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BAYESIAN AND CLASSICAL INFERENCE FOR TOPP-LEONE INVERSE WEIBULL DISTRIBUTION BASED ON TYPE-II CENSORED DATA

ZAHRA SHOKOOH GHAZANI

Abstract. This paper delves into an examination of both non-Bayesian and Bayesian estimation techniques for determining the Topp-leone inverse Weibull distribution parameters based on progressive Type-II censoring. The first approach employs expectation maximization (EM) algorithms to derive maximum likelihood estimates for these variables. Subsequently, Bayesian estimators are obtained by utilizing symmetric and asymmetric loss functions such as Squared error and Linex loss functions. The Markov chain Monte Carlo method is invoked to obtain these Bayesian estimates, solidifying their reliability in this framework.

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1. Introduction

In numerous investigations of lifetime and reliability, it is common for researchers to lack complete information regarding the failure times of all units under study. Situations may arise where specific units are deliberately removed from the experiment or unintentionally lost altogether. Thus, censored data often emerges from such experiments. This occurrence is well-known as type-I and type-II censoring, representing two prevalent types of censoring schemes.

These two schemes share a characteristic that precludes the removal of units before the final termination point of testing. However, a hybrid scheme, mixture-type censoring, combines elements from type-I and type-II schemes. The introduction of this mixed arrangement can be attributed to [1]and has since gained significant popularity in the realm of reliability and life-testing experiments.

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In the realm of longevity studies, one must take into account situations where the failure of a studied unit can be attributed to various components of the same type, which may not be completely visible. In such cases, the time at which the unit fails is documented and analyzed based on information gathered through observation. Furthermore, this analysis considers the component with the shortest lifespan among those contributing to the failure.

This approach is taken because it is often impractical for researchers to examine all factors contributing to a unit's failures; these factors are concealed or regarded as supplementary risks. For instance, when assessing reliability in second-series systems, researchers focus on identifying the component with the shortest lifespan among all influential components as it leads to failure and can be observed. Should you wish for further exploration of this subject matter, see [2, 3, 4].

The phenomenon of censoring occurs when test subjects, despite being healthy, are lost or excluded from the study. This research focuses on a specific type of censoring where the test is ongoing until the r^{th} unit fails and the r value is predetermined. In such cases, we can determine the probability function of observations $t_1 < \cdots < t_r$ based on an assumed density function.

$$L(\omega; t) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} g(t_i, \omega) \left[1 - G(t_r; \omega)\right]^{n-r}. \tag{1}$$

The ω denotes the distribution parameters, while t_r refers to the specific failure time observed in the r^{th} test unit.

In their scholarly pursuit, [5]delved into the intricate domain of incremental censoring data, specifically focusing on estimating of parameters pertaining to the inverse Weibull distribution. Notably, similar investigations were undertaken by researchers such as [6, 7, 8]respectively, as well as [9].However, their explorations encompassed alternative censored models including exponential, gamma-exponential, Poisson inverse exponential distribution, and Weibull-Poisson distribution correspondingly.

2. The Topp-leone Inverse Weibull Distribution

In this Section, we shall derive three parameters for the Topp-Leone Inverse Weibull distribution. To construct the density and distribution function, let us consider that the probability density function (pdf) and cumulative distribution function (cdf) of the Inverse Weibull distribution are provided by

$$F_{IW}(t; \beta, \theta) = e^{-\frac{\beta}{t^\theta}}, \qquad \beta, \theta > 0,\ t > 0, \tag{2}$$

$$f_{IW}(t; \beta, \theta) = \frac{\beta\theta}{t^{\theta+1}} e^{-\frac{\beta}{t^\theta}}, \tag{3}$$

the probability density function (pdf) and cumulative distribution function (cdf) of the Topp-Leone family are provided by

$$G_{TL}(t; \alpha) = [F(t)]^\alpha [2 - F(t)]^\alpha = [1 - (\bar{F}(t))^2]^\alpha, \quad \alpha > 0,\ t \in \mathbb{R}, \tag{4}$$

$$g_{TL}(t;\alpha) = 2\alpha f(t)\bar{F}(t)[F(t)]^{\alpha-1}[2-F(t)]^{\alpha-1}, \quad \alpha > 0, \tag{5}$$

where $\bar{F}(t) = 1 - F(t)$.

