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A STUDY ON $(\in, \in \vee q)$-FUZZY CONGRUENCE ON RING

N. PRADIPKUMAR AND O. RATNABALA DEVI*

Abstract. The purpose of this paper is to introduce the concept of $(\in, \in \vee q)$-fuzzy congruence relation over ring and discuss some properties of the $(\in, \in \vee q)$-fuzzy congruence relation. We also establish a brief relation between $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy congruence relation. The image and preimage of $(\in, \in \vee q)$-fuzzy congruence are also studied under the so called semibalanced map.

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1. Introduction

In 1965, Zadeh [29] introduced the concept of fuzzy set. Using this concept in 1971, Rosenfeld [25] introduced the concept of fuzzy subgroup. Countless studied have been carried out by many mathematician to study fuzzy algebraic structure such as group, ring, near-ring and so on. Das [10] characterized fuzzy subgroups by their level subgroup. In 1982, Liu [16] applied the concept of fuzzy sets to the theory of rings and introduced the notion of fuzzy ideals of a ring.

The idea of a fuzzy point, its belongingness to and quasi-coincidence which is mentioned in Ming and Ming [19] played a vital role in the inception of various fuzzy algebraic structures. In 1992, Bhakat and Das [4] introduced a new type of fuzzy subgroup (viz $(\in, \in \vee q)$-fuzzy subgroup by using the combined notion of "belongingness" and "quasi coincidence" of fuzzy point and fuzzy sets. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In 1996, Bhakat and Das [6] further defined $(\in, \in \vee q)$-fuzzy subrings and ideals of a ring.

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Since the introduction of fuzzy relation on a set by Zadeh [29, 30], there is a remarkable growth of research in this area. Many authors like Chakraborty, Das, Nemitz, Murali etc [8, 9, 23, 21, 22] studied fuzzy equivalence relation. In 1989 and 1991, Murali [22] proposed the concept of fuzzy congruence relations which led to the concept of fuzzy normal subgroups on a group as proposed by Kuroki [15]. Later on, Kim and Bae [14], Samhan and Ahsanullah [27] and others further studied fuzzy congruence relation in groups, rings and near rings. In 2011, M. Ali et al. [2] proposed the concept of $(\in, \in \vee q)$-fuzzy equivalence relations and indistinguishability relations and proved various important results.

In this paper, we further proceed to study of $(\in, \in \vee q)$-fuzzy congruence relation on a ring following the concept of $(\in, \in \vee q)$-fuzzy equivalence relation initiated by M. Ali et al. [2]. We also establish a series of properties of the generalized $(\in, \in \vee q)$-fuzzy congruence relation on ring. The image and pre image of $(\in, \in \vee q)$-fuzzy congruence are also studied under the so called semi balanced map.

2. Preliminaries

This section contains some basic definitions and preliminary results that will be needed in the sequel. Let R be a ring and X be any non-empty set.

Definition 2.1. [16] A fuzzy subset μ of a ring R is called fuzzy subring of R, if for all $x, y \in R$

(i) $\mu(x-y) \geqslant \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\}$.

Definition 2.2. [16] A fuzzy subset μ of a ring R is called fuzzy ideal of R, if for all $x, y \in R$

(i) $\mu(x-y) \geqslant \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\}$,
(iii) $\mu(xy) \geqslant \max\{\mu(x), \mu(y)\}$ ($\mu(xy) \geqslant \mu(y)$ and $\mu(\text{xy}) \geqslant \mu(\text{x})$).

Definition 2.3. [19] Let $t \in (0,1]$, $x \in X$, a fuzzy subset x_t of X defined by

$$x_t(y) = \begin{cases} t & if\ x = y, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for all } y \in X$$

is called a fuzzy point. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy subset μ of X if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), written as "$x_t \in \mu$" (resp. "$x_t \, q \, \mu$"). If "$x_t \in \mu$" or "$x_t \, q \, \mu$",it is denoted by "$x_t \in \vee \, q \, \mu$". The notation $x_t \,\bar{\in}\, \mu$, $x_t \,\bar{q}\, \mu$, and $x_t \,\overline{\in \vee q}\, \mu$ will mean that $x_t \in \mu$, $x_t \, q \, \mu$ and $x_t \in \vee \, q \, \mu$ respectively do not hold.

Definition 2.4. [6] A fuzzy subset μ of ring R is said to be an $(\in, \in \vee q)$-fuzzy subring of R if for all $x, y \in R$ and $t, r \in (0,1]$,

(i) $x_t, y_r \in \mu \Rightarrow (x+y)_{\min(t,r)} \in \vee q \ \mu$,
(ii) $x_t \in \mu \Rightarrow (-x)_t \in \vee q \ \mu$,
(iii) $x_t, y_r \in \mu \Rightarrow (xy)_{\min(t,r)} \in \vee q \ \mu$.

Remark 2.1. Condition (i) is equivalent to (i) $\mu(x+y) \geqslant \min(\mu(x), \mu(y), 0.5)$ $\forall\, x, y \in R$ and

condition (ii) is equivalent to (ii) $\mu(-x) \geqslant \min(\mu(x), 0.5)\ \forall\, x \in R$.

Definition 2.5. [6] Let R be a ring and μ be a fuzzy subset of R. Then μ is said to be an $(\in, \in \vee q)$-fuzzy ideal of R if

(i) μ is an $(\in, \in \vee q)$-fuzzy subring of R,
(ii) $x_t \in \mu$ and $y \in R \Rightarrow (xy)_t, (yx)_t \in \vee q\ \mu$.

Theorem 2.6. [6] *μ is an $(\in, \in \vee q)$-fuzzy ideal of R if and only if*

(i) $\mu(x-y) \geqslant min\{\mu(x), \mu(y), 0.5\}$,
(ii) $\mu(xy), \mu(yx) \geqslant min\{\mu(x), 0.5\}\ \forall\, x, y \in R$.

Definition 2.7. [15] A fuzzy subset set μ of $R \times R$ is called a fuzzy relation on R. A fuzzy relation μ on R is called a fuzzy equivalence relation on R if satisfies the following conditions

(i) $\mu(x, x) = 1$ for all $x \in R$,
(ii) $\mu(y, x) = \mu(x, y)$ for all $x, y \in R$,
(iii) $\mu(x, z) \geqslant \bigvee_{y \in R}\{\mu(x, y), \mu(y, z)\}$ for all $x, y, z \in R$.

Definition 2.8. [27] A fuzzy relation μ on R is called a fuzzy left compatible relation if $\mu(x+a, x+b) \geqslant \mu(a, b)$ and $\mu(xa, xb) \geqslant \mu(a, b)$ for all $x, a, b \in R$ and is called a fuzzy right compatible if $\mu(a+x, b+x) \geqslant \mu(a, b)$ and $\mu(ax, bx) \geqslant \mu(a, b)$ for all $x, a, b \in R$.

