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FUZZY STONE SPACE OF AN ADL

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Abstract. The set of all prime L-fuzzy ideals of an ADL A with truth values in a frame L is topologized and the resulting space is denoted by $\mathcal{F}_L$ *Spec* (A), called fuzzy Stone space of A. Certain properties of the space $\mathcal{F}_L$ *Spec* (A) are discussed, and it is proved that $\mathcal{F}_L$ *Spec* (A) is homeomorphic with the product space *Spec* $(A) \times$ *Spec* (L).

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1. Introduction

Zadeh [20] introduced the notion of a fuzzy subset of a set X as a function from X into $I = [0, 1]$. Since Rosenfeld [10] introduced and developed the theory of fuzzy sets in realm of group theory, many authors are engaged in fuzzyfying various concepts of abstract algebras, such as groups, rings, vector spaces, lattices, etc. (for, refer $[1, 2, 3, 4, 5, 6, 8, 9]$).

The concept of prime ideals is very crucial in the study of structure theory of distributive lattices in general and of Boolean algebras in particular. M.H. Stone [14] proved that any distributive lattice (Boolean algebra) isomorphic to the sublattice (subalgebra, respectively) of a lattice (algebra) of subsets of the set of all prime ideals of the corresponding lattice, and he established a duality between Boolean algebras and Boolean spaces (i.e., compact Haussdorff and totally disconnected spaces). For this, he constructed *Spec* (L), the space of prime ideals of a bounded distributive lattice L.

Swamy and Rao [18] have introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of the ring theoretic and Lattice theoretic generalizations of Boolean rings and Boolean algebras. Also, they extended

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the concept of prime ideals to ADLs and obtained a topological space by Stone topology on the set of prime ideals of an ADL A, resulting space is called Stone space or prime spectrum, denoted by $Spec\ (A)$. Further, Swamy, Sundar Raj and Natneal, [11, 12, 15] have introduced the concepts of L-fuzzy ideals, prime L-fuzzy ideals and maximal L-fuzzy ideals, of an ADL A with the truth values in a complete lattice L satisfying the infinite meet distributive law:

$$a \wedge \big(\bigvee_{b \in S} b \big) = \bigvee_{b \in S} (a \wedge b)$$

for any $S \subseteq L$ and $a \in L$ such a lattices are called frames. Also, Natnael, Srikanya and Santhi Sundar Raj [7] have defined L-fuzzy prime spectrum of ADLs, and Srikanya, Praksh Babu, Ramanuja Rao and Santhi Sundar Raj [13] have introduced the concepts of fuzzy initial and final segments of an ADL. An element $p \neq 1$ in L is called meet - prime or prime if for any $a, b \in L, a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$. An element $m \neq 1$ in L is called a dual atom (maximal) if there is no $a \in L$ such that $m < a < 1$.
Throughout this paper, $A = (A, \wedge, \vee, 0)$ stands for an ADL with a maximal element and $L = (L, \wedge, \vee, 0)$ stands for a frame. In section 2, we recall certain definitions and results to be used in the sequel. In section 3, a topology is defined on the set of prime L-fuzzy ideals of an ADL A and the resulting space is called fuzzy Stone space, denoted by $\mathcal{F}_L\ Spec\ (A)$. Here it is shown that $\mathcal{F}_L\ Spec\ (A)$ is T_0-space. A base for the topolgy of $\mathcal{F}_L\ Spec\ (A)$ is obtained and studied the properties of basic open sets. For any meet prime element α in L, it is proved that the subspace

$$Y_\alpha = \big\{ \lambda \in \mathcal{F}_L\ Spec\ (A)\ :\ Im\ \lambda = \{1, \alpha\} \big\}$$

is compact. Also, it is proved that this space Y_α is T_1 if and only if every element of Y_α is a maximal L-fuzzy ideal of A. Further, we established a set of equivalent conditions for $\mathcal{F}_L\ Spec\ (A)$ to be T_1. Finally, it is proved that $\mathcal{F}_L\ Spec\ (A)$ is homeomorphic with product space of $Spec\ (A)$ and $Spec\ (L)$.

2. Preliminaries

In this section, we present some basic definitions and results which will be useful later on.

Definition 2.1. [18] An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(6) $(a \vee b) \wedge b = b$

Any bounded below distributive lattice is an ADL. Any nonempty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0 & \text{if } a = 0 \\ b & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL.