Upon incorporating equations (2) and (3) into equations (4) and (5), we obtain the cumulative distribution function and probability density function of the envisioned model as follows.

$$G_{TLIW}(t;\alpha,\beta,\theta) = \left[1 - \left\{1 - e^{-\frac{\beta}{t^\theta}}\right\}^2\right]^\alpha, \quad \alpha,\beta,\theta > 0, \tag{6}$$

$$g_{TLIW}(t;\alpha,\beta,\theta) = \frac{2\alpha\beta\theta}{t^{\theta+1}} e^{-\frac{\beta}{t^\theta}} \left(1 - e^{-\frac{\beta}{t^\theta}}\right) \left[1 - \left\{1 - e^{-\frac{\beta}{t^\theta}}\right\}^2\right]^{\alpha-1}, \tag{7}$$

by substituting equations (6) and (7) in equation (1), we can derive the Topp–Leone inverse Weibull likelihood function, specifically tailored for type II censored data.

$$\begin{aligned} L(\omega;t) =& \frac{n!}{(n-r)!}(2\alpha\beta\theta)^r \left(\prod_{i=1}^r t_i^{-(\theta+1)}\right) \left(e^{-\beta\sum_{i=1}^r t_i^{-\theta}}\right) \left(\prod_{i=1}^r \left(1 - e^{-\frac{\beta}{t_i^\theta}}\right)\right) \\ &\times \left(\prod_{i=1}^r \left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]^{\alpha-1}\right) \left[1 - \left(1 - \left\{1 - e^{-\frac{\beta}{t_r^\theta}}\right\}^2\right)^\alpha\right]^{n-r}. \end{aligned}$$

3. Maximum likelihood estimation of the Topp-leone Inverse Weibull Distribution

Consider the censored data $t_1 < \cdots < t_r$ from the Topp-leone Inverse Weibull density function. The logarithm of the likelihood function is as follows.

$$\begin{aligned} l(\omega;t) =& \ln\left(\frac{n!}{(n-r)!}\right) + r\ln 2 + r\ln\alpha + r\ln\beta + r\ln\theta - (\theta+1)\sum_{i=1}^r \ln t_i - \beta\sum_{i=1}^r t_i \\ &+ \sum_{i=1}^r \ln\left(1 - e^{-\frac{\beta}{t_r^\theta}}\right) + \sum_{i=1}^r (\alpha-1)\ln\left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right] \\ &+ (n-r)\ln\left[1 - \left(1 - \left\{1 - e^{-\frac{\beta}{t_r^\theta}}\right\}^2\right)^\alpha\right]. \end{aligned} \tag{8}$$

The partial derivatives of $l(\omega;t)$ with respect to the parameters are:

$$\frac{\partial l(\omega;t))}{\partial\alpha} = \frac{r}{\alpha} + \sum_{i=1}^r \ln\left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}\right] + \frac{(n-r)\left(\ln(1-\xi_r^2)\right)\left(1-\xi_r^2\right)^\alpha}{\left[1 - (1-\xi_r^2)^\alpha\right]} = 0,$$

$$\frac{\partial l(\omega;t))}{\partial\beta} = \frac{r}{\beta} - \sum_{i=1}^r t_i^{-\theta} + \sum_{i=1}^r \frac{e^{-\frac{\beta}{t_i^\theta}}}{t_i^\theta\left(1 - e^{-\frac{\beta}{t_i^\theta}}\right)} - 2\sum_{i=1}^r \frac{(\alpha-1)e^{-\frac{\beta}{t_i^\theta}}\left(1 - e^{-\frac{\beta}{t_i^\theta}}\right)}{t_i^\theta\left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]}$$

$$+\frac{(n-r)\alpha(1+\xi_r)\left(1-\xi_r^2\right)^{\alpha-1}}{t_i^{\theta}\left[1-(1-\xi_r^2)^{\alpha}\right]}=0,$$

and

$$\begin{aligned}\frac{\partial l(\omega;t))}{\partial\theta}=&\frac{r}{\theta}-\sum_{i=1}^{r}\ln t_i+\beta\sum_{i=1}^{r}t_i^{-\theta}\ln t_i-\sum_{i=1}^{r}\frac{\beta\ln t_i e^{-\frac{\beta}{t_i^{\theta}}}t_i^{-\theta}}{1-e^{-\frac{\beta}{t_i^{\theta}}}}\\&+2\sum_{i=1}^{r}\frac{\beta(\alpha-1)\ln t_i\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)t_i^{-\theta}}{1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2}\\&+\frac{2\alpha\beta(n-r)\left(1-\xi_r^2\right)^{\alpha-1}\xi_r(\ln t_i)t_i^{-\theta}}{1-(1-\xi_r^2)^{\alpha}}=0,\end{aligned}$$

where $\xi_r=1-e^{-\frac{\beta}{t_r^{\theta}}}$.

Since these equations are not explicit functions in terms of parameters, numerical methods should be used to solve them. [10] showed that using the EM algorithm will give better results in cases where we have censored data than Newton Raphson's method (see [12]).

4. EM Algorithm

The expectation–maximization (EM) algorithm presents itself as an iterative approach, seeking to unveil (local) maximum likelihood or maximum a posteriori (MAP) estimations of parameters within statistical models. This endeavor is especially pertinent when the model's foundation relies upon latent variables that elude direct observation and have been introduced by [11].The EM algorithm entails a meticulous sequence of two steps. The initial step, the E-step, involves determining the mathematical expectation value of a pseudo-logarithmic function. Subsequently, in the second step, the M-step, the function obtained in the preceding step is optimized by maximizing the model's parameters. To employ the EM algorithm, we initially acquire the joint probability distribution function for each (n_i,t_i), where i varies from one to n.

$$f(n_i;t_i;\omega)=(\alpha\beta\theta)^n t_i^{-(\theta+1)}e^{-\frac{\beta}{t_i^{\theta}}}\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\left[1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right]^{\alpha-1},\qquad(9)$$

where $t_i>0$, $\alpha,\beta,\theta>0$ and $n_i=1,2,\ldots$.

The resultant expression manifests as the log- likelihood function.

$$\begin{aligned}&l_c(\omega;t)\\&=n\ln\alpha+n\ln\beta+n\ln\theta-(\theta+1)\sum_{i=1}^{r}\ln t_i-\beta\sum_{i=1}^{r}t_i^{-\theta}+\sum_{i=1}^{r}\ln\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\end{aligned}$$

$$+ (\alpha-1)\sum_{i=1}^{r}\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]-(\theta+1)\sum_{i=r+1}^{n}\ln t_i-\beta\sum_{i=r+1}^{n}t_i^{-\theta}$$
$$+\sum_{i=r+1}^{n}\ln\left(1-e^{-\frac{\beta}{t_i^\theta}}\right)+(\alpha-1)\sum_{i=r+1}^{n}\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right].$$

In step E, the expectation of the log- likelihood function for censored data is as follows.

$$\begin{aligned}
&E(l_c(\omega;t))\\
=&n\ln\alpha+n\ln\beta+n\ln\theta-(\theta+1)\sum_{i=1}^{r}\ln t_i-\beta\sum_{i=1}^{r}t_i^{-\theta}+\sum_{i=1}^{r}\ln\left(1-e^{-\frac{\beta}{t_i^\theta}}\right)\\
&+(\alpha-1)\sum_{i=1}^{r}\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]-(\theta+1)\sum_{i=r+1}^{n}e_{1i}(\omega)-\beta\sum_{i=r+1}^{n}e_{2i}(\omega)\\
&+\sum_{i=r+1}^{n}e_{3i}(\omega)+(\alpha-1)\sum_{i=r+1}^{n}e_{4i}(\omega),
\end{aligned}\tag{10}$$

where

$$e_{1i}(\omega)=E\left(\ln T_i\mid T_i>t_r\right)=\int_{t_r}^{\infty}\ln t_i\cdot g(t_i\mid T_i>t_r)dt_i,$$

and we can write

$$g(t_i\mid T_i>t_r)=\frac{g(t_i;\omega)}{P(T_i>t_r)}=\frac{2\alpha\beta\theta}{t_i^{\theta+1}}\;\frac{e^{-\frac{\beta}{t_i^\theta}}\left(1-e^{-\frac{\beta}{t_i^\theta}}\right)\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]^{\alpha-1}}{1-\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]^{\alpha}}.\tag{11}$$

By substitution equation (11) and taking $x=\frac{t_r}{t_i}$, we have

$$e_{1i}(\omega)=(2\alpha\beta\theta)t_r^{-\theta}\int_0^1 x^{\theta-1}\ln\left(\frac{t_r}{x}\right)\frac{(\eta_r-\eta_r^2)(1-\eta_r^2)^{\alpha-1}}{1-[1-\eta_r^2]^\alpha}dx,$$

where $\eta_r=\left(1-e^{-\frac{\beta x^\theta}{t_r^\theta}}\right)$.

We have

$$e_{2i}(\omega)=E\left(T_i^{-\theta}\mid T_i>t_r\right)=\int_{t_r}^{\infty}t_i^{-\theta}g(t_i\mid T_i>t_r)dt_i.$$

By applying equation (11) and taking $x=\frac{t_r}{t_i}$, we have

$$e_{2i}(\omega)=\frac{(2\alpha\beta\theta)}{t_r^{2\theta-2}}\int_0^1 x^{2\theta-3}\frac{\eta_r(1-\eta_r)[1-\eta_r^2]^{\alpha-1}}{1-[1-\eta_r^2]^\alpha},$$

and

$$e_{3i}(\omega) = E\left(\ln\left(1-e^{-\frac{\beta}{T_i^\theta}}\right) \mid T_i > t_r\right) = \int_{t_r}^{\infty} \ln\left(1-e^{-\frac{\beta}{t_i^\theta}}\right) g(t_i \mid T_i > t_r)dt_i.$$

Using equation (11) we can write

$$e_{3i}(\omega) = \frac{2\alpha\beta\theta}{t_r^\theta}\int_0^1 x^{-\theta-3}\frac{(\eta_r-\eta_r^2)[1-\eta_r^2]^{\alpha-1}\ln\eta_r}{1-[1-\eta_r^2]^\alpha}dx,$$

and

$$\begin{aligned} e_{4i}(\omega) =& E\left(\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right] \mid T_i > t_r\right) \\ &= \int_{t_r}^{\infty}\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right] g(t_i \mid T_i > t_r)dt_i. \end{aligned}$$

By using equation (11), we have

$$e_{3i}(\omega) = (2\alpha\beta\theta)\int_0^1 x^{\theta-1}t_r^{-\theta}\eta_r(1-\eta_r)\left[1-[1-\eta_r\}^2\right]^{\alpha-1}\frac{\ln\left[1-\{1-\eta_r\}^2\right]}{1-[1-\{1-\eta_r\}^2]}dx.$$

In step M, we obtain the derivative of equation (10) with respect to the parameters and set them equal to zero.

$$\frac{\partial E(l_c(\omega;t))}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right] + \sum_{i=r+1}^{n} e_{4i}(\omega) = 0.$$

Hence we have

$$\alpha^{(k+1)} = n\left\{-\sum_{i=1}^{r}\ln\left[1-\left\{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right\}^2\right] - \sum_{i=r+1}^{(k)} e_{4i}^{(k)}(\omega)\right\}^{-1},$$

and

$$\frac{\partial E(l_c(\omega;t))}{\partial\beta} = 0 \Rightarrow$$

$$\frac{n}{\beta} - \sum_{i=1}^{r} t_i^{-\theta} + \sum_{i=1}^{r}\frac{1}{t_i^\theta}\frac{e^{-\frac{\beta}{t_i^\theta}}}{1-e^{-\frac{\beta}{t_i^\theta}}} + 2(\alpha-1)\sum_{i=1}^{r}\frac{e^{-\frac{\beta}{t_i^\theta}}\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2}{t_i^\theta\left[1-\left\{1-e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]} - \sum_{i=r+1}^{n} e_{2i}(\omega) = 0.$$