It is called compatible relation if
$\mu(a+c, b+d) \geqslant \mu(a, b) \wedge \mu(c, d)$ and $\mu(ac, bd) \geqslant \mu(a, b) \wedge \mu(c, d)$ for all $a, b, c, d \in R$.

Definition 2.9. [27] A fuzzy relation on R is called a fuzzy compatible relation if and only if it is both a left and a right fuzzy compatible relation on R.

Proposition 2.10. [27] *Let α and β be any two fuzzy compatible relations on R. Then $\alpha o \beta$ is also a fuzzy compatible relation on R.*

Definition 2.11. [27] A fuzzy equivalence relation μ on R is called a fuzzy congruence relation if the following conditions are satisfied for all $x, y, z, t \in R$

(i) $\mu(x+y, z+t) \geqslant \min\{\mu(x, z), \mu(y, t)\}$,
(ii) $\mu(xy, zt) \geqslant \min\{\mu(x, z), \mu(y, t)\}$.

Definition 2.12. [2] A fuzzy subset $(x, y)_t$ of $X \times Y$ given by

$$(x, y)_t(a, b) = \begin{cases} t & if\ x = a, y = b, \\ 0 & \text{otherwise}, \end{cases} \text{ for all } (a, b) \in X \times Y$$

is called a fuzzy ordered pair or simply a fuzzy pair.

A fuzzy pair $(x, y)_t$ is said to belong to ([resp. be quasi coincident with]) a fuzzy relation μ, written as $(x, y)_t \in \mu$ (resp. $(x, y)_t\, q\, \mu$) if $\mu(x, y) \geqslant t$ (resp. $\mu(x, y)$ $+t > 1$). If $(x, y)_t \in \mu$ or $(x, y)_t\, q\, \mu$, then it is written as $(x, y)_t \in \vee\, q\, \mu$.

Definition 2.13. [2] A fuzzy relation μ on a set X is said to be $(\in, \in \vee q)$-fuzzy reflexive if for $x, y \in X, (x,y)_t \in \mu$ implies $(a,a)_t \in \vee\ q\ \mu\ \forall\, a \in X,\ t \in (0,1]$; μ is $(\in, \in \vee q)$-fuzzy symmetric if $(x,y)_t \in \mu$ implies $(y,x)_t \in \vee\ q\ \mu\ \forall\, x, y \in X,\ t \in (0,1]$; $(\in, \in \vee q)$-fuzzy transitive if $(x,y)_{t_1} \in \mu$ and $(y,z)_{t_2} \in \mu$ implies $(x,z)_{\min(t_1,t_2)} \in \vee\ q\ \mu\ \forall\, x, y, z \in X, t_1, t_2 \in (0,1]$. μ is called $(\in, \in \vee q)$-fuzzy equivalence relation if μ is $(\in, \in \vee q)$-fuzzy reflexive, $(\in, \in \vee q)$-fuzzy symmetric and $(\in, \in \vee q)$-fuzzy transitive.

Proposition 2.14. [2] *A fuzzy relation μ on a set X is an $(\in, \in \vee q)$-fuzzy equivalence relation if and only if it satisfies the following conditions :*

(i) $\mu(a,a) \geqslant min\{\mu(x,y), 0.5\},\ \forall\, a, x, y \in X,$
(ii) $\mu(y,x) \geqslant min\{\mu(x,y), 0.5\},\ \forall\, x, y, \in X$
(iii) $\mu(x,z) \geqslant min\{\mu(x,y), \mu(y,z), 0.5\}\ \forall\, x, y, z \in X.$

Definition 2.15. [2] If μ is a fuzzy relation on X, then the subset μ_0 of $X \times X$ defined as $\mu_0 = \{(x,y) \in X \times X, \mu(x,y) > 0\}$ is called the support of μ.

Proposition 2.16. [2] *Let μ be an $(\in, \in \vee\ q)$-fuzzy equivalence relation on a set X. Then the support μ_0 of μ is equivalence relation on X.*

Each $(\in, \in \vee\ q)$-fuzzy equivalence relation μ on X can be characterized by its level relation

$$\mu_t = \{(x,y) \in X \times X \mid\ (x,y) \geqslant t\}\ \text{and}$$
$$\bar{\mu}_t = \{(x,y) \in X \times X,\ \mu(x,y) \geqslant t\}\ \cup\ \{(x,y) \in X \times X \mid\ \mu(x,y) + t > 1\}$$
$$= \{(x,y) \in X \times X \mid\ (x,y)_t \in \vee\ q\, \mu\}.$$

Proposition 2.17. [2] *A fuzzy relation μ on X is an $(\in, \in \vee\ q)$-fuzzy equivalence relation on X if and only if $\mu_t \neq \phi$ is an equivalence relation on X for all $t \in (0, 0.5]$.*

Proposition 2.18. [2] *A fuzzy relation μ on X is an $(\in, \in \vee\ q)$-fuzzy equivalence relation on X if and only if $\bar{\mu}_t \neq \phi$ is an equivalence relation on X for all $t \in (0, 1]$.*

Definition 2.19. [24] A fuzzy relation μ on a group G is an $(\in, \in \vee q)$-fuzzy compatible relation if $(x,y)_t \in \mu,\ (a,b)_s \in \mu$ implies $(xa, yb)_{\min\{t,s\}} \in \vee\ q\ \mu$ for all $x, y, a, b \in G, t, s \in (0,1]$. μ is said to be $(\in, \in \vee q)$-fuzzy congruence relation if μ is an $(\in, \in \vee q)$-fuzzy equivalence relation and $(\in, \in \vee q)$-fuzzy compatible relation on a group G.

Proposition 2.20. [24] *A fuzzy relation μ on a group G is an $(\in, \in \vee\ q)$-fuzzy congruence relation on G if and only if $\bar{\mu}_t \neq \phi$ is a congruence relation for all $t \in (0, 1]$.*

3. $(\in, \in \vee\ q)$-Fuzzy Congruence on Ring

In this section, we shall introduce the concept of $(\in, \in \vee q)$-fuzzy congruence relation on ring and study some fundamental properties.

Definition 3.1. A fuzzy relation μ on a ring R is $(\in, \in \vee q)$-fuzzy compatible relation if $(x,y)_t \in \mu, (a,b)_s \in \mu$ implies $(x+a, y+b)_{\min\{t,s\}} \in \vee q\ \mu$ and $(xa, yb)_{\min\{t,s\}} \in \vee q\ \mu$ for all $x, y, a, b \in R$ and $t, s \in (0,1]$.

