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WORKOUT FOR α-ψ-φ-CONTRACTIONS IN GENERALIZED TRIPLED METRIC SPACE WITH APPLICATION

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Abstract. In this paper, by using fixed point techniques, we establish some common fixed point theorems for mappings satisfying an α-ψ-φ-contractive condition in generalized tripled metric space. Finally, we give an example to illustrate our main outcome.

1. Introduction and Preliminaries

It is also known that common fixed point theorems are generalizations of fixed point theorems. Thus, over the past few decades, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. In this paper, we prove some common fixed point theorems for a larger class of α-ψ-φ-contractions in generalized tripled metric spaces.

Definition 1.1. Let Y be a non-empty set and $d : Y \times Y \to \mathbb{R}^+$ be a mapping such that, for all $x, y \in Y$ and for all distinct points $u, v \in Y$ each of them different from x and y, we have

(1) $d(x, y) = 0$ if and only if $x = y$;
(*ii*) $d(x, y) = d(y, x)$;
(*iii*) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then (Y, d) is called a *generalized metric space* or shortly GMS.

Definition 1.2. Let (Y, d) be a GMS, $\{y_n\}$ be a sequence in Y and $y \in Y$. Then

(*i*) We say that $\{y_n\}$ is *GMS convergent* to y if and only if $d(y_n, y) \to 0$ as $n \to \infty$;

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(ii) We say that $\{y_n\}$ is a *GMS Cauchy sequence* if and only if for each $\varepsilon > 0$, there exists a natural number N such that $d(y_n, y_m) < \varepsilon$ for all $n > m \geq N$;

(iii) (Y, d) is called *GMS complete* if every GMS Cauchy sequence is GMS convergent in Y.

We denote by Ψ the set of function $\psi : [0, \infty) \to [0, \infty)$ satisfying the following hypothesis

(ψ_1) ψ is continuous and nondecreasing;

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

we denote by Φ the set of function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following hypothesis

(φ_1) φ is lower semi-continuous;

(φ_2) $\varphi(t) = 0$ if and only if $t = 0$.

Lakzian ans Samet established the following fixed point theorem involving a pair of altering distance functions in a generalized complete metric space.

Theorem 1.3 ([12]). *Let (Y, d) be a Hausdorff and complete GMS and let $T : Y \to Y$ be a self-mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

for all $x, y \in Y$, where $\psi \in \Psi$ and $\varphi : [0, \infty) \to [0, \infty)$ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Let (Y, d) be a non-empty set and $T, f : Y \to Y$. The mappings T and f are said to be *weakly compatible* if they commute at their concidence points such that $Tfx = fTx$. A point $y \in Y$ is called *point of the coincidence* of T and f, if there exists a point $x \in Y$ such that $y = Tx = fx$.

Theorem 1.4 ([13]). *Let (Y, d) be a Hausdorff GMS, and let T and f be a self-mappings on Y, such that $TY \subset fY$. Assume that (fY, d) is a complete GMS and that the following conditions holds:*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \varphi(d(fx, fy))$$

for all $x, y \in Y$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T and f have a unique point of coincidence in Y. Moreover if T and f are weakly compatible, then T and f have a unique common fixed point.

2. Main Results

In this section, first we define a GMS tripled metric space and we prove some common fixed point results for two self-mapping satisfying an α-φ-ψ-contraction condition.

Definition 2.1. Let Y be a non-empty set and $S : Y \times Y \times Y \to \mathbb{R}^+$ be a mapping such that for all $x, y, z \in Y$ and for all distinct $u, v, w \in Y$ each of them difference from x, y and z, such that, S satisfies on following conