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DOMINATING ENERGY AND DOMINATING LAPLACIAN ENERGY OF HESITANCY FUZZY GRAPH

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Abstract. This article introduces the concepts of Energy and Laplacian Energy (LE) of Domination in Hesitancy fuzzy graph (DHFG). Also, the adjacency matrix of a DHFG is defined and proposed the definition of the energy of domination in hesitancy fuzzy graph, and Laplacian energy of domination in hesitancy fuzzy graph is given.

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1. Introduction

In 1965, L.A. Zadeh [1] established the foundation of fuzzy sets (FSs) and fuzzy relations (FRs). It has been used in the evaluation of cluster patterns. Rosen feld [2] established the structure of fuzzy graphs (FGs) by considering FRs on FSs. Somasundaram A and Somasundaram S [7] studied the concept of domination in FGs. The Laplacian matrix and energy of a fuzzy graph are defined by Sadegh Rahimi Sharbaf and Fatmeh Fayazi [8]. Some conclusions on Laplacian energy limits of the fuzzy graphs are given. A fuzzy graph's energy and certain limitations on its energy are examined in [4 and 5]. Ore and Berge [19,20] developed the notion of dominant sets in graphs. In IFG, R. Parvathi and G. Thamizhendhi [9,10] established the dominating set, dominating number, independent set, and total dominating number. The study of domination concepts in IFG is more efficient than FGs, which is beneficial for traffic density and communications systems. Kartheek E and Sharief Basha.S [11] expanded the notion of the minimum dominant energy (MDE) to FGs. They also provide

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the two boundaries of the MDE of FGs. R. Parvathi and M.G. Karunambigai [3] were introducing a new concept of intuitionistic fuzzy graphs (IFGs) and also provide properties of IFGs with suitable illustrations. Kalimulla A, Vijayaragavan R, and Sharief Basha.S proposed the concept of Dominating Energy (DE) in various operations such as union, join, Cartesian product, composition, and complement, between two IFGs. Two IFGs were investigated in respect to their dominant energy. The DE of several operations of an IFG is also examined. Laplacian Energy of IFGs was established by Sharief Basha.S and Kartheek E [14]. The Signless Laplacian energy of IFGS was established by Obbu Ramesh and S.Sharief Basha [6]. And the two boundaries are also given with suitable examples. R. Vijayaragavan, A. Kalimulla, and S. Sharief Basha [12] discussed the importance of dominating Laplacian energy (DLE) of IFG in Cartesian product and Tensor product. The idea of DLE in various products of IFGs is expanded. They explored the DLE in products of IFGs and also the DLE in the products of two IFGs is described.

The Hesitant fuzzy sets (HFSs) concept is expanded from the concepts of FSs and IFSs. V Torra [15] was first introduced the HFSs concept and also given basic some basic properties of HFSs. HFSs have been expanded by Xu Z. and Zhu B [16, 17] from various perspectives, including quantitative and qualitative. The notion of dominance in Hesitancy fuzzy graphs is extremely rich, both theoretically developments and practically. Hesitancy fuzzy graphs (HFG) are developed to capture the common complication that happens during the decision of an entity's grade of membership from a set of alternative values, which causes one to hesitate. HFG are used to select a time-Minimised emergency route (TiMER) for the transportation of accident sufferers. Pathinathan T, Jon Arockiaraj J and Jesintha Rosline J [18] were introduced the new idea of FGs called the HFGs from the IFGs and "Intuitionistic Double Layered FGs" and also, a number of related outcomes have been examined and solved. Shakthivel R, Vikramaprasad R, and Vinothkumar N were investigating the dominating notion in HFGs and certain properties of domination in Hesitancy Fuzzy Graph (DHFG), as well as DHFGs products such as "union, join, cartesian product, and composition". The energy of hesitancy fuzzy graphs are extended from the concepts of intuitionistic fuzzy graphs by Rajagopal Reddy and Sharief Basha [21]. They are introduced the properties of energy of HFGs with suitable example.