The fuzzy relation μ is said to be $(\in, \in \vee q)$-fuzzy congruence relation on R if and only if μ is $(\in, \in \vee q)$-fuzzy equivalence relation and $(\in, \in \vee q)$-fuzzy compatible relation on R.

Example 3.2. Let Z be the set of all integers. Then Z is a ring with respect to the usual addition and multiplication of numbers. The fuzzy relation μ on Z defined by

$$\mu(x,y) = \begin{cases} 1 & if\ x = y, \\ 0.6 & if\ x \neq y \text{ and both } x, y \text{ are even or odd }, \\ 0 & \text{otherwise} \end{cases}$$

is a $(\in, \in \vee q)$-fuzzy congruence relation on Z as well as fuzzy congruence relation on Z.

Proposition 3.3. *A fuzzy congruence relation on a ring R is an $(\in, \in \vee q)$-fuzzy congruence relation.*

Proof. Suppose μ is fuzzy congruence relation on R.
For any $x \in R$, $\mu(x,x) = 1$ imply $\mu(x,x) \geqslant t$ or $\mu(x,x) + t > 1$ for all $t \in (0,1]$. This implies $(x,x)_t \in \vee\ q\ \mu$ which means μ is an $(\in, \in \vee\ q)$-fuzzy reflexive relation. For any $x, y \in R$, $\mu(x,y) = \mu(y,x)$. Suppose $(x,y)_t \in \mu, \forall\ t \in (0,1]$. Then $\mu(x,y) \geqslant t \Rightarrow (y,x)_t \in \mu$. So, $(y,x)_t \in \vee\ q\ \mu\ \forall\ t \in (0,1]$. Thus, μ is an $(\in, \in \vee q)$-fuzzy symmetric relation.
Suppose $(x,y)_t \in \mu$ and $(y,z)_s \in \mu$, we have $\mu(x,y) \geqslant t, \mu(y,z) \geqslant s$, implies that $\mu(x,z) \geqslant \min\{\text{t,s}\}$.So $(x,z)_{\min\{\text{t,s}\}} \in \mu$ which means $(x,z)_{\min\{\text{t,s}\}} \in \vee\ q\ \mu$. Thus, μ is an $(\in, \in \vee q)$-fuzzy transitive relation.

Further, for any $(x,y)_t \in \mu, (a,b)_s \in \mu$, we have $\mu(x,y) \geqslant t, \mu(a,b) \geqslant s$. As μ is fuzzy compatible, $\mu(x+a, y+b) \geqslant \min\{\mu(x,y), \mu(a,b)\} \geqslant \min\{\text{t,s}\}$ and $\mu(xa, yb) \geqslant \min\{\mu(x,y), \mu(a,b)\} \geqslant \min\{\text{t,s}\}$.
So, $(x+a, y+b)_{\min\{\text{t,s}\}} \in \vee\ q\ \mu$ and $(xa, yb)_{\min\{\text{t,s}\}} \in \vee\ q\ \mu$. Thus, μ is an $(\in, \in \vee q)$-fuzzy congruence relation on R. □

Example 3.4. Let Z be the set of all integers. Then Z is a ring with respect to the usual addition and multiplication of numbers. The fuzzy relation μ on Z is defined by

$$\mu(x,y) = \begin{cases} 0.7 & if\ x = y, \\ 0.4 & if\ x \neq y \text{ and both } x, y \text{ are even or odd },, \\ 0 & \text{otherwise} \end{cases}$$

is a $(\in, \in \vee q)$-fuzzy congruence relation on Z but not a fuzzy congruence relation on Z as μ is not fuzzy reflexive relation.

Theorem 3.5. *Let R be a ring and μ be a fuzzy relation i,e a function $\mu : R \times R \to [0,1]$. Then, μ is an $(\in, \in \vee\, q)$-fuzzy equivalence on R if for all $a, x, y, z \in R$*

(i) $\mu(a,a) \geqslant min\{\mu(x,y), 0.5\}$ *(Fuzzy reflexive),*
(ii) $\mu(y,x) \geqslant min\{\mu(x,y), 0.5\}$ *(Fuzzy symmetric),*
(iii) $\mu(x,z) \geqslant\ min\{\mu(x,y), \mu(y,z), 0.5\}$ *(Fuzzy transitive).*

Proof. It follows from Proposition 2.14 .

□

Theorem 3.6. *Let μ be an $(\in, \in \vee q)$-fuzzy relation on a ring R, then μ is an $(\in, \in \vee q)$-fuzzy compatible relation if and only if it satisfies the following conditions for all $t, x, y, z \in R$*

(i) $\mu(x+y, z+t) \geqslant min\{\mu(x,z), \mu(y,t), 0.5\}$,
(ii) $\mu(xy, zt) \geqslant min\{\mu(x,z), \mu(y,t), 0.5\}$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy compatible relation and $t, x, y, z \in R$ such that $\mu(x+y, z+t) < \min\{\mu(x,z), \mu(y,t), 0.5\}$ or $\mu(xy, zt) < \min\{\mu(x,z), \mu(y,t), 0.5\}$.
First we consider the case for $\min\{\mu(x,z), \mu(y,t)\} < 0.5$. Then $\mu(x+y, z+t) < \min\{\mu(x,z), \mu(y,t)\}$ or $\mu(xy, zt) < \min\{\mu(x,z), \mu(y,t)\}$. Thus there exist some $r \in (0,1]$ such that $\mu(x+y, z+t) < r < \min\{\mu(x,z), \mu(y,t)\} < 0.5$ or $\mu(xy, zt) < r < \min\{\mu(x,z), \mu(y,t)\} < 0.5$. This implies $(x+y, z+t)_r \overline{\in \vee q}\ \mu$ or $(xy, zt)_r \overline{\in \vee q}\ \mu$ which is a contradiction as $(x,z)_r \in \mu$ and $(y,t)_r \in \mu$.
Again, consider the case for $\min\{\mu(x,z), \mu(y,t)\} \geqslant 0.5$. Then, $\mu(x+y, z+t) < 0.5$ or $\mu(xy, zt) < 0.5$. Since $\min\{\mu(x,z), \mu(y,t)\} \geqslant 0.5$, we have, $(x,z)_{0.5} \in \mu$ and $(y,t)_{0.5} \in \mu$ but $(x+y, z+t)_{0.5} \overline{\in \vee q}\ \mu$ or $(xy, zt)_{0.5} \overline{\in \vee q}\ \mu$ which is a contradiction.

Hence, $\mu(x+y, z+t) \geqslant \min\{\mu(x,z), \mu(y,t), 0.5\}$ and $\mu(xy, zt) \geqslant \min\{\mu(x,z), \mu(y,t), 0.5\}$ for all $t, x, y, z \in R$.