Definition 2.2. [18] Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define $a \leq b$ if $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.3. [18] *The following hold for any a, b and c in an ADL A.*

(1) $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $a \wedge b \leq b \leq b \vee a$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$
(5) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e., $\wedge$ is associative)*
(6) $a \vee (b \vee a) = a \vee b$
(7) $a \leq b \Rightarrow a \wedge b = a = b \wedge a$ *and* $a \vee b = b = b \vee a$
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$ *whenever* $a \wedge b = 0$
(11) $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$ *(and hence A is a distributive lattice), whenever $a \leq x$ and $b \leq x$ for some $x \in A$.*
(12) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$
(13) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \vee b = \sup\{a, b\}$.
(14) *The set $\{y \in A : y \leq a\}$ is a bounded distributive lattice under the induced operations $\wedge$ and $\vee$ with 0 is the least element and a is the largest element.*
(15) *m is maximal in $(A, \leq) \Leftrightarrow m \wedge a = a$ ($\Leftrightarrow m \vee a = m$) for all $a \in A$.*

Definition 2.4. [18] Let I be a non empty subset of an ADL A. Then I is called an ideal of A if $a, b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$. As a consequence, for any ideal I of A, $x \wedge a \in I$ for all $a \in I$ and $x \in A$.

Definition 2.5. [18] A prime ideal P of an ADL A, we mean that $P \neq A$, and for any $a, b \in A$, the condition that $a \wedge b \in P$ implies that either $a \in P$ or $b \in P$. An ideal $M \neq A$ is said to be maximal if there is no ideal I of A such that $M \subsetneqq I \subsetneqq A$.

Definition 2.6. [11] An L -fuzzy subset λ of an ADL A is called an L -fuzzy ideal of A, if

(1) $\lambda(0) = 1$ and
(2) $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$.

Theorem 2.7. [11] *An L-fuzzy subset λ of an ADL A is an L-fuzzy ideal of A iff each α-cut, $\lambda_\alpha = \{x \in A \ : \ \lambda(x) \geq \alpha\}$, $\alpha \in L$, is an ideal of A iff the following hold:*
(1) $\lambda(0) = 1$
(2) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$ *and*
(3) $\lambda(x \wedge y) \geq \lambda(x) \vee \lambda(y)$, *for all* $x, y \in A$.

Definition 2.8. [11] A proper L-fuzzy ideal λ of an ADL A is called a prime L-fuzzy ideal if, for any L-fuzzy ideals ν and μ of A, $\nu \wedge \mu \leq \lambda$ implies either $\nu \leq \lambda$ or $\mu \leq \lambda$.

Definition 2.9. A proper L-fuzzy ideal λ of an ADL A is said to be maximal if λ is a maximal element in the set of all proper L-fuzzy ideals of A with respect to pointwise ordering.

3. Fuzzy Stone space

Let X denote the set of all prime L-fuzzy ideals of an ADL A. For any L-fuzzy subset μ of A, let
$V(\mu) = \{\lambda \in X : \mu \leq \lambda\}$and
$X(\mu) = \{\lambda \in X : \mu \nleq \lambda\}$.
It can be easily seen that, for any L-fuzzy subset μ of A,
$V(\mu) = V(\bar{\mu})$ and $X(\mu) = X(\bar{\mu})$,
where $\bar{\mu}$ is the L-fuzzy ideal of A generated by μ.

Theorem 3.1. *Let $\mathcal{T} = \{X(\mu) : \ \mu \ is\ an\ L-fuzzy\ ideal\ of\ A\}$. Then the pair $(X, \mathcal{T})$ is a topological space.*

Proof. Consider the L-fuzzy ideals μ and ν of A defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and $\nu(x) = 1$

for all $x \in A$. Then $V(\mu) = X$ and $V(\nu) = \emptyset$ and hence $X(\mu) = \emptyset$ and $X(\nu) = X$, so that μ, $\nu \in \mathcal{T}$. Next, let μ and ν be any two L-fuzzy ideals of A. Then
$\lambda \in V(\mu) \cup V(\nu) \Rightarrow \mu \leq \lambda$ or $\nu \leq \lambda$
$\Rightarrow \mu \wedge \nu \leq \lambda$
$\Rightarrow \lambda \in V(\mu \wedge \nu)$
and
$\lambda \in V(\mu \wedge \nu) \Rightarrow \mu \wedge \nu \leq \lambda$
$\Rightarrow \mu \leq \lambda$ or $\nu \leq \lambda$ ($\because \lambda$ is a prime L-fuzzy ideal of A)
$\Rightarrow \lambda \in V(\mu) \cup V(\nu)$
Hence $V(\mu) \cup V(\nu) = V(\mu \vee \nu)$ and thus
$X(\mu) \wedge X(\nu) = X(\mu \wedge \nu)$.