According to the following, this document is arranged. In section 2, defines the fundamental definitions of domination in HFG. In section 3, focuses some related definitions of Energy of HFG and Dominating Energy of HFG and also find the numerical values of Energy of DHFG. In section 4, proposes the related definitions of Laplacian Energy of HFG and Dominating Laplacian Energy of HFG and also find the numerical values of Laplacian Energy of DHFG. Finally, the paper's conclusion is presented in Section 5.

2. Preliminaries

Definition 2.1. A HFG is of the form $HG = (V, E, \mu, \gamma, \beta)$, where

- $V = \{v_1, v_2, ..., v_p\}$ such that $\mu_1 : V \to [0, 1]$, $\gamma_1 : V \to [0, 1]$ and $\beta_1 : V \to [0, 1]$ are denotes the degree of membership, nonmembership and hesitant of the elements $v_i \in V$ and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$, where
$$\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]\,. \tag{1}$$
- $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0, 1], \gamma_2 : V \times V \to [0, 1]$ and $\beta_2 : V \times V \to [0, 1]$ are such that,
$$\mu_2(v_i, v_j) \le min[\mu_1(v_i), \mu_1(v_j)] \tag{2}$$
$$\gamma_2(v_i, v_j) \le max[\gamma_1(v_i), \gamma_1(v_j)] \tag{3}$$
$$\beta_2(v_i, v_j) \le min[\beta_1(v_i), \beta_1(v_j)]and \tag{4}$$
$$0 \le \mu_2(v_i, v_j) + \mu_2(v_i, v_j) + \mu_2(v_i, v_j) \le 1,\ \forall (v_i, v_j) \in E. \tag{5}$$

Definition 2.2. Consider $HG = (V, E, \mu, \gamma, \beta)$ defines a HFG, then the arc (v_i, v_j) of HFG is called a strong arc if
$$\mu_2(v_i, v_j) \le \mu_1(v_i) \wedge \mu_1(v_j), \tag{6}$$
$$\gamma_2(v_i, v_j) \le \gamma_1(v_i) \wedge \gamma_1(v_j), and \tag{7}$$
$$\beta_2(v_i, v_j) \le \beta_1(v_i) \wedge \beta_1(v_j), \tag{8}$$

Definition 2.3. Assume that $HG = (V, E, \mu, \gamma, \beta)$ is a HFG, $u, v \in V$ and u dominates v in HG if a strong arc exists between them. A subset $D \subseteq V$ is called as a dominating set in an HFG HG, if there exists u in D such that u dominates for every $v \in V - D$.

3. The Energy of HFG and Dominating Energy of HFG

Definition 3.1. Suppose $HG = (V, E, \mu, \gamma, \beta)$ is an HFG, and also consider the DHFG is $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$, then the dominating hesitancy fuzzy adjacency matrix (DHFAM) $D(A(HG)) = [d_{ij}]$ where
$$p_{ij} = \begin{cases} (\mu_{ij}, \gamma_{ij}, \beta_{ij}) & if\ (v_i, v_j) \in E \\ (1, 1, 1) & if\ i = j\ and\ v_i \in D \\ 0 & otherwise \end{cases}$$

Now the DHFAM D(HG) can be written as
$$D(HG) = (\mu_D(HG), \gamma_D(HG), \beta_D(HG))$$

Where
$$\mu_D(HG) = \begin{cases} \mu_{ij} & if\ (v_i, v_j) \in E \\ 1 & if\ i = j\ and\ v_i \in D \\ 0 & otherwise \end{cases}$$

$$\gamma_D(HG) = \begin{cases} \gamma_{ij} & if\ (v_i, v_j) \in E \\ 1 & if\ i = j\ and\ v_i \in D \\ 0 & otherwise \end{cases}$$

$$\beta_D(HG) = \begin{cases} \beta_{ij} & if\ (v_i, v_j) \in E \\ 1 & if\ i = j\ and\ v_i \in D \\ 0 & otherwise \end{cases}$$

Definition 3.2. Consider $HG = (V, E, \mu, \gamma, \beta)$ be an HFG, then Eigen roots of DHFAM $D(HG)$ is defined as (Z, Y, X) where κ is the set of Eigen roots of $\mu_D(HG)$, α is the set of Eigen roots of $\gamma_D(HG)$ and λ is the set of Eigen roots of $\beta_D(HG)$. The energy of a dominating hesitancy fuzzy graph (DHFG) $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$ is defined as

$$\left(\sum_{\kappa_i \in Z} |\kappa_i|, \sum_{\alpha_i \in Y} |\alpha_i|, \sum_{\lambda_i \in X} |\lambda_i| \right)$$

Where $\sum_{\kappa_i \in Z} |\kappa_i|$ is the summation of the absolute values of the Eigen roots of $\mu_D(HG)$ and $E(\mu_D(HG))$ is defined as the energy of a membership matrix, $\sum_{\alpha_i \in Y} |\alpha_i|$ is the summation of the absolute values of the Eigen roots of $\gamma_D(HG)$ and $E(\gamma_D(HG))$ is defined as the energy of a nonmembership matrix and $\sum_{\lambda_i \in Y} |\lambda_i|$ is the summation of the absolute values of the Eigen roots of $\beta_D(HG)$ and the energy of a hesitant matrix is defined as $E(\beta_D(HG))$.

Example 1.
From figure 1, we obtain

$$A(HG) = \begin{bmatrix} (0,0,0) & (0.4,0.3,0.2) & (0.3,0.4,0.2) & (0.2,0.5,0.3) \\ (0.4,0.3,0.2) & (0,0,0) & (0.4,0.4,0.1) & (0.2,0.5,0.2) \\ (0.3,0.4,0.2) & (0.4,0.4,0.1) & (0,0,0) & (0.2,0.5,0.1) \\ (0.2,0.5,0.3) & (0.2,0.5,0.2) & (0.2,0.5,0.1) & (0,0,0) \end{bmatrix}$$

First we find the Energy of $A(HG)$ of HFG and then the Dominating Energy of $D(A(HG))$ of HFG
(i) We find the Energy of $A(HG)$ of HFG.
The Energy of $A(HG)$ of HFG $HG = (V, E, \mu, \gamma, \beta)$ is

$$E(HG) = \sum_{i=1}^{n} |\kappa_i| \tag{9}$$

Where κ_i is the eigenroots of $A(HG)$ of HFG
Now we define the matrix $A(HG)$ into three matrices $A_\mu(HG)$, $A_\gamma(HG)$ and $A_\beta(HG)$ are

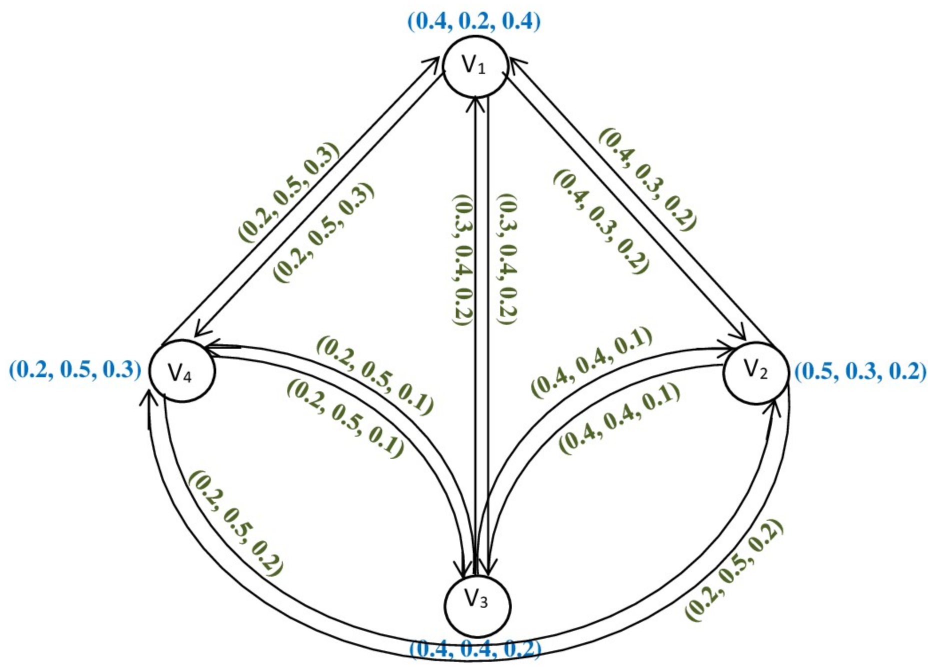

FIGURE 1. Hesitancy Fuzzy Graph with four Vertices

$$A_\mu(HG) = \begin{bmatrix} 0 & 0.4 & 0.3 & 0.2 \\ 0.4 & 0 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix}, \quad A_\gamma(HG) = \begin{bmatrix} 0 & 0.3 & 0.4 & 0.5 \\ 0.3 & 0 & 0.4 & 0.5 \\ 0.4 & 0.4 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}$$

$$A_\beta(HG) = \begin{bmatrix} 0 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0 \end{bmatrix}$$

Calculating the Eigenroots of the above three matrices and subtituiting in equation 9 we get
$E(A_\mu(HG)) = 1.7451, \quad E(A_\gamma(HG)) = 2.6160$, and $E(A_\beta(HG)) = 1.1345$
Therefore, the Energy of $A(HG)$ of HFG is $(1.7451, 2.6160, 1.1345)$

$$i.e., \; E(HG) = (1.7451, 2.6160, 1.1345)$$

(ii) We find the Dominating Energy of $D(A(HG))$ of HFG
Assume $HG = (V, E, \mu, \gamma, \beta)$ be an HFG, then the vertex set $V = \{v_1, v_2, v_3, v_4\}$, and $E = \{(v_1v_2), (v_1v_3), (v_1v_4), (v_2v_1), (v_2v_3), (v_2v_4), (v_3v_1), (v_3v_2), (v_3v_4), (v_4v_1),$

$(v_4v_2), (v_4v_3)\}$ be the edge set.
Assume a DHFG $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$ then $V = \{v_1, v_2, v_3, v_4\}$, and $(\mu_1, \gamma_1, \beta_1)$ are given by $\mu_1 : V \to [0, 1]$, $\gamma_1 : V \to [0, 1]$ and $\beta_1 : V \to [0, 1]$ Where

$$\mu_1(v_1) = max_{v_k}(\mu(v_1, v_k))$$
$$\gamma_1(v_1) = max_{v_k}(\gamma(v_1, v_k))$$
$$\beta_1(v_1) = max_{v_k}(\beta(v_1, v_k))$$

Now,

$$\begin{aligned}\mu_1(v_1) &= max_{v_k}(\mu(v_1, v_k)),\ \forall\ k = 1, 2, 3, 4\\ &= max(\mu(v_1, v_2), \mu(v_1, v_3), \mu(v_1, v_4))\\ &= max(0.4, 0.3, 0.2)\\ \mu_1(v_1) &= 0.4\end{aligned}$$

We calculate the remaining values $\mu_1(v_2), \mu_1(v_3)$, and $\mu_1(v_4)$ as same as above

$$\mu_1(v_2) = max(0.4, 0.4, 0.2) = 0.4$$
$$\mu_1(v_3) = max(0.3, 0.4, 0.2) = 0.4$$
$$\mu_1(v_4) = max(0.2, 0.2, 0.2) = 0.2$$

$$\begin{aligned}\gamma_1(v_1) &= min_{v_k}(\gamma(v_1, v_k)),\ \forall\ k = 1, 2, 3, 4\\ &= min(\gamma(v_1, v_2), \gamma(v_1, v_3), \gamma(v_1, v_4))\\ &= min(0.3, 0.4, 0.5)\\ \gamma_1(v_1) &= 0.3\end{aligned}$$

We calculate the remaining values $\gamma_1(v_2), \gamma_1(v_3)$, and $\gamma_1(v_4)$ as same as above

$$\gamma_1(v_2) = min(0.3, 0.4, 0.5) = 0.3$$
$$\gamma_1(v_3) = min(0.4, 0.4, 0.5) = 0.4$$
$$\gamma_1(v_4) = min(0.5, 0.5, 0.5) = 0.5$$

$$\begin{aligned}\beta_1(v_1) &= max_{v_k}(\beta(v_1, v_k)),\ \forall\ k = 1, 2, 3, 4\\ &= max(\beta(v_1, v_2), \beta(v_1, v_3), \beta(v_1, v_4))\\ &= max(0.2, 0.2, 0.3)\\ \beta_1(v_1) &= 0.3\end{aligned}$$

We calculate the remaining values $\beta_1(v_2), \beta_1(v_3)$, and $\beta_1(v_4)$ as same as above

$$\beta_1(v_2) = max(0.2, 0.1, 0.2) = 0.2$$
$$\beta_1(v_3) = max(0.2, 0.1, 0.1) = 0.2$$
$$\beta_1(v_4) = max(0.3, 0.2, 0.1) = 0.3$$

According to definition 2.2, v_1 dominates v_3 *and* v_4, and v_2 dominates v_3 *and* v_4, because

$$\mu_2(v_1, v_3) \leq \mu_1(v_1) \wedge \mu_1(v_3)$$
$$\gamma_2(v_1, v_3) \leq \gamma_1(v_1) \wedge \gamma_1(v_3)$$
$$\beta_2(v_1, v_3) \leq \beta_1(v_1) \wedge \beta_1(v_3)$$
$$\mu_2(v_1, v_4) \leq \mu_1(v_1) \wedge \mu_1(v_4)$$
$$\gamma_2(v_1, v_4) \leq \gamma_1(v_1) \wedge \gamma_1(v_4)$$
$$\beta_2(v_1, v_4) \leq \beta_1(v_1) \wedge \beta_1(v_4)$$

and

$$\mu_2(v_2, v_3) \leq \mu_1(v_2) \wedge \mu_1(v_3)$$
$$\gamma_2(v_2, v_3) \leq \gamma_1(v_2) \wedge \gamma_1(v_3)$$
$$\beta_2(v_2, v_3) \leq \beta_1(v_2) \wedge \beta_1(v_3)$$
$$\mu_2(v_2, v_4) \leq \mu_1(v_2) \wedge \mu_1(v_4)$$
$$\gamma_2(v_2, v_4) \leq \gamma_1(v_2) \wedge \gamma_1(v_4)$$
$$\beta_2(v_2, v_4) \leq \beta_1(v_2) \wedge \beta_1(v_4)$$

Therefore, the DHFG $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$ is $D = \{v_1, v_3\}$ then $V - D = \{v_3, v_4\}$ where $V = \{v_1, v_2, v_3, v_4\}$.
Now $|D| = 2 =$ total of the diagonal elements.
According to definition 2.3, we define the DHFAM $D(HG)$ of HFG is

$$D(HG) = \begin{bmatrix} (1,1,1) & (0.4,0.3,0.2) & (0.3,0.4,0.2) & (0.2,0.5,0.3) \\ (0.4,0.3,0.2) & (1,1,1) & (0.4,0.4,0.1) & (0.2,0.5,0.2) \\ (0.3,0.4,0.2) & (0.4,0.4,0.1) & (0,0,0) & (0.2,0.5,0.1) \\ (0.2,0.5,0.3) & (0.2,0.5,0.2) & (0.2,0.5,0.1) & (0,0,0) \end{bmatrix}$$