Conversely, assume that $\mu(x+y, z+t) \geqslant \min\{\mu(x,z), \mu(y,t), 0.5\}$ and $\mu(xy, zt) \geqslant \min\{\mu(x,z), \mu(y,t), 0.5\}$. Suppose $(x,z)_r \in \mu$ and $(y,t)_s \in \mu$ be such that $r < s$. Then $\mu(x,z) \geqslant r$ and $\mu(y,t) \geqslant s$. If $\min\{r,s\} \leqslant 0.5$, then $\mu(x+y, z+t) \geqslant \min\{r,s\}$ and $\mu(xy, zt) \geqslant \min\{r,s\}$. This implies $(x+y, z+t)_{\min\{r,s\}} \in \mu$ and $(xy, zt)_{\min\{r,s\}} \in \mu$. If $\min\{r,s\} > 0.5$, then $\mu(x+y, z+t) > 0.5$ and $\mu(xy, zt) > 0.5$. So, $\mu(x+y, z+t) + \min\{r,s\} > 1$ and $\mu(xy, zt) + \min\{r,s\} > 1$ i.e $(x+y, z+t)_{\min\{r,s\}} q\ \mu$ and $(xy, zt)_{\min\{r,s\}} q\ \mu$.

Hence, $(x+y, z+t)_{\min\{r,s\}} \in \vee q\ \mu$ and $(xy, zt)_{\min\{r,s\}} \in \vee q\ \mu$. □

Theorem 3.7. *A fuzzy relation μ on a ring R is an $(\in, \in \vee q)$-fuzzy congruence relation on a ring R if and only if it satisfies the following conditions for all $t, x, y, z \in R$*

(i) $\mu(t,t) \geqslant min\{\mu(x,y), 0.5\}$,
(ii) $\mu(y,x) \geqslant min\{\mu(x,y), 0.5\}$,
(iii) $\mu(x,z) \geqslant\ min\{\mu(x,y), \mu(y,z), 0.5\}$,

(iv) $\mu(x+y, z+t) \geqslant min\{\mu(x,z), \mu(y,t), 0.5\}$,
(v) $\mu(xy, zt) \geqslant min\{\mu(x,z), \mu(y,t), 0.5\}$.

Proof. It follows from Theorem 3.1 and 3.2. □

Proposition 3.8. *Let μ be an $(\in, \in \vee q)$-fuzzy congruence relation on R. Then for all $x, y, z \in R$ we have the following results:*
(i) $\mu(x, y) \geqslant min\{\mu(x+z, y+z), 0.5\}$
(ii) $\mu(-x, -y) \geqslant min\{\mu(x,y), 0.5\}$.

Proof. (i) $\mu(x,y) = \mu(x+z-z, y+z-z) \geqslant \min\{\mu(x+z, y+z), \mu(-z,-z), 0.5\} \geqslant \min\{\mu(x+z, y+z), 0.5\}$

(ii) $\mu(-x,-y) \geqslant \min\{\mu(x-x, x-y), 0.5\} \geqslant \min\{\mu(0, x-y), 0.5\} \geqslant \min\{\mu(y-y, x-y), 0.5\} \geqslant \min\{\mu(y,x), 0.5\} \geqslant \min\{\mu(x,y), 0.5\}$
□

Theorem 3.9. *A fuzzy relation μ on a ring R is an $(\in, \in \vee q)$-fuzzy congruence relation on R if and only if $\bar{\mu}_s \neq \phi$ is a congruence relation for all $s \in (0,1]$.*

Proof. It follows from Proposition 2.6 . □

We next see some properties of the set of all $(\in, \in \vee q)$-fuzzy congruences on a ring R. Let μ and τ be two fuzzy relations on a ring R. Then the product $\mu \circ \tau$ is defined by the following

$$\mu \circ \tau(a,b) = \sup_{x \in R} \min\{\mu(a,x), \tau(x,b)\}$$

for all $(a,b) \in R \times R$. We use $\wedge$ for min and $\vee$ for sup operators.

Lemma 3.10. *Let μ and τ be two $(\in, \in \vee q)$-fuzzy reflexive relations on a ring R. Then $\mu \circ \tau$ is also an $(\in, \in \vee q)$-fuzzy reflexive relations on R.*

Proof. For any $x, z \in R$,

$$\begin{aligned} \mu \circ \tau(x,z) \wedge 0.5 &= \bigvee_{y \in R} \{\mu(x,y) \wedge \tau(y,z)\} \wedge 0.5 \\ &= \bigvee_{y \in R} \{\mu(x,y) \wedge 0.5\} \wedge \{\mu(y,z) \wedge 0.5\} \\ &\leqslant \mu(a,a) \wedge \tau(a,a) \\ &\leqslant \bigvee_{t \in R} \{\mu(a,t) \wedge \tau(t,a)\} \\ &= \mu \circ \tau(a,a) \quad \forall\ a \in R. \end{aligned}$$

That means by Proposition 2.2, $\mu \circ \tau$ is an $(\in, \in \vee q)$-fuzzy reflexive relation. □

Lemma 3.11. *Let μ and τ be two $(\in, \in \vee q)$-fuzzy compatible relations on a ring R. Then $\mu \circ \tau$ is also an $(\in, \in \vee q)$-fuzzy compatible relation on R.*

Proof. For any $x, z, a, b \in R$,

$$\begin{aligned}
\mu \circ \tau(x+a, y+b) &= \bigvee_{t \in R} \{\mu(x+a, t) \wedge \tau(t, y+b)\} \\
&= \bigvee_{c,d \in R} \{\mu(x+a, c+d) \wedge \tau(c+d, y+b)\} \\
&\geqslant \bigvee_{c,d \in R} \{\mu(x, c) \wedge \mu(a, d) \wedge 0.5\} \wedge \{\tau(c, y) \wedge \tau(d, b) \wedge 0.5\} \\
&= \bigvee_{c,d \in R} \mu(x, c) \wedge \mu(a, d) \wedge \tau(c, y) \wedge \tau(d, b) \wedge 0.5 \\
&\geqslant \bigvee_{c \in R} \{\mu(x, c) \wedge \tau(c, y)\} \wedge \bigvee_{d \in R} \{\mu(a, d) \wedge \tau(d, b)\} \wedge 0.5 \\
&= \mu \circ \tau(x, y) \wedge \mu \circ \tau(a, b) \wedge 0.5
\end{aligned}$$

Similarly, $\mu \circ \tau(xa, yb) \geqslant \mu \circ \tau(x, y) \wedge \mu \circ \tau(a, b) \wedge 0.5$

By Theorem 3.2, $\mu \circ \tau$ is an $(\in, \in \vee\, q)$-fuzzy compatible. □

Lemma 3.12. *Let μ and τ be two $(\in, \in \vee\, q)$-fuzzy transitive and symmetric relations on ring R. Then $\mu \circ \tau$ is also an $(\in, \in \vee\, q)$-fuzzy transitive relation.*

Proof. For any $x, z \in R$, $\mu \circ \tau(x, z) = \bigvee_{y \in R} \{\mu(x, y) \wedge \tau(y, z)\}$.