This shows that $\mathcal{T}$ is closed under finite intersections.
Finally, let $\{\ \mu_i\ :\ i \in \Delta\}$ be any family of L-fuzzy ideals of A. Then, it can be easily verified that

$$\bigcap \{V(\mu_i)\ :\ i \in \Delta\} = V(\overline{\bigvee_{i\in\Delta} \mu_i}),$$

so that

$$\bigcup \{X(\mu_i)\ :\ i \in \Delta\} = X(\overline{\bigvee_{i\in\Delta} \mu_i}).$$

Hence $\mathcal{T}$ is closed under arbitrary unions. Thus $\mathcal{T}$ is a topology on X. □

Definition 3.2. Let $x \in A$ and $\beta \in L$, define $x_\beta : A \to L$ by

$$x_\beta(y) = \begin{cases} \beta & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

for all $y \in A$, is called a L-fuzzy point corresponding to x and β.

In the following, we discuss the space X in terms of α- level L-fuzzy ideals and L-fuzzy points of A. First let us recall from [11] that for any ideal I of A and $\alpha \in L$, the L-fuzzy ideal α_I of A defined by

$$\alpha_I(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases}$$

and that α_I is called the α- level L-fuzzy ideal corresponding to an ideal I of A. Also, it is proved that α_I is a prime (maximal) L-fuzzy ideal of A if and only if I is a prime (maximal) ideal of A and α is a meet prime element (dual atom) in L. Moreover, it is prove that, every prime (maxmal) L-fuzzy ideal of A is of the form α_P for some prime (maximal) ideal P of A and meet prime element (dual atom) α in L. In particular, if $0 \in L$ is meet prime, then the characterstic function χ_P corresponding to a prime ideal P of A defined by

$$\chi_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{otherwise} \end{cases}$$

is a prime L-fuzzy ideal of A.

Theorem 3.3. *For any $x \in A$ and $\beta \in L$,*

$$X(x_\beta) = \{\alpha_P \in X\ :\ x \notin P \text{ and } \beta \nleq \alpha\}.$$

Proof. $\lambda \in X(x_\beta) \Rightarrow \lambda = \alpha_P$ for some prime ideal P of A and a meet prime element α in L such that $x_\beta \nleq \alpha_P$.
$\Rightarrow\ x_\beta(y) \nleq \alpha_P(y)$ for some $y \in A$
$\Rightarrow\ \beta \nleq \alpha_P(y)$ and $y = x$
$\Rightarrow\ x \notin P,\ \beta \nleq \alpha$ and $\lambda = \alpha_P$.
On other hand, let $\alpha_P \in X$ such that $x \notin P$ and $\beta \nleq \alpha$. Then $\alpha_P(x) = \alpha$, so that $x_\beta \nleq \alpha_P$. □

From [11] recall that, for any ideal I of A and $\beta \leq \alpha$ in L, the L- fuzzy ideal $< \alpha, \beta >_I$ defined by

$$< \alpha, \beta >_I (x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } 0 \neq x \in I \\ \beta & \text{if } x \notin I. \end{cases}$$

Theorem 3.4. (1) *For any $x \in A$ and $\alpha \in L$, $X\big(x_\alpha\big) = X(\lambda)$ for some L-fuzzy ideal λ of A.*
(2) *The subfamily $\{X(x_\beta) \ : \ x \in A$ and $\beta \in L\}$ of $\mathcal{T}$ is a base for $\mathcal{T}$.*

Proof. (1). It follows by the following facts:
$\overline{x_\alpha} = \langle \alpha, 0 \rangle_{(x]}$
and $X\big(x_\alpha\big) = X\big(\overline{x_\alpha}\big)$.
(2). By (1), $\{X(x_\beta) \ : \ x \in A$ and $\beta \in L\}$ is a subfamily of $\mathcal{T}$. Let $X(\mu) \in \mathcal{T}$ and $\lambda \in X(\mu)$. Then $\mu \nleq \lambda$ and hence there exist $x \in A$ such that $\mu(x) \nleq \lambda(x)$. Let $\mu(x) = \beta$. Then, $x_\beta(x) \nleq \lambda(x)$, so that $x_\beta \nleq \lambda$ and hence $\overline{x_\beta} \nleq \lambda$. Therefore, $\lambda \in X(\overline{x_\beta}) = X(x_\beta)$.
Now $V(\mu) \subseteq V(x_\beta)$; for,
$\sigma \in V(\mu) \Rightarrow \mu \leq \sigma$
$\Rightarrow \mu(x) \leq \sigma(x)$
$\Rightarrow \beta \leq \sigma(x)$
$\Rightarrow x_\beta(x) \leq \sigma(x)$
$\Rightarrow x_\beta \leq \sigma$
$\Rightarrow \sigma \in V(x_\beta)$.
Hence $X(x_\beta) \subseteq X(\mu)$. Thus $\lambda \in X(x_\beta) \subseteq X(\mu)$. □

Definition 3.5. Let X be set of all prime L-fuzzy ideals of an ADL A. Then, by theroem 3.4(2), the class $\{X(x_\beta) \ : \ x \in A$ and $\beta \in L\}$ forms a base for the topology on X is called the fuzzy stone toplogy, in honour of M.H. Stone [14] and X together with the fuzzy stone topology is called the fuzzy stone space of A and is denoted by $\mathcal{F}_L$ *Spec* (A) (the fuzzy spectrum of A).

Next we prove some properties of basic open sets of X.

Theorem 3.6. *The following hold for any $x, y \in A$ and $0 \neq \alpha \in L$.*

(1) $X(x_\alpha) \subseteq X(y_\alpha)$ *if and only if* $(x] \subseteq (y]$
(2) $X(x_\alpha) = X(y_\alpha)$ *if and only if* $(x] = (y]$
(3) $X(x_\alpha) \cap X(y_\alpha) = X\big((x \wedge y)_\alpha\big)$
(4) $X(x_\alpha) = \emptyset$ *if and only if* $x = 0$, *where* $0 \in L$ *is meet-prime*
(5) $X(x_\alpha) = X$ *if and only if x is maximal, where* $0 \in L$ *is meet-prime.*

Proof. (1). Assume that $(x] \subseteq (y]$ and
$\mu \in X(x_\alpha) \Rightarrow x_\alpha \nleq \mu$
$\Rightarrow \alpha \nleq \mu(x)$
$\Rightarrow x \notin \mu_\alpha$
$\Rightarrow (x] \nsubseteq \mu_\alpha$

$\Rightarrow (y] \nsubseteq \mu_\alpha$
$\Rightarrow \alpha \nleq \mu(y)$
$\Rightarrow y_\alpha \nleq \mu$
$\Rightarrow \mu \in X(y_\alpha)$
Conversely, assume that $X(x_\alpha) \subseteq X(y_\alpha)$. Then $V(y_\alpha) \subseteq V(x_\alpha)$. For any $z \notin (y]$, there exists a prime ideal P of A such that $z \notin P$ and $y \in P$. Now $\chi_P \in X$ and $\chi_P \in V(y_\alpha)$. This implies $\chi_P \in V(x_\alpha)$ and hence $x_\alpha \leq \chi_P$ so that $\alpha \leq \chi_P(x)$ and hence $x \in P$. Therefore $z \notin P$ and $x \in P$ so that $z \notin (x]$. Thus $(x] \subseteq (y]$.
(2). It is clear by (1).
(3). Let $\mu \in X(x_\alpha) \cap X(y_\alpha)$. Then $x_\alpha \nleq \mu$ and $y_\alpha \nleq \mu$. This implies $\beta \nleq \mu(x)$ and $\beta \nleq \mu(y)$ so that x and $y \notin \mu_1 = \{a \in A : \mu(a) = 1\}$. Since μ_1 is a prime ideal of A, $x \wedge y \notin \mu_1$. Therefore $\mu(x) = \mu(y) = \mu(x \wedge y)$, since $|Im\ \mu| = 2$. This implies $(x \wedge y)_\alpha \nleq \mu$ so that $\mu \in X\big((x \wedge y)_\alpha\big)$. On other hand, if $\mu \in X\big((x \wedge y)_\alpha\big)$ then $(x \wedge y)_\alpha \nleq \mu$. This implies $\alpha \nleq \mu(x \wedge y) = \mu(y \wedge x)$ so that $\alpha \nleq \mu(x)$ and $\alpha \nleq \mu(y)$, since μ is an antitone. Therefore $\mu \in X(x_\alpha) \cap X(y_\alpha)$.
(4). Let P be a prime ideal of A and $\mu = \chi_P$, the characteristic function of P. Then χ_P is a prime L-fuzzy ideal of A, since $0 \in L$ is meet prime so that $\mu \in X$ and that $\mu_1 = P$. Next, if $X(x_\alpha) = \emptyset$ then $V(x_\alpha) = X$ which implies that $x_\alpha \leq \mu$ and therefore $\alpha \leq \mu(x)$ so that $\mu(x) = 1$ and thus $x \in P$. Hence $x \in \bigcap\{P\ :\ P \text{ is a prime ideal of } A\} = \{0\}$ and thus $x = 0$.
Conversely, assume that $x = 0$. Let $\mu \in X$. Then the 1-cut μ_1 is a prime ideal of A and so $0 \in \mu_1$. Therefore $\mu(x) = 1$. Hence $x_\alpha(x) \leq \mu(x)$. Therefore $x_\alpha \leq \mu$ so that $\mu \in V(x_\alpha)$ for all $\mu \in X$. Thus $V(x_\alpha) = X$, i.e. $X(x_\alpha) = \emptyset$.
(5). Let P be a prime ideal of A and $\mu = \chi_P$. Then $\mu \in X$. Assume that $X(x_\alpha) = X$. Then $X_\alpha \nleq \mu$ and hence $\alpha \nleq \mu(x)$ so that $\mu(x) = 0$ and therefore $x \notin P$. Therefore $x \notin \bigcup\{P\ :\ P \text{ is a prime ideal of } A\}$. Consequently x is maximal.
Conversely, suppose x is maximal. Then there is no prime ideal of A contains x. Let $\mu \in X$. Then μ_1 is a prime ideal of A so that $x \notin \mu_1$ and hence $\mu(x) = 0$. Therefore $x_\alpha(x) \nleq \mu(x)$ and hence $\mu \in X(x_\alpha)$. Thus $X(x_\alpha) = X$. □

Theorem 3.7. *Let $\alpha \in L$ be a meet prime element in L and let $Y_\alpha = \big\{\lambda \in X\ :\ Im\ \lambda = \{1, \alpha\}\big\}$. Then the subspace Y_α is compact.*

Proof. As $\{X(x_\beta)\ :\ x \in A \text{ and } 0 \neq \beta \in L\}$ is a base for the topology $\mathcal{T}$ on X, it can be easily seen that the family

$$\{X(x_\beta) \cap Y_\alpha\ :\ x \in A,\ \beta \nleq \alpha\}$$

constitutes a base for the subspace topology on Y.
Now, let

$$\big\{X\big((x_i)_{\beta_i}\big) \cap Y_\alpha\ :\ i \in \Delta,\ x_i \in A \text{ and } \beta_i \nleq \alpha\big\}$$

be any open cover of Y_α by its basic open sets. Let $\beta = Sup\{\beta_i\ :\ i \in \Delta\}$. Then the family

$$\big\{X\big((x_i)_\beta\big) \cap Y_\alpha\ :\ i \in \Delta \text{ and } \beta \nleq \alpha\big\}$$

also covers Y_α. Now,
$Y_\alpha = \bigcup \{X((x_i)_\beta) \cap Y_\alpha \ : \ i \in \Delta \text{ and } \beta \nleq \alpha\}$
$= \left(\bigcup \{X((x_i)_\beta) : \ i \in \Delta \text{ and } \beta \nleq \alpha\}\right) \bigcap Y_\alpha$
$= \left(X - V\left(\bigcup \{(x_i)_\beta \ : \ i \in \Delta \text{ and } \beta \nleq \alpha\}\right)\right) \bigcap Y_\alpha$
$= Y_\alpha - \left(V\left(\bigcup \{(x_i)_\beta \ : \ i \in \Delta \text{ and } \beta \nleq \alpha\}\right) \bigcap Y_\alpha\right)$.
This implies that
$V\left(\bigcup \{(x_i)_\beta \ : \ i \in \Delta \text{ and } \beta \nleq \alpha\}\right) \bigcap Y_\alpha = \emptyset$
Let P be a prime ideal of A and define a prime L-fuzzy ideal μ of A by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in P \\ \alpha & \text{if } x \notin P \end{cases}$$

Then $\mu = \alpha_P$ so that $Im\mu = \{1, \alpha\}$ and hence $\mu \in Y_\alpha$.
Clearly $\mu \notin \ V\left(\bigcup \{(x_i)_\beta \ : \ i \in \Delta \text{ and } \beta \nleq \alpha\}\right)$. Hence there exists $i \in \Delta$ such that $(x_i)_\beta \nleq \mu$ and hence $\beta \nleq \mu(x_i)$. Consequently, $x_i \notin P$. Hence there is no prime ideal of A containing the set $\{x_i \in A \ : \ i \in \Delta\}$. Put $I = \{x_i \in A \ : \ i \in \Delta\}$. Then $(I] = A$. In particular $m \in (I]$, where m is a maximal element in A. Then $m = \left(\bigvee_{i=1}^{n} x_i\right) \wedge y$ for some $x_1, x_2 \ ... \ x_n \in I$ and $y \in A$. Now
$V\left(\bigcup\{(x_i)_\beta \ : \ i = 1, 2, \ ... \ n \text{ and } \beta \nleq \alpha\}\right) \cap Y_\alpha = \emptyset$;
Otherwise if
$\mu \in V\left(\bigcup\{(x_i)_\beta \ : \ i = 1, 2, \ ... \ n \text{ and } \beta \nleq \alpha\}\right) \cap Y_\alpha$
then $\bigcup\{(x_i)_\beta \ : \ i = 1, 2, \ ... \ n \text{ and } \beta \nleq \alpha\} \nleq \mu$ and $Im\ \mu = \{1, \alpha\}$ which implies $\beta = (x_i)_\beta(x_i) \leq \mu(x_i)$ for all $i = 1, 2, \ ... \ n$ so that $\mu(x_i) = 1$ for all $i = 1, 2 \ ... \ n$. Since $\beta \nleq \alpha$, $x_i \in P$ for all $i = 1, 2, \ ... \ n$, so that $\bigvee_{i=1}^{n} x_i \in P$ and hence
$m = \left(\bigvee_{i=1}^{n} x_i\right) \wedge y \in P$;
which is a contradiction.
Therefore $\mu \notin V\left(\bigcup\{(x_i)_\beta \ : \ i = 1, 2, \ ... \ n \text{ and } \beta \nleq \alpha\}\right)$. This shows that $\mu \in X(\bigcup\{(x_i)_\beta \ : \ i = 1, 2, \ ... \ n \text{ and } \beta \nleq \alpha\}$ and hence there exists $j \in \{1, 2 \ ... \ n\}$ such that

$$(x_j)_\beta \nleq \mu \text{ and hence } \beta \nleq \mu(x_j)$$

so that $x_j \notin P$. This shows that $\mu \in X\left((x_j)_\beta\right)$ and so $\mu \in \bigcup_{i=1}^{n} X\left((x_i)_\beta\right)$.
Therefore $Y_\alpha \subseteq \bigcup_{i=1}^{n} \left\{X\left((x_i)_\beta\right) \cap Y_\alpha \ \ : \ \ \beta \nleq \alpha\right\}$. Thus $\left\{X\left((x_i)_\beta\right) \cap Y_\alpha \ \ : \ \ i = 1, 2, \ ... \ n\right\}$ covers Y_α. Hence Y_α is compact. □

Recall that a topological space S is called T_0-space, if for any $x \neq y \in S$, either there exists an open set containing x but not y or else there is an open set containing y but not x. Also recall that a topological space S is T_1 if and only if every subset of S consisting of a single point is closed.

Theorem 3.8. (1) *The space X is T_0.*
(2) $V(\lambda) = \overline{\{\lambda\}}$*, the closure of λ in X, where $\lambda \in X$.*
(3) $\mu \in \overline{\{\lambda\}} \Leftrightarrow \lambda \leq \mu$*, where* $\lambda,\ \mu \in X$
(4) *Let α be a meet prime element in L and let $Y_\alpha = \big\{\lambda \in X\ :\ Im\ \lambda = \{1, \alpha\}\big\}$. Then Y_α is a T_1-space if and only if every element of Y_α is a maximal L-fuzzy ideal of A.*

Proof. (1). Let λ and $\mu \in X$ such that $\lambda \neq \mu$. Then either $\lambda \nleq \mu$ or $\mu \leq \lambda$. Let $\lambda \nleq \mu$. Then $\mu \in X(\lambda)$. Also $\lambda \notin X(\lambda)$ and $X(\lambda)$ is a open set in X. So, X is a T_0-space.
(2). Let $\lambda \in X$. Then $V(\lambda)$ is a closed set in X containing λ and hence $\overline{\{\lambda\}} \subseteq V(\lambda)$. On the other hand, if $\mu \notin \overline{\{\lambda\}}$ then there exists an open set $X - V(\theta)(= X(\theta))$ such that $\mu \in X - V(\theta)$ but $\lambda \notin X - V(\theta)$. Therefore $\theta \nleq \mu$ but $\theta \leq \lambda$; and so $\mu \notin V(\lambda)$. Thus $V(\lambda) \subseteq \overline{\{\lambda\}}$ and the equality follows.
(3). Let $\lambda, \mu \in X$. By (2), $V(\lambda) = \overline{\{\lambda\}}$. Now,
$\mu \in \overline{\{\lambda\}} \Leftrightarrow \mu \in V(\lambda) \Leftrightarrow \lambda \leq \mu$.
(4) Suppose Y_α is a T_1-space and α is a dual atom. Let $\lambda \in Y_\alpha$. Then $\{\lambda\}$ is closed in Y_α.
So, by (2)
$V(\lambda) \cap Y_\alpha = \{\lambda\}$.
Since $\lambda \in Y_\alpha$, we have $Im\ \lambda = \{1, \alpha\}$ and the ideal $\lambda_1 = \{x \in A\ :\ \lambda(x) = 1\}$ is prime. In order to prove that λ is maximal L-fuzzy ideal, we show that λ_1 is a maximal ideal of A. For this, it is sufficient to show that there is no prime ideal of A properly containing λ_1. Let P be a prime ideal of A such that $\lambda_1 \subsetneqq P$. Now define $\theta : A \to L$ by