Here, we write the three matrices are membership matrix $\mu_D(HG)$, nonmembership $\gamma_D(HG)$ and hesitant $\beta_D(HG)$

$$\mu_D(HG) = \begin{bmatrix} 1 & 0.4 & 0.3 & 0.2 \\ 0.4 & 1 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix}, \ \gamma_D(HG) = \begin{bmatrix} 1 & 0.3 & 0.4 & 0.5 \\ 0.3 & 1 & 0.4 & 0.5 \\ 0.4 & 0.4 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}$$

$$\beta_D(HG) = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.3 \\ 0.2 & 1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0 \end{bmatrix}$$

The Eigen roots of each adjacency matrix $\mu_D(HG)$, $\gamma_D(HG)$ and $\beta_D(HG)$ are calculated as follows
The Eigen roots of $\mu_D(HG) = \{-0.2288, -0.0018, 0.6055, 1.6251\}$
The Eigen roots of $\gamma_D(HG) = \{-0.5099, -0.0779, 0.7000, 1.8878\}$

The Eigen roots of $\beta_D(HG) = \{-0.1183, -0.0289, 0.8090, 1.3381\}$
By the definition of Energy of DHFAM is

$$E\left(A(D(HG))\right) = \sum_{\kappa_i \in Z} |\kappa_i|$$

we determine the Energy of $\mu_D(HG)$ is

$$\begin{aligned} E\left(\mu_D(HG)\right) &= \sum_{i=1}^{4} |\kappa_i| \\ &= |\kappa_1| + |\kappa_2| + |\kappa_3| + |\kappa_4| \end{aligned}$$

By substituting needed values in above equation and calculating weget the energy of DHFAM
$E\left(\mu_D(HG)\right) = 2.4612.$
The energy of DHFAMs $\gamma_D(HG)$ and $\beta_D(HG)$ are calculated as the same as above
$\gamma_D(HG) = 3.1756$ and $\beta_D(HG) = 2.2943.$
$\therefore$ The Energy of DHFG $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$ is $(2.4612, 3.1756, 2.2943)$
$\left(\sum_{\kappa_i \in Z} |\kappa_i|, \sum_{\alpha_i \in Z} |\alpha_i|, \sum_{\lambda_i \in Z} |\lambda_i|\right) = (2.4612, 3.1756, 2.2943).$

4. The Laplacian Energy of HFG and the Dominating Laplacian Energy of HFG

In this section, First we find the Laplacian Energy of A(HG) of HFG and then the Dominating Laplacian Energy of D(A(HG)) of HFG $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$.

Definition 4.1. Consider $A(HG)$ be an adjacency matrix of HFG and the degree of the matrix of an HFG $HG = (V, E, \mu, \gamma, \beta)$ is defined as $D(HG) = [d_{ij}]$, then the Hesitancy fuzzy Laplacian matrix (HFLM) of HFG is defined as

$$L(HG) = D(HG) - A(HG)$$

.

Definition 4.2. Suppose $D(A(HG))$ be a dominating hesitancy fuzzy adjacency matrix of DHFG and $D(HG) = [d_{ij}]$ be a degree matrix of $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$. The matrix $L(D(HG)) = D(HG) - D(A(HG))$ is defined as the dominating hesitancy fuzzy Laplacian matrix (DHFLM) of DHFG.

Definition 4.3. Assume $HG = (V, E, \mu, \gamma, \beta)$ be an HFG with $|V| = r$ nodes then the Laplacian Energy (LE) of an HFG is denoted as

$$LE(\beta_{ij}(HG)) = \sum_{i=1}^{r} \left| \lambda_i - \frac{2 \sum_{1 \le i \le j \le r} \beta(v_i, v_j)}{r} \right| \tag{10}$$

Where λ_i be the Eigen root of adjacency matrices of HFG.
Therefore, $[LE(\mu_{ij}(HG)), LE(\gamma_{ij}(HG)), LE(\beta_{ij}(HG))]$ be the LE of an HFG $HG = (V, E, \mu, \gamma, \beta)$