So, for any $y \in R$

$$\begin{aligned}
\mu \circ \tau(x, z) &\geqslant \mu(x, y) \wedge \tau(y, z) \\
&\geqslant \bigvee_{u_1, u_2 \in R} \{\mu(x, u_1) \wedge \mu(u_1, y) \wedge 0.5\} \wedge \{\tau(y, u_2) \wedge \tau(u_2, z) \wedge 0.5\} \\
&\quad [\mu \text{ and } \tau \text{ are } (\in, \in \vee \text{ q}) - \text{fuzzy transitive relation}] \\
&= \bigvee_{u_1, u_2 \in R} \{\mu(x, u_1) \wedge \tau(y, u_2)\} \wedge \{\mu(u_1, y) \wedge \tau(u_2, z)\} \wedge 0.5 \\
&\geqslant \bigvee_{u \in R} \{\mu(x, u) \wedge \tau(y, u)\} \wedge \{\mu(u, y) \wedge \tau(u, z)\} \wedge 0.5 \\
&\geqslant \bigvee_{u \in R} \{\mu(x, u) \wedge \tau(u, y)\} \wedge \{\mu(y, u) \wedge \tau(u, z)\} \wedge 0.5 \\
&\quad [\mu \text{ and } \tau \text{ are } (\in, \in \vee\, q) \text{ fuzzy symmetric relation}] \\
&= \mu \circ \tau(x, y) \wedge \mu \circ \tau(y, z) \wedge 0.5
\end{aligned}$$

By Proposition 2.2, $\mu \circ \tau$ is an $(\in, \in \vee\, q)$-fuzzy transitive relation on R. □

Theorem 3.13. *Let μ and τ be two $(\in, \in \vee\, q)$-fuzzy congruences on ring R. Then the following are equivalent:*

(i) *$\mu \circ \tau$ is $(\in, \in \vee\, q)$-fuzzy congruence,*
(ii) *$\mu \circ \tau$ is $(\in, \in \vee\, q)$-fuzzy equivalence,*
(iii) *$\mu \circ \tau$ is $(\in, \in \vee\, q)$-fuzzy symmetric.*

Proof. It is clear that $(i) \implies (ii)$ and $(ii) \implies (iii)$.
Assume that (iii) holds. By Lemma (3.10) and (3.12), $\mu\circ\tau$ is $(\in,\in\vee q)$-fuzzy reflexive and $(\in,\in\vee q)$-fuzzy transitive. Also, by Lemma (3.11), $\mu\circ\tau$ is an $(\in,\in\vee q)$-fuzzy compatible. Thus $\mu\circ\tau$ is $(\in,\in\vee q)$-fuzzy congruence. Hence $(iii) \implies (i)$.

□

Theorem 3.14. *Let μ and τ be two $(\in,\in\vee q)$-fuzzy symmetric relations on a ring R. Then $\mu\circ\tau$ is an $(\in,\in\vee q)$-fuzzy symmetric if and only if $\tau\circ\mu$ is an $(\in,\in\vee q)$-fuzzy symmetric.*

Proof. Assume that $\mu\circ\tau$ is $(\in,\in\vee q)$-fuzzy symmetric. Then for any $x, y \in R$,

$$\begin{aligned}\tau\circ\mu(x,y) &= \bigvee_{z\in R}\{\tau(x,z)\wedge\mu(z,y)\}\\ &\geqslant \bigvee_{z\in R}\mu(y,z)\wedge\tau(z,x)\wedge 0.5 \text{ [As } \mu \text{ and } \tau \text{ are } (\in,\in\vee \text{q})-\text{fuzzy}\\ &\qquad\qquad\text{symmetric relation]}\\ &= \mu\circ\tau(y,x)\wedge 0.5\\ &\geqslant \{\mu\circ\tau(x,y)\wedge 0.5\}\wedge 0.5 \ \text{ [since } \mu\circ\tau \text{ is } (\in,\in\vee \text{q})-\text{fuzzy symmetric]}\\ &= \bigvee_{u\in R}\{\mu(x,u)\wedge\tau(u,y)\}\wedge 0.5\\ &\geqslant \bigvee_{u\in R}\{\tau(y,u)\wedge\mu(u,x)\}\wedge 0.5\\ &= \tau\circ\mu(y,x)\wedge 0.5\end{aligned}$$

So, $\tau\circ\mu$ is $(\in,\in\vee q)$-fuzzy symmetric.
Converse is similarly proved by interchanging μ and τ. □

Lemma 3.15. *Let μ and τ be two $(\in,\in\vee q)$-fuzzy congruence relations on a ring R.*
Then $\mu\circ\tau(x,y) \geqslant \tau\circ\mu(x,y)\wedge 0.5,\ \forall\ x, y \in R$.

Proof. For $x, y \in R$,

$$\begin{aligned}\mu\circ\tau(x,y) &= \bigvee_{t\in R}\{\mu(x,t)\wedge\tau(t,y)\}\\ &= \bigvee_{z\in R}\{\mu(x,y-z+x)\wedge\tau(y-z+x,y)\}\\ &= \bigvee_{z\in R}\{\mu(z-z+x,y-z+x)\wedge\tau(y-z+x,y-z+z)\}\\ &\geqslant \bigvee_{z\in R}\{\mu(z,y)\wedge\tau(x,z)\}\wedge 0.5 \ \text{ [since } \mu \text{ and } \tau \text{ are } (\in,\in\vee \text{q})-\text{fuzzy}\\ &\qquad\qquad\text{reflexive and } (\in,\in\vee \text{q})-\text{fuzzy compatible]}\end{aligned}$$

$$=\tau \circ \mu(x, y) \wedge 0.5$$

□

Theorem 3.16. *Let μ and τ be two $(\in, \in \vee q)$-fuzzy congruences on a ring R. Then $\mu \circ \tau$ is also an $(\in, \in \vee q)$-fuzzy congruence on R.*

Proof. By Theorem 3.13, we need to prove that $\mu \circ \tau(x, y)$ is $(\in, \in \vee q)$-fuzzy symmetric.
For $x, y \in R$,

$$\begin{aligned}
\mu \circ \tau(x, y) &= \bigvee_{t \in R} \{\mu(x, t) \wedge \tau(t, y)\} \\
&\geqslant \bigvee_{t \in R} \mu(t, x) \wedge \tau(y, t) \wedge 0.5 \\
&\geqslant \bigvee_{t \in R} \{\tau(y, t) \wedge \mu(t, x)\} \wedge 0.5 \\
&= \tau \circ \mu(y, x) \wedge 0.5 \\
&\geqslant \mu \circ \tau(y, x) \wedge 0.5 \quad \text{[by Lemma 3.15]}
\end{aligned}$$

So, $\mu \circ \tau(x, y)$ is $(\in, \in \vee q)$-fuzzy symmetric. □

4. $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$fuzzy congruence relation

In this section, we discuss the relationship between $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy congruence on ring.