$$\theta(x) = \begin{cases} 1 & \text{if } x \in P \\ \alpha & \text{otherwise.} \end{cases}$$

Then clearly $\theta = \alpha_P$ so that $\theta \in Y_\alpha$. As $\lambda_1 \subsetneqq P$, there exists $a \in P - \lambda_1$ such that $\lambda(a) = \alpha < 1 = \theta(a)$. Therefore $\lambda \lneqq \theta$. This contradicts the fact that $V(\lambda) \cap Y_\alpha = \{\lambda\}$.
Conversely let $\lambda \in Y_\alpha$ such that λ is a maximal L-fuzzy ideal of A. Then $\lambda = \alpha_M$, where α is a dual atom in L and $M = \lambda_1 = \{x \in A\ :\ \lambda(x) = 1\}$ is a maximal ideal of A. Now we prove that $V(\lambda) \cap Y_\alpha = \{\lambda\}$. Clearly, $\{\lambda\} \subseteq V(\lambda) \cap Y_\alpha$. On the other hand, if $\mu \in V(\lambda) \cap Y_\alpha$, then $\lambda \leq \mu$ and $Im\ \lambda = Im\ \mu = \{1, \alpha\}$. Now $\lambda_1 \subseteq \mu_1$ and by the maximality of λ_1, we get $\lambda_1 = \mu_1$ and consequently $\lambda = \mu$. Therefore $V(\lambda) \cap Y_\alpha = \{\lambda\}$, this shows that $\{\lambda\}$ is a closed subset of Y_α and hence Y_α is a T_1-space. □

Theorem 3.9. *For an ADL A the following statements are equivalent:*
(1) *The space X is T_1*
(2) *Every prime ideal of A is maximal and every meet prime element in L is dual atom.*
(3) *Every prime $L-$fuzzy ideal of A is maximal.*

Proof. (1) $\implies$ (2). Let P be a prime ideal of A and α a meet prime element in L. Then $\lambda = \alpha_p$ is a prime L-fuzzy ideal of A, so that $\lambda \in X$. By (1), $\{\lambda\}$ is closed in X and hence $\overline{\{\lambda\}} = \{\lambda\}$. But we have $\overline{\{\lambda\}} = V(\lambda)$.
Therefore $V(\lambda) = \{\lambda\}$. In order to prove that P is maximal and α is dual atom, it is sufficient to prove that there is no prime ideal of A properly containing P, and no meet prime element bigger than α in L. Let Q be a prime ideal of A and β a meet prime element in L such that $P \subset Q$ and $\alpha < \beta$. Now define $\mu : A \to L$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in Q \\ \beta & \text{otherwise.} \end{cases}$$

Then $\mu = \beta_Q$, so that $\mu \in X$. As $P \subset Q$, there exists $a \in Q$ such that $a \notin P$.
Now $\lambda \leq \mu$ and
$\lambda(a) = \alpha < 1 = \mu(a)$.
Therefore $\lambda \lneqq \mu$. This contradicts the fact that $V(\lambda) = \{\lambda\}$.
(2) $\implies$ (3). It follows by (2).
(3) $\implies$ (1). It follows by (3) and from the fact that, for any $\lambda \in X$, $\overline{\{\lambda\}} = V(\lambda)$. □

We shall conclude the paper with the following. First let us recall [18] that, for any non-trivial ADL A, the class $\{Y(S) : S \subseteq A\}$, where $Y(S) = \{P \in Y : S \nsubseteq P\}$, is a topology on the set Y of all prime ideals of A and is called the hull-kernel topology. Also, the class $\{Y_a : a \in A\}$, where $Y_a = \{P \in Y : a \notin P\}$ form a base for the hull-kernel topology on Y. The set Y together with the hull-kernel topology is called the spectrum of A and is denoted by *Spec A*. It is proved that *Spec A* is a T_0-space and *SpecA* is T_1 if and only if A is relatively complemented if and only if every prime ideal of A is a maximal ideal. Any two topological spaces T_1 and T_2 are said to be homeomorphic if there exists a homeomorphism (i.e. a bijection $g : T_1 \to T_2$ which is a continuous and open mapping) between them.

