. Using the Hölder inequality, we obtain from (3.10) that

$$z(t)\le a(t)+b(t)\left[\int_0^t (t-s)^{p(\beta-1)}s^{-p\gamma}ds\right]^{\frac{1}{p}}\left[\int_0^t l^q(s)g\left(z\left(s\right)\right)^q ds\right]^{\frac{1}{q}}. \tag{3.11}$$

Since $(A+B)^n\le 2^{n-1}\left(A^n+B^n\right)$ holds for any $A\ge0, B\ge0$ and using Lemma 1.13,

$$\int_0^t (t-s)^{p(\beta-1)}s^{-p\gamma}ds=t^\theta\beta\left[-p\gamma+1,\ p(\beta-1)+1\right]\ ,\ t\in\mathbb{R}_+, \tag{3.12}$$

we obtain from (3.11) that

$$z^q\left(t\right)\le 2^{q-1}a(t)+2^{q-1}(t^\theta\beta\left[-p\gamma+1,\ p(\beta-1)+1\right])^{\frac{q}{p}}b^q(t)\int_0^t l^q(s)g^q\left(z(s)\right)ds,$$

where $\theta=p\left[(\beta-1)-\gamma\right]+1$.

Then the above inequality can be reformulated as

$$z^q\left(t\right)\le\widetilde{a}(t)+\widetilde{b}(t)\int_0^t l^q(s)g^q\left(z(s)\right)ds. \tag{3.13}$$

Let $t^*\in[0,t]$ be a positive constant chosen, we get

$$z^{q}\left(t\right)\leq\widetilde{a}(t^{*})+\widetilde{b}(t^{*})\int_{0}^{t}l^{q}(s)g^{q}\left(z(s)\right)ds. \tag{3.14}$$

Let $G(t)$ be the right-hand side of the inequality (3.14) ,then $z\left(t\right)\leq G^{\frac{1}{q}}\left(t\right)$ and this yields $g^{q}\left(z(t)\right)\leq g^{q}\left(G^{\frac{1}{q}}\left(t\right)\right)$. It is clear that

$$\frac{G'(t)}{g^{q}\left(G^{\frac{1}{q}}\left(t\right)\right)}=\frac{\widetilde{b}(t^{*})l^{q}(t)g^{q}\left(z(t)\right)}{g^{q}\left(G^{\frac{1}{q}}\left(t\right)\right)},$$

i.e.,

$$\frac{d}{dt}\int_{0}^{G(t)}\frac{d\sigma}{g^{q}\left(\sigma^{\frac{1}{q}}\right)}\leq\widetilde{b}(t^{*})l^{q}(t),$$

or

$$\frac{d}{dt}\Omega_{q}\left(G\left(t\right)\right)\leq\widetilde{b}(t^{*})l^{q}(t). \tag{3.15}$$

where Ω_q is defined by (3.9).

Integrating the inequality(3.15) from 0 to t, we obtain

$$\Omega_{q}\left(z\left(t\right)^{q}\right)\leq\Omega_{q}(\widetilde{a}(t^{*}))+\widetilde{b}(t^{*})\int_{0}^{t}l^{q}(s)ds, \tag{3.16}$$

Letting $t=t^{*}$ in (3.16) and considering $t^{*}>0$ is arbitary, after substituting t^{*} with t, we get

$$\Omega_{q}\left(z\left(t\right)^{q}\right)\leq\Omega_{q}(\widetilde{a}(t))+\widetilde{b}(t)\int_{0}^{t}l^{q}(s)ds. \tag{3.17}$$

Then

$$z(t)\leq\left\{\Omega_{q}^{-1}\left[\Omega_{q}(\widetilde{a}(t))+\widetilde{b}(t)\int_{0}^{t}l^{q}(s)ds\right]\right\}^{\frac{1}{q}}. \tag{3.18}$$

This completes the proof of Theorem. □

Inspired by the concept of inequality (3.7), one can derive a bound of an fractional integral inequality in the next corollary using functions of class $\mathfrak{F}$ (introduced in Section 1).

Corollary 3.3. *Let $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,T)$ $(0<T\leq+\infty)$, $l(t)\in L^{q}_{Loc}[0,T)(q>1)$.Let $g\in C[0,+\infty[$ belongs to class $\mathfrak{F}$ (see Definition 1.9), and $a(t)\neq 0$ be nondecreasing function in $[0,X)$ and $u(t)$ be a continuous and nonnegative function on $[0,T)$ .If*

$$u(t)\leq a(t)+b(t)\int_{0}^{t}\left(t-s\right)^{\beta-1}s^{-\gamma}l(s)g\left(u\left(s\right)\right)ds,\quad t\in[0,\infty). \tag{3.19}$$

Then

$$u(t)\leq a(t)\left\{\Omega_{q}^{-1}\left[\Omega_{q}(2^{q-1})+\widetilde{b}(t)\int_{0}^{t}l^{q}(s)ds\right]\right\}^{\frac{1}{q}}\quad t\in\left[0,T_{1}\right]. \tag{3.20}$$

where $\widetilde{b}(t), \Omega_q$ *are defined as in Theorem 3.2.*

Proof. The inequality (3.19) can be rewritten as

$$\frac{u(t)}{a(t)} \leq 1 + \frac{1}{a(t)} b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} g\left(u\left(s\right)\right) ds \quad . \tag{3.21}$$