Definition 4.1. Let λ be an $(\in, \in \vee q)$-fuzzy ideal of R. Define a fuzzy relation μ_λ on R by $\mu_\lambda(x, y) = \min\{\lambda(x - y), 0.5\}$ for all $x, y \in R$. Then μ_λ is called the fuzzy relation induced by $(\in, \in \vee q)$-fuzzy ideal λ of R.

Theorem 4.2. *Let λ be an $(\in, \in \vee q)$-fuzzy ideal of R. Then the fuzzy relation μ_λ on R induced is an $(\in, \in \vee q)$-fuzzy congruence relation on R.*

Proof. For any $a \in R$, we have

$$\begin{aligned}
\mu_\lambda(a, a) &= \min\{\lambda(a - a), 0.5\} \\
&= \min\{\lambda(0), 0.5\} \\
&\geqslant \min\{\lambda(x), 0.5\} \quad \text{for all } x \in R
\end{aligned}$$

Then , for any $x, y \in R$,

$$\begin{aligned}
\mu_\lambda(a, a) &\geqslant \min\{\lambda(x - y), 0.5\} \\
&= \min\{\mu_\lambda(x, y), 0.5\}
\end{aligned}$$

Thus, μ_λ is an $(\in, \in \vee q)$-fuzzy reflexive relation.
Also,

$$\begin{aligned}
\mu_\lambda(x, y) &= \min\{\lambda(x - y), 0.5\} \\
&\geqslant \min\{\min\{\lambda(y - x), 0.5\}, 0.5\}
\end{aligned}$$

$$= \min\{\mu_\lambda(y,x), 0.5\} \text{ for all } x, y \in R$$

Thus, μ_λ is an $(\in, \in \vee\, q)$-fuzzy symmetric relation.
For any $x, y, z \in R$,

$$\begin{aligned}
\mu_\lambda(x,z) &= \min\{\lambda(x-z), 0.5\} \\
&= \min\{\lambda(x-y+y-z), 0.5\} \\
&\geqslant \min\{\min\{\lambda(x-y), \lambda(y-z), 0.5\}, 0.5\} \\
&= \min\{\min\{\lambda(x-y), 0.5\}, \min\{\lambda(y-z), 0.5\}, 0.5\} \\
&= \min\{\mu_\lambda(x,y), \mu_\lambda(y,z), 0.5\}
\end{aligned}$$

So, μ_λ is an $(\in, \in \vee\, q)$-fuzzy transitive relation. Thus, μ_λ is an $(\in, \in \vee\, q)$-fuzzy equivalence relation on R.
Further, for any $a, b, x, y \in R$,

$$\begin{aligned}
\mu_\lambda(x+a, y+b) &= \min\{\lambda(\overline{x+a} - \overline{y+b}), 0.5\} \\
&= \min\{\lambda(x+a-y-b), 0.5\} \\
&= \min\{\lambda(x-y+a-b), 0.5\} \\
&\geqslant \min\{\min\{\lambda(x-y), \lambda(a-b), 0.5\}, 0.5\} \\
&\geqslant \min\{\min\{\lambda(x-y), 0.5\}, \min\{\lambda(a-b), 0.5\}, 0.5\} \\
&= \min\{\mu_\lambda(x,y), \mu_\lambda(a,b), 0.5\}
\end{aligned}$$

Similarly, $\mu_\lambda(xa, yb) \geqslant \min\{\mu_\lambda(x,y), \mu_\lambda(a,b), 0.5\}$.
Hence, μ_λ is an $(\in, \in \vee\, q)$-fuzzy congruence relation. In fact, μ_λ is an $(\in, \in \vee\, q)$-fuzzy congruence relation induced by $(\in, \in \vee\, q)$-fuzzy ideal λ of R. □

Theorem 4.3. *Let μ be an $(\in, \in \vee q)$-fuzzy congruence on a ring R. Then the fuzzy subset λ_μ of R defined by $\lambda_\mu(x) = min\{\mu(x,0), 0.5\}$, $\forall\, x \in R$ is $(\in, \in \vee\, q)$-fuzzy ideal of R.*

Proof. First we take $x, y \in R$

$$\begin{aligned}
\lambda_\mu(x-y) =& \min\{\mu(x-y, 0), 0.5\} \\
=& \min\{\mu(x-y, 0-0), 0.5\} \\
\geqslant& \min\{\mu(x,0), \mu(-y,0), 0.5\} \\
\geqslant& \min\{\mu(x,0), \mu(y,0), 0.5\} \text{ [using prop. 3.2}(ii)] \\
=& \min\{\lambda_\mu(x), \lambda_\mu(y), 0.5\}
\end{aligned}$$

Again,

$$\begin{aligned}
\lambda_\mu(xy) =& \min\{\mu(xy, 0), 0.5\} \\
=& \min\{\mu(xy, 0y), 0.5\} \\
\geqslant& \min\{\mu(x,0), \mu(y,y), 0.5\} \\
\geqslant& \min\{\mu(x,0), 0.5\} \text{ [using reflexivity]} \\
=& \min\{\lambda_\mu(x), 0.5\}
\end{aligned}$$

Similarly,

$$\lambda_\mu(yx) \geqslant \min\{\lambda_\mu(x), 0.5\}$$

Thus, λ_μ is an $(\in, \in \vee q)$-fuzzy ideal of R. □

Definition 4.4. Let μ be an $(\in, \in \vee q)$-fuzzy congruence on a ring R. A fuzzy subset of R defined by $\mu_x : R \to [0,1]$ by $\mu_x(a) = \min\{\mu(x,a), 0.5\}$, $a \in R$ is defined as fuzzy congruence class of x determined by μ in R. The set of all fuzzy congruence class is denoted by R/μ known as the fuzzy quotient of R over μ.