Then

$$\beta^{(k+1)} =$$

$$n\left\{\sum_{i=1}^{r} t_i^{-\theta^{(k)}} - \sum_{i=1}^{r}\frac{e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}}{t_i^{\theta^{(k)}}\left(1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right)} - 2(\alpha-1)\sum_{i=1}^{r}\frac{e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\left\{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right\}}{t_i^{\theta^{(k)}}\left[1-\left\{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right\}\right]} + \sum_{i=r+1}^{n} e_{2i}(\omega)\right\}^{-1},$$

and

$$\frac{\partial E(l_c(\omega;t))}{\partial\theta}=0\Rightarrow$$

$$\frac{n}{\theta}+\sum_{i=1}^{r}\ln t_i+\beta\sum_{i=1}^{r}t_i^{-\theta}\ln t_i+\sum_{i=1}^{r}\frac{\beta t_i^{-\theta}\ln t_i}{1-e^{-\frac{\beta}{t_i^{\theta}}}}+2(\alpha-1)\sum_{i=1}^{r}\beta t_i^{-\theta}\ln t_i e^{-\frac{\beta}{t_i^{\theta}}}\frac{\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}}{1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2}$$

$$-\sum_{i=r+1}^{n}e_{1i}(\omega)=0.$$

So we have

$$\theta^{(k+1)}=$$

$$n\Bigg\{\sum_{i=1}^{r}\ln t_i-\beta^{(k)}\sum_{i=1}^{r}t_i^{-\theta^{(k)}}\ln t_i-\sum_{i=1}^{r}\frac{\beta^{(k)}t_i^{-\theta^{(k)}}\ln t_i}{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}}-2\left(\alpha^{(k)}-1\right)\sum_{i=1}^{r}\beta^{(k)}t_i^{-\theta^{(k)}}\ln t_i e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}$$

$$\times\frac{\left\{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right\}}{1-\left\{1-e^{-\frac{\beta^{(k)}}{t_i^{\theta^{(k)}}}}\right\}^2}+\sum_{i=r+1}^{n}e_{1i}^{(k)}(\omega)\Bigg\}^{-1}.$$

(See [13]).

5. Bayesian Estimation

Suppose that we are presented with a sample t, denoted as $t=(t(1),t(2),\ldots,t(r))$, which has been observed from the Topp-Leone inverse Weibull distribution under type-II censoring. We assume that all the unknown parameters α, β and θ follow independent gamma priors. Probability density functions of a particular form can be describe these priors densities

$$\pi_1(\alpha)\propto\alpha^{m_1-1}e^{-n_1\alpha},$$
$$\pi_2(\beta)\propto\beta^{m_2-1}e^{-n_2\beta},$$
$$\pi_3(\theta)\propto\theta^{m_3-1}e^{-n_3\theta}.$$

Moreover, the collective posterior density of (α,β,θ) concerning the observed data t can be derived.

$$\pi(\alpha,\beta,\theta\mid t)\propto L(t\mid\alpha,\beta,\theta)\cdot\pi_1(\alpha)\cdot\pi_2(\beta)\cdot\pi_3(\theta),$$

$$\pi(\alpha,\beta,\theta\mid t)\propto\alpha^{r+m_1-1}\cdot\beta^{r+m_2-1}\cdot\theta^{r+m_3-1}\cdot e^{-\theta\left(\sum_{i=1}^{r}\ln t_i+n_3\right)}\cdot e^{-\beta\left(\sum_{i=1}^{r}t_i^{-\theta}+n_2\right)}$$
$$\times e^{-\alpha\left(n_1-\sum_{i=1}^{r}\left[1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right]\right)}\exp\left(\sum_{i=1}^{r}\ln\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\right)$$

$$\times \left[1-\left(1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right)^{\alpha}\right]^{n-r}. \tag{12}$$

The squared error and Linex loss function have been employed Bayesian parameter estimation. These particular loss functions are delineated as follows, with $\hat{\delta}(\omega)$ denoting the estimated value for the $\delta(\omega)$ parameter.

$$\begin{aligned} L_{SB}\left(\delta(\omega),\hat{\delta}(\omega)\right) &= \left(\hat{\delta}(\omega)-\delta(\omega)\right)^2, \\ L_{LB}\left(\delta(\omega),\hat{\delta}(\omega)\right) &= e^{\left(\hat{\delta}(\omega)-\delta(\omega)\right)-c\left[\left(\hat{\delta}(\omega)-\delta(\omega)\right)-1\right]}. \end{aligned}$$

The Bayesian estimator of $\delta(\omega)$ under linex loss is given by

$$\hat{\delta}(\omega) = \frac{-1}{k}\ln\left(E\left(e^{-k\delta(\omega)} \mid t\right)\right), \quad k \neq 0.$$