Theorem 3.10. *The space X is homeomorphic with the product space Spec $A \times$ Spec L.*

Proof. Define $g : Spec\ A \times Spec\ L \to X$ by
$g(P, \alpha) = \alpha_P$
for any $P \in Spec\ A$ and $\alpha \in Spec\ L$. Then by g is a bijection. Let $X(x_\alpha)$ be a basic open set in X. Then
$g^{-1}\big(X(x_\alpha)\big) = \{(P, \beta) \in Spec\ A \times Spec\ L\ ,\ \beta_P \in X(x_\alpha)\}$
$= \{(P, \beta) \in Spec\ A \times Spec\ L\ , x \notin P \text{ and } \alpha \nleq \beta\}$
$= Y_x \times Z_\alpha$.
which is a basic open set in *Spec A* $\times$ *Spec L*, since $Y_x = \{P \in Y : x \notin P\}$ and $Z_\alpha = \{\beta \in L : \alpha \nleq \beta\}$ are the basic open sets of *Spec A* and *Spec L* respectively. Therefore g is continuous. Also, $g(Y_x \times Z_\alpha) = X(x_\alpha)$ and hence g is open mapping, Thus g is homeomorphism. □

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References

1. J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145-174.
2. J.K. Kohli, Rajesh Kumar, *Fuzzy prime spectrum of a ring II*, Fuzzy Sets and Systems **59** (1993), 223–230.
3. N. Kuroki, *On fuzz ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems **5** (1981), 203-215.
4. W.J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy sets and systems **8** (1982), 133-139.
5. D.S. Malik, J.N. Mordeson, *Extensions of fuzzy subrings and fuzzy ideals*, Fuzzy Sets and Systems **45** (1992), 245-251.
6. T.K. Mukhergee, M.K. Sen, *On fuzzy ideals of a ring*, Fuzzy Sets and Systems **21** (1987), 99-104.
7. Natnael Teshale Amare, Srikanya Gonnabhaktula and Ch. Santhi Sundar Raj, *L-fuzzy prime spectrum of ADL's*, Advances in Fuzzy systems **2021** (2021), Article ID 5520736.
8. Y.S. Pawar, S.S. Khopade *Spectrum of L-fuzzy prime ideals of a distributive lattice*, Fuzzy sets and Systems **27**, Article ID 1001-7402(2013) 01-0012-08.
9. Rajesh Kumar, *Fuzzy prime spectrum of a ring*, Fuzzy Sets and Systems **46** (1992), 147–154.
10. A. Rosenfield, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.
11. Ch. Santhi Sundar Raj, A. Natnael Teshale and U.M. Swamy, *Fuzzy Prime ideals of ADLs*, International Journal of Computing Science and Applied Mathematics **4** (2018), 32-36.
12. Ch. Santhi Sundar Raj, A. Natnael Teshale and U.M. Swamy, *Fuzzy Maximal Ideals of ADL'S*, International Journal of Mathematics and its Applications **6** (2018), 819-824.
13. G. Srikanya, G. Prakasam Babu, K. Ramanuja Rao and Ch. Santhisundar Raj, *Fuzzy Initial and Final Segments in ADL's*, International Journal of Analysis and Applications **21** (2023), 67.
14. M.H. Stone, *The theory of representations for Boolean algebras*, Trans. Amer. math. Soc. **40** (1936), 37-111.
15. U.M. Swamy, Ch. Santhi Sundar Raj and A. Natnael Teshale, *Fuzzy Ideals of almost distributive lattices*, Annals of Fuzzy Mathematics and Informatics **14** (2017), 371-379.
16. U.M. Swamy and D.V. Raju, *Algebraic Fuzzy systems*, Fuzzy sets and Systems **41** (1991), 187-194.
17. U.M. Swamy and D.V. Raju, *Fuzzy ideals and congruences of lattices*, Fuzzy Sets and Systems **95** (1998), 249-253.
18. U.M. Swamy and G.C. Rao, *Almost distributive lattice*, J. Australian Math. Soc. (Ser. A) **31** (1981), 77-91.
19. U.M. Swamy and K.L.N. Swamy, *Fuzzy Prime Ideals of Rings*, J. Math. Anal. Appl. **134** (1988), 94-103.
20. L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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