Theorem 4.5. *Let μ be an $(\in, \in \vee q)$-fuzzy congruence on a ring R. The set R/μ of all fuzzy congruence classes forms a ring with the operations defined by*

$$\mu_x + \mu_y = \mu_{x+y} \text{ and } \mu_x * \mu_y = \mu_{xy} \ \forall\ x, y \in R.$$

Proof. We first prove that $+$ and $*$ are well defined.
Let $\mu_x = \mu_a,\ \mu_y = \mu_b$, so $\min\{\mu(x,g), 0.5\} = \min\{\mu(a,g), 0.5\}$ and
$\min\{\mu(y,h), 0.5\} = \min\{\mu(b,h), 0.5\}\ \forall\ g, h \in R$
Putting $g = a,\ h = b$ in the above equations,
$\min\{\mu(x,a), 0.5\} = \min\{\mu(a,a), 0.5\}$
and $\min\{\mu(y,b), 0.5\} = \min\{\mu(b,b), 0.5\}$
Now,

$$\begin{aligned}
\mu_{x+y}(g) &= \min\{\mu(x+y,g), 0.5\} \\
&\geqslant \min\{\mu(x+y,a+b), \mu(a+b,g), 0.5\} \\
&\geqslant \min\{\mu(x,a), \mu(y,b), \mu(a+b,g), 0.5\} \\
&= \min\{\mu(a,a), \mu(b,b), \mu(a+b,g), 0.5\} \\
&\geqslant \min\{\mu(a+b,g), 0.5\} = \mu_{a+b}(g)
\end{aligned}$$

Similarly, $\mu_{a+b}(g) \geqslant \mu_{x+y}(g)$ for all $g \in R$. This means $\mu_{a+b}(g) = \mu_{x+y}(g)$
Again,

$$\begin{aligned}
\mu_{xy}(g) &= \min\{\mu(xy,g), 0.5\} \\
&\geqslant \min\{\mu(xy,ab), \mu(ab,g), 0.5\} \\
&\geqslant \min\{\mu(x,a), \mu(y,b), \mu(ab,g), 0.5\} \\
&= \min\{\mu(a,a), \mu(b,b), \mu(ab,g), 0.5\} \\
&\geqslant \min\{\mu(ab,g), 0.5\} = \mu_{ab}(g)
\end{aligned}$$

Similarly, $\mu_{ab}(g) \geqslant \mu_{xy}(g)$ for all $g \in R$. This means $\mu_{ab} = \mu_{xy}$.

Thus, it proves that addition and multiplication are well defined. The rest of the proof is a routine matter for verification and we omit its proof.

Hence, R/μ forms a ring under the binary operations defined above. □

Remark 4.1. If μ is a $(\in, \in \vee q)$-fuzzy congruence on a ring R, then R/μ is a ring which has the zero element μ_0.

5. Image and preimage of $(\in, \in \vee q)$-fuzzy congruence on a ring

Definition 5.1. [25] Let f be a mapping from a set X into another set Y. If ν is a fuzzy subset of X, the image $f(\nu)$ of ν is the fuzzy subset of Y defined by

$$f(\nu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \nu(x) & \text{if } \mathrm{f}^{-1}(\mathrm{y}) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

The preimage $f^{-1}(\nu')$ of a fuzzy subset ν' of Y is the fuzzy subset of X defined by

$$f^{-1}(\nu')(x) = \nu'(f(x)),\ x \in X$$

.

Definition 5.2. [13] Let X and Y be two nonempty sets. A mapping $f : X \times X \to Y \times Y$ is called a semibalanced mapping if

(i) given $a \in X$, there exist $u \in Y$ such that $f(a, a) = (u, u)$;

(ii)$f(a, a) = (u, u)$ and $f(b, b) = (v, v)$, where $a, b \in X, u, v \in Y$ implies that $f(a, b) = (u, v)$.

Definition 5.3. [11] A mapping $f : X \times X \to Y \times Y$ is called a balanced mapping if

(i) $f(a, b) = (u, u) \implies a = b$,

(ii) $f(a, b) = (u, v) \implies f(b, a) = (v, u)$,

(iii) $f(a, a) = (u, u)$ and $f(b, b) = (v, v) \implies f(a, b) = (u, v)$ for all $a, b \in X$ and $u, v \in Y$.

A mapping $f : X \times X \to Y \times Y$ is a balanced mapping if and only if it is a one-to-one semibalanced mapping.

Theorem 5.4. *If ν' is an $(\in, \in \vee q)$-fuzzy compatible relation on a ring $R^{'}$ and f is a ring homomorphism from $R \times R$ to $\mathrm{R}^{'} \times \mathrm{R}^{'}$, then $f^{-1}(\nu')$ is also an $(\in, \in \vee q)$-fuzzy compatible relation on R.*

Proof. Let $x, y, a, b \in R$. Let $f(x, y) = (u, v)$ and $f(a, b) = (s, t)$ for some $u, v, s, t \in R^{'}$.
Then, $f(x, y) + f(a, b) = (u, v) + (s, t) = (u + s, v + t)$.
Now,

$$\begin{aligned} f^{-1}(\nu')(x + a, y + b) &= \nu'\{f(x + a, y + b)\} \\ &= \nu'\{f(x, y) + f(a, b)\} \\ &= \nu'(u + s, v + t) \\ &\geqslant \min\{\nu'(u, v), \nu'(s, t), 0.5\} \\ &\geqslant \min\{\nu'(f(x, y)), \nu'(f(a, b)), 0.5\} \\ &= \min\{f^{-1}(\nu')(x, y), f^{-1}(\nu')(a, b), 0.5\} \end{aligned}$$

Similarly,$f^{-1}(\nu')(xa, yb) \geqslant \min\{f^{-1}(\nu')(x, y), f^{-1}(\nu')(a, b), 0.5\}$.
Hence, $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy compatible relation on R. □

Theorem 5.5. *If ν is an $(\in, \in \vee\, q)$-fuzzy compatible relation on a ring R and f is a ring homomorphism from $R \times R$ into $R^{'} \times R^{'}$. Then $f(\nu)$ is also an $(\in, \in \vee\, q)$-fuzzy compatible relation on $R^{'}$.*

Proof. Let $a, b, c, d \in R$ and $u, v, w, r \in R^{'}$. Then we have,

$$\begin{aligned}
f(\nu)(u+w, v+r) &= \bigvee_{(x,y) \in f^{-1}(u+w, v+w)} \nu(x,y) \\
&\geqslant \bigvee_{(a+c, b+d) \in f^{-1}(u+w, v+r)} \nu(a+c, b+d) \\
&= \bigvee_{f(a+c, b+d) = (u+w, v+r)} \nu(a+c, b+d) \\
&\geqslant \bigvee_{f(a,b)=(u,v), f(c,d)=(w,r)} \min\{\nu(a,b), \nu(c,d), 0.5\} \\
&= \min\{ \bigvee_{f(a,b)=(u,v)} \nu(a,b), \bigvee_{f(c,d)=(w,r)} \nu(c,d),\ 0.5\} \\
&= \min\{f(\nu)(u,v), f(\nu)(w,r), 0.5\}
\end{aligned}$$