According to equation (12), the Bayesian estimation of the parameters under the error squared loss function is as follows.

$$\begin{aligned} \hat{\alpha}_{SB} =& E(\alpha \mid t) \propto \int_0^\infty\int_0^\infty\int_0^\infty \alpha\pi(\alpha,\beta,\theta \mid t)d\alpha d\beta d\theta \\ & \propto \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{r+m_1}\beta^{r+m_2-1}\theta^{r+m_3-1}h(\alpha,\beta,\theta \mid t)d\alpha d\beta d\theta, \\ \hat{\beta}_{SB} =& E(\beta \mid t) \propto \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{r+m_1-1}\beta^{r+m_2}\theta^{r+m_3-1}h(\alpha,\beta,\theta \mid t)d\alpha d\beta d\theta, \\ \hat{\theta}_{SB} =& E(\theta \mid t) \propto \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{r+m_1-1}\beta^{r+m_2-1}\theta^{r+m_3}h(\alpha,\beta,\theta \mid t)d\alpha d\beta d\theta, \end{aligned}$$

where

$$\begin{aligned} h(\alpha,\beta,\theta \mid t) =& e^{-\theta\left(\sum_{i=1}^r \ln t_i+n_3\right)-\beta\left(\sum_{i=1}^r t_i^{-\theta}+n_2\right)-\alpha\left(n_1-\sum_{i=1}^r\left[1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right]\right)} \\ & \times \exp\left(\sum_{i=1}^r \ln\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\right)\left[1-\left(1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right)^{\alpha}\right]^{n-r}. \end{aligned}$$

The Bayesian estimation of the parameters under the Linex loss function is as follows

$$\begin{aligned} \hat{\delta}(\omega) &= \frac{-1}{k}\ln\left(E\left(e^{-k\delta(\omega)} \mid t\right)\right), \\ \hat{\alpha}_{LB}(\omega) &= \frac{-1}{k}\ln\left(E\left(e^{-k\alpha} \mid t\right)\right), \\ \hat{\beta}_{LB}(\omega) &= \frac{-1}{k}\ln\left(E\left(e^{-k\beta} \mid t\right)\right), \\ \hat{\theta}_{LB}(\omega) &= \frac{-1}{k}\ln\left(E\left(e^{-k\theta} \mid t\right)\right). \end{aligned}$$

Then

$$E\left(e^{-k\alpha} \mid t\right)$$

$$\propto \int_0^\infty \int_0^\infty \int_0^\infty e^{-\theta\left(\sum_{i=1}^r \ln t_i + n_3\right)} e^{-\beta\left(\sum_{i=1}^r t_i^{-\theta} + n_2\right)} e^{-\alpha\left(k + n_1 - \sum_{i=1}^r \left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]\right)}$$
$$q(\alpha, \beta, \theta \mid t) d\alpha d\beta d\theta,$$

$$E\left(e^{-k\beta} \mid t\right)$$

$$\propto \int_0^\infty \int_0^\infty \int_0^\infty e^{-\theta\left(\sum_{i=1}^r \ln t_i + n_3\right)} e^{-\beta\left(k + n_2 + \sum_{i=1}^r t_i^{-\theta}\right)} e^{-\alpha\left(n_1 - \sum_{i=1}^r \left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]\right)}$$
$$q(\alpha, \beta, \theta \mid t) d\alpha d\beta d\theta,$$

$$E\left(e^{-k\theta} \mid t\right)$$

$$\propto \int_0^\infty \int_0^\infty \int_0^\infty e^{-\theta\left(k + n_3 + \sum_{i=1}^r \ln t_i\right)} e^{-\beta\left(n_2 + \sum_{i=1}^r t_i^{-\theta}\right)} e^{-\alpha\left(n_1 - \sum_{i=1}^r \left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]\right)}$$
$$q(\alpha, \beta, \theta \mid t) d\alpha d\beta d\theta,$$

where

$$q(\alpha, \beta, \theta)$$
$$= \alpha^{r+m_1-1} \beta^{r+m_2-1} \theta^{r+m_3-1} \exp\left(\sum_{i=1}^r \ln\left(1 - e^{-\frac{\beta}{t_i^\theta}}\right)\right) \left[1 - \left(1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right)^\alpha\right]^{n-r}.$$