Definition 4.4. Consider $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$ be a DHFG with $|V| = r$ nodes and $[LE(D(\mu_{ij}(HG))), LE(D(\gamma_{ij}(HG))), LE(D(\beta_{ij}(HG)))]$ be the LE of an DHFG and denoted as follows

$$LE(D(\mu_{ij}(HG))) = \sum_{i=1}^{r} \left| \kappa_i - \frac{2 \sum\limits_{1 \le i \le j \le r} \mu(v_i, v_j)}{r} \right|$$

$$LE(D(\gamma_{ij}(HG))) = \sum_{i=1}^{r} \left| \delta_i - \frac{2 \sum\limits_{1 \le i \le j \le r} \gamma(v_i, v_j)}{r} \right|$$

$$LE(D(\beta_{ij}(HG))) = \sum_{i=1}^{r} \left| \lambda_i - \frac{2 \sum\limits_{1 \le i \le j \le r} \beta(v_i, v_j)}{r} \right|$$

Where κ_i, δ_i and λ_i are the Eigen roots of domination of hesitancy fuzzy adjacency matrices $D(mu_{ij}(HG)), D(\gamma_{ij}(HG))$ and $D(\beta_{ij}(HG))$ of DHFG.

(i) We find the Laplacian Energy of $A(HG)$ of HFG Now we define the Laplacian matrix of $A(HG)$ is

$$L(A(HG)) = \begin{bmatrix} (0.9, 1.2, 0.7) & -(0.4, 0.3, 0.2) & -(0.3, 0.4, 0.2) & -(0.2, 0.5, 0.3) \\ -(0.4, 0.3, 0.2) & (1.0, 1.2, 0.5) & -(0.4, 0.4, 0.1) & -(0.2, 0.5, 0.2) \\ -(0.3, 0.4, 0.2) & -(0.4, 0.4, 0.1) & (0.9, 1.3, 0.4) & -(0.2, 0.5, 0.1) \\ -(0.2, 0.5, 0.3) & -(0.2, 0.5, 0.2) & -(0.2, 0.5, 0.1) & (0.6, 1.5, 0.6) \end{bmatrix}$$

The Laplacian matrix of A(HG)into three matrices $L(A_\mu(HG))$, $L(A_\gamma(HG))$ and $L(A_\beta(HG))$ are

$$L(A_\mu(HG) = \begin{bmatrix} 0.9 & -0.4 & -0.3 & -0.2 \\ -0.4 & 1.0 & -0.4 & -0.2 \\ -0.3 & -0.4 & 0.9 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.6 \end{bmatrix}$$

$$L(A_\gamma(HG) = \begin{bmatrix} 1.2 & -0.3 & -0.4 & -0.5 \\ -0.3 & 1.2 & -0.4 & -0.5 \\ -0.4 & -0.4 & 1.3 & -0.5 \\ -0.5 & -0.5 & -0.5 & 1.5 \end{bmatrix}$$

and

$$L(A_\beta(HG) = \begin{bmatrix} 0.7 & -0.2 & -0.2 & -0.3 \\ -0.2 & 0.5 & -0.1 & -0.2 \\ -0.2 & -0.1 & 0.4 & -0.1 \\ -0.3 & -0.2 & -0.1 & 0.6 \end{bmatrix}$$

Calculating the Eigenroots of $L(A_\mu(HG))$, $L(A_\gamma(HG))$ and $L(A_\beta(HG))$ and then subtituiting in equation (10) we get
The Laplacian Energy of $A(HG)$ is $(1.8000, 2.6000, 1.1806)$
Therefore, $LE(A(HG)) = (1.8000, 2.6000, 1.1806)$
(ii) We find the Dominating Laplacian Energy of an HFG
In this portion, we define the definition of Laplacian energy of DHFG and also find the laplacin energy of DHFG using the DHFG $HG = (V, E, \mu, \gamma, \beta, \mu_1, \gamma_1, \beta_1)$.
The Laplacian matrix $L(D(HG)) = D(HG) - D(A(HG))$ is Using the matrix 4, we define the Laplacian Matrix of DHFG is