Similarly, $f(\nu)(uw, vr) \geqslant \min\{f(\nu)(u,v), f(\nu)(w,r), 0.5\}$
Hence, $f(\nu)$ is an $(\in, \in \vee\, q)$-fuzzy compatible relation on $R^{'}$. □

Theorem 5.6. *If f is a semibalanced map from $R \times R$ into $R^{'} \times R^{'}$ and ν' is an $(\in, \in \vee q)$-fuzzy equivalence relation on $R^{'}$, then $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy equivalence relation on R.*

Proof. Let $a, b, c \in R$, then there exist $u, v, w \in R'$ such that $f(a,a) = (u,u)$, $f(b,b) = (v,v)$ and $f(c,c) = (w,w)$ so that $f(a,b) = (u,v)$ and $f(b,a) = (v,u)$. Then

$$\begin{aligned}
f^{-1}(\nu')(c,c) &= \nu'(f(c,c)) = \nu'(w,w) \\
&\geqslant \min\{\nu'(u,v), 0.5\} \\
&= \min\{\nu'(f(a,b), 0.5\} \\
&= \min\{f^{-1}(\nu')(a,b), 0.5\}
\end{aligned}$$

Thus, $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy reflexive relation.
Now,

$$\begin{aligned}
f^{-1}(\nu')(b,a) &= \nu'(f(b,a)) = \nu'(v,u) \\
&\geqslant \min\{\nu'(u,v), 0.5\} \\
&= \min\{\nu'(f(a,b), 0.5\} \\
&= \min\{f^{-1}(\nu')(a,b), 0.5\}
\end{aligned}$$

Thus, $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy symmetric relation.
Further,

$$\begin{aligned} f^{-1}(\nu')(a,c) &= \nu'(f(a,c)) = \nu'(u,w) \\ &\geqslant \min\{\nu'(u,v), \nu'(v,w), 0.5\} \\ &= \min\{\nu(f(a,b), f(b,c), 0.5\} \\ &= \min\{f^{-1}(\nu')(a,b), f^{-1}(\nu')(b,c), 0.5\} \end{aligned}$$

Thus, $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy transitive relation.
Hence, $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy equivalence relation on R. □

Theorem 5.7. *If ν' is an $(\in, \in \vee q)$-fuzzy congruence relation on a ring $R^{'}$ and f is a ring homomorphism from $R \times R$ into $R^{'} \times R^{'}$, which is a semibalanced map, then $f^{-1}(\nu')$ is an $(\in, \in \vee q)$-fuzzy congruence relation on R.*

Proof. It follows from Theorem 5.1 and 5.3. □

Theorem 5.8. *Let f be balanced mapping from $R \times R$ onto $R^{'} \times R^{'}$ which is a ring homomorphism. If ν is an $(\in, \in \vee q)$-fuzzy congruence relation on R then $f(\nu)$ is an $(\in, \in \vee q)$-fuzzy congruence relation on $R^{'}$.*

Proof. Let $u, v, w, r \in R^{'}$.
Since f is one-one and onto, there exist $a, b, c, d \in R\ s.t. f(a,b) = (u,v), f(c,c) = (w,w), f(d,d) = (w,w)$ and so on.

$$f(\nu)(u,v) = \sup_{(x,y)\ \in\ f^{-1}(u,v)} \nu(x,y) = \nu(a,b)$$

Then,

$$\begin{aligned} f(\nu)(w,w) = \nu(c,c) &\geqslant \min\{\nu(a,b), 0.5\} \\ &= \min\{f(\nu)(u,v), 0.5\} \end{aligned}$$

This implies $f(\nu)$ is an $(\in, \in \vee q)$-fuzzy reflexive relation.
Now,

$$\begin{aligned} f(\nu)(v,u) &= \sup_{(y,x)\ \in\ f^{-1}(v,u)} \nu(y,x) = \nu(b,a) \geqslant \min\{\nu(a,b), 0.5\} \\ &= \min\{f(\nu)(u,v), 0.5\}. \end{aligned}$$

This implies $f(\nu)$ is an $(\in, \in \vee q)$-fuzzy symmetric relation.
Also,

$$\begin{aligned} f(\nu)(u,w) &= \sup_{(x,z)\ \in\ f^{-1}(u,w)} \nu(x,z) = \nu(a,c) \\ &\geqslant \min\{\nu(a,b), \nu(b,c), 0.5\} \\ &= \min\{f(\nu)(u,v), f(\nu)(v,w), 0.5\} \\ &[\text{ as } \nu(a,b) = f(\nu)(u,v) \text{ and } \nu(b,c) = f(\nu)(v,w)]. \end{aligned}$$

This imply $f(\nu)$ is an $(\in, \in \vee q)$-fuzzy transitive relation.

Using Theorem 5.2 we get,

$$f(\nu)(u+w, v+r) \geqslant \min\{f(\nu)(u,v), f(\nu)(w,r), 0.5\}$$

and

$$f(\nu)(uw, vr) \geqslant \min\{f(\nu)(u,v), f(\nu)(w,r), 0.5\}$$

.

Hence, $f(\nu)$ is an $(\in, \in \vee q)$-fuzzy congruence relation on $R^{'}$. □

6. Conclusion

The notion of $(\in, \in \vee q)$-fuzzy congruence relation over ring is introduced in this paper and the algebraic properties of $(\in, \in \vee q)$-fuzzy congruence relation are studied. Additionally, we established a brief relation between $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy congruence relation. It is also shown that the concept of $(\in, \in \vee q)$-fuzzy congruence relation is preserved by the image and preimage under a balanced and semibalanced map respectively. This work could be expanded upon by investigating the properties of $(\in, \in \vee q)$-fuzzy congruence relation on other algebraic structure such as semi rings, near-rings and near-ring modules.

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Ningthoujam Pradipkumar received M.Sc. Annamalai University and Currently undergoing Ph.D. at Manipur University. His research interests is Fuzzy Algebra.

Department of Mathematics, Manipur University, Imphal, 795001, Manipur, India.
e-mail: `npradip30@manipuruniv.ac.in`

Okram Ratnabala Devi received M.Sc. from Pune University, and Ph.D. from Manipur University. She is currently working as Professor at Manipur University since 2015. Her research interests are fuzzy algebra, ring theory and coding.

Department of Mathematics, Manipur University, Imphal, 795001, Manipur, India.
e-mail: ratnabala@manipuruniv.ac.in