The calculation of Bayesian estimates in their current form presents particular challenges, necessitating the application of a numerical method put forward by [14, 15].This method expertly utilizes the Markov chain Monte Carlo (MCMC) approach to address the problem. An intricate utilization of Gibbs sampling via the Metropolis-Hastings algorithm generates a sample from the posterior distribution, leading to the subsequent estimation of parameters. Within this Gibbs sampling methodology, the complete conditional posterior distribution is as follows.

$$pi_1^*(\alpha \mid \beta, \theta, t)$$
$$= \alpha^{r+m_1-1} e^{-\alpha\left(n_1 - \sum_{i=1}^r \left[1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right]\right)} \left[1 - \left(1 - \left\{1 - e^{-\frac{\beta}{t_i^\theta}}\right\}^2\right)^\alpha\right]^{n-r}, \tag{13}$$

$$\pi_2^*(\beta \mid \alpha, \theta, t)$$

$$= \beta^{r+m_2-1} e^{-\alpha\left(n_1-\sum_{i=1}^{r}\left[1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right]\right)} \exp\left(\sum_{i=1}^{r} \ln\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\right)$$

$$\times \left[1-\left(1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right)^{\alpha}\right]^{n-r}, \tag{14}$$

$$\pi_3^*(\theta \mid \alpha, \beta, t)$$

$$= \theta^{r+m_3-1} e^{-\theta\left(\sum_{i=1}^{r} \ln t_i + n_3\right)} e^{-\beta\left(\sum_{i=1}^{r} t_i^{-\theta}+n_2\right)} e^{-\alpha\left(n_1-\sum_{i=1}^{n}\left[1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right]\right)}$$

$$\times \exp\left(\sum_{i=1}^{r} \ln\left(1-e^{-\frac{\beta}{t_i^{\theta}}}\right)\right)\left[1-\left(1-\left\{1-e^{-\frac{\beta}{t_i^{\theta}}}\right\}^2\right)^{\alpha}\right]^{n-r}. \tag{15}$$

According to equations (13) to (15), Bayesian estimation of parameters (α, β, θ) is based on the samples produced by the MCMC method according to the following steps:

Step 1: $(\alpha_0, \beta_0, \theta_0)$ are determined as initial values.

Step 2: With the Metropolis-Histings algorithm, the value of α_i from $\pi_1^*(\alpha \mid \beta_{i-1}, \theta_{i-1}, t)$ distribution and β_i from $\pi_2^*(\beta \mid \alpha_i, \theta_{i-1}, t)$ distribution and θ_i from $\pi_3^*(\theta|\alpha_i, \beta_i, t)$ distribution are produced.

Step 3: Step 2 is repeated for the value of $i = 1, \ldots, N$.

Step 4: The Bayesian estimation of α, β, θ is calculated under the squared Loss function as $\hat{H}_{SB} = \frac{1}{N}\sum_{i=1}^{n} \hat{\omega}_i$, and the Linex Loss function is calculated as follows.

$$\hat{H}_L = \frac{-1}{k}\left(\frac{1}{N}\sum_{i=1}^{N} e^{-k\hat{\omega}_i}\right).$$

6. Conclusion

The present study delves into the problem of determining unknown parameters for a Topp-leone inverse Weibull distribution within the framework of PCS-II, examining it from both BE and Non-BE standpoints. We derived Maximum Likelihood Estimates (MLE) through meticulous analysis of the parameters characterizing a Topp-leone inverse Weibull distribution. Moreover, our investigation entailed performing Bayesian Estimation (BE), utilizing Markov Chain Monte Carlo (MCMC), while considering both symmetric and asymmetric loss functions to yield comprehensive estimates.

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Zahra Shokooh Ghazani is an Assistant Professor in Statistics at Central Tehran Branch, Islamic Azad University, Tehran, Iran. She got her B.Sc., in Applied Mathematics from Mazandaran university, in1993. She received his M.Sc. in Statistics from Ferdowsi University of Mashhad, in 1995 and she received his Ph.D. degree in Statistics from Science and Research Branch, IAU, in 2003. She is a member Iran Statistical Society. Her research interests are in the areas of distribution theory and probability theory.

Department of Statistics and Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
e-mail: Zah.Shokooh_Ghazani@iauctb.ac.ir