$$L(D(HG)) = \begin{bmatrix} (1.9, 2.2, 1.7) & -(0.4, 0.3, 0.2) & -(0.3, 0.4, 0.2) & -(0.2, 0.5, 0.3) \\ -(0.4, 0.3, 0.2) & (2.0, 2.2, 1.5) & -(0.4, 0.4, 0.1) & -(0.2, 0.5, 0.2) \\ -(0.3, 0.4, 0.2) & -(0.4, 0.4, 0.1) & (0.9, 1.3, 0.4) & -(0.2, 0.5, 0.1) \\ -(0.2, 0.5, 0.3) & -(0.2, 0.5, 0.2) & -(0.2, 0.5, 0.1) & (0.6, 1.5, 0.6) \end{bmatrix}$$

The Laplacian Energy of DHFG is

$$[LE(D(\mu_{ij}(HG))), LE(D(\gamma_{ij}(HG))), LE(D(\beta_{ij}(HG)))]$$

The laplacian energy of $D(\mu_{ij}(HG))$ is

$$LE(D(\mu_{ij}(HG))) = \left| \kappa_i - \frac{2 \sum\limits_{1 \le i \le j \le r} \mu(v_i, v_j)}{r} \right| \tag{11}$$

The Laplacian Matrix of $D(\mu_{ij}(HG))$ is

$$L(D(HG)) = \begin{bmatrix} 1.9 & -0.4 & -0.3 & -0.2 \\ -0.4 & 2.0 & -0.4 & -0.2 \\ -0.3 & -0.4 & 0.9 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.6 \end{bmatrix}$$

The Eigen roots of the above Laplacian matrix are
$\kappa_1 = 0.2972,\ \kappa_2 = 0.9141,\ \kappa_3 = 1.8267$, and $\kappa_4 = 2.3620$
By substituiting these Eigen roots in equation (11) and calculating weget
$LE(D(\mu_{ij}(HG))) = 2.9774$
We find the LE of $LE(D(\gamma_{ij}(HG)))$ and $LE(D(\beta_{ij}(HG)))$ is calculated as the above, we get
$LE(D(\gamma_{ij}(HG))) = 2.8608$ and $LE(D(\beta_{ij}(HG))) = 1.8738$.
$\therefore [LE(D(\mu_{ij}(HG))), LE(D(\gamma_{ij}(HG))), LE(D(\beta_{ij}(HG)))] = (2.9774, 2.8608, 1.8738)$
Hence, the LE of domination in HFG is

$$LE(D(HG)) = (2.9774, 2.8608, 1.8738)$$

In the table 1, the results of the Energy and Laplacian energy of an HFG and the Dominating Energy and Dominating Laplacian Energy of an HFG are compared as follows

Table 1. The results of Energy and Laplacian Energy of HFG and the results of DOminating Energy and Dominating Laplacian Enenrgy of HFG .

E of $A(HG)$ of HFG	$(1.7451, 2.6061, 1.1345)$
D E of $D(A(HG))$ of HFG	$(2.4612, 3.1756, 2.2943)$
LE of $A(HG)$ of HFG	$(1.8000, 2.6000, 1.1806)$
DLE of $D(A(HG))$ of HFG	$(2.3620, 2.9774, 1.8738)$

We observe that the above table, the Energy of $A(HG)$ of an HFG is less than the Dominating Energy of $D(A(HG))$ of an HFG.
Therefore, $E(HG) < E(D(HG))$
Likewise, the Laplacian Energy of $A(HG)$ of an HFG is less than the Dominating Laplacian Energy of $D(A(HG))$ of an HFG.
Therefore, $LE(HG) < LE(D(HG))$

5. Conclusion

In this article, we have defined the adjacency matrix of dominating Hesitancy fuzzy graph (DHFG) and the Energy of a DHFG. Also, introduced the concept of the Laplacian Energy of a DHFG. The results of the Energy and Laplacian energy of an HFG are compared with the results of the Dominating Energy and Dominating Laplacian Energy of an HFG. Further explanations and examples of the other valid theoretical ideas will be included in a future article on this subject.

Conflicts of interest : The authors declare no conflict of interest.

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