Since $a(x)$ is nondecreasing function, we get

$$\frac{u(t)}{a(t)} \leq 1 + b(t) \int_0^t \frac{1}{a(s)} \left[(t-s)^{\beta-1} s^{-\gamma} l(s) g\left(u\left(s\right)\right)\right] ds. \tag{3.22}$$

Let $z(t) = \frac{u(t)}{a(t)}$.Since g belongs to class $\mathfrak{F}$, one has

$$z(t) \leq 1 + b(t) \int_0^t \left[(t-s)^{\beta-1} s^{-\gamma} l(s) g\left(z\left(s\right)\right)\right] ds, \tag{3.23}$$

the rest of the proof is identical to the proof of the Theorem 3.2. □

4. Applications

In this section, we present an alternative condition, distinct from that described in [16], to investigate the existence and uniqueness of solutions for the initial value problem (1.17).

Theorem 4.1. *If* $f : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, and*

$$|f(t,x) - f(t,y)| \leq l(t) g\left(|x-y|\right), \tag{4.1}$$

for all $x, y \in \mathbb{R}$ *and* $t \in (0, +\infty)$, *where* g *is defined as in Corollary* 2.2 *such that* $g(0) = 0$, $l(t) \in C(0, +\infty) \cap L^q_{Loc}[0, +\infty)$ *and* $|f(t, 0)| \in L^q_{Loc}[0, +\infty)$ $(q > \frac{1}{\beta})$.*Then equation* (1.17)*has a unique global solution on* $(0, +\infty)$.

Proof. We know

$$|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq l(t) g(|x|) + |f(t,0)|.$$

Applying the mean value Theorem for the function g, then for every $|x| > 0$, there exists $c \in]\ 0, |x|[$such that

$$g(|x|) - g(0) = g^{'}(c)(|x| - 0) \leq g^{'}(0)(|x| - 0),$$

then

$$|f(t,x)| \leq l(t) g^{'}(0)|x| + |f(t,0)|.$$

By Lemma 1.11, the equation (1.17)has at least one global solution .

Now, suppose $x_1(t), x_2(t)$ are two global solutions of equation (1.17). Then

$$\begin{aligned} |\ x_1(t) - x_2(t)\ | &= \left| \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (f(s, x_1(s)) - f(s, x_2(s))) ds \right| \\ &\leq \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\beta-1} s^{1-\beta} l(s) g(|\ x_1(s) - x_2(s)\ |) ds. \end{aligned} \tag{4.2}$$

Let $u(t) =|\ x_1(s) - x_2(s)\ |, L(t) = s^{1-\beta} l(s)$,then

$$u(t) \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\beta-1} L(s) g(u(s)) ds. \tag{4.3}$$

By Corollary 2.2, we can get $x_1(t) = x_2(t)$. Thus the proof is complete. □

Example 4.2.

$$\begin{cases} D_r^{\frac{3}{5}} x(t) = t^{\frac{-7}{12}} \arctan(x) + t^{\frac{-1}{2}}, \\ \lim_{t \to 0^+} t^{\frac{2}{5}} x(t) = 1. \end{cases} \tag{4.4}$$

We know that

$$\begin{aligned} |\, f(t,x) - f(t,y) \,| &= \left| t^{\frac{-7}{12}} \arctan(x) - t^{\frac{-7}{12}} \arctan(y) \right| \\ &= t^{\frac{-7}{12}} \left| \arctan(\frac{x-y}{1+xy}) \right| \leq t^{\frac{-7}{12}} \arctan(|\, x-y \,|), \end{aligned}$$

where $x, y \in (0, +\infty)$. Since $t^{\frac{-7}{12}} \in C(0,+\infty) \cap L^q_{Loc}[0,+\infty)$ and $t^{\frac{-1}{2}} \in C(0,+\infty) \cap L^q_{Loc}[0,+\infty)$ $(q > \frac{5}{3})$, then from Theorem 4.1, equation (4.4)has a unique global solution on $(0, +\infty)$.

5. Conclusion

In this work, several new fractional integral inequalities were derived. They can be considered as generalizations and refinements of many existing results. These nequalities help in the investigation of qualitative properties of certain classes of fractional equations.

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Fatma Said is currently Ph.d. student of mathematics at University of Annaba, Algeria. His research interests include in differential equations, dynamical systems and integral inequalities.

Department of Mathematics, Laboratory Lanos, Badji Mokhtar-Annaba University, Annaba, Algeria.
e-mail: `saidfatma2323@gmail.com`

Brahim Kilani is currently assistant professor of mathematics at University of Annaba, Algeria. His research interests include differential equations, dynamical systems, partial differential equations and integral inequalities.

Department of Mathematics, Laboratory of Mathematics, Dynamics and Modelization, Badji Mokhtar-Annaba University, Annaba, Algeria.
e-mail: `kilbra2000@yahoo.fr`

Khaled Boukerrioua is currently professor of mathematics at University of Annaba. His research interests include differential equations, dynamical systems, partial differential equations and integral inequalities.

Department of Mathematics, Laboratory Lanos, Badji Mokhtar-Annaba University, Annaba, Algeria.
e-mail: `khaledv2004@yahoo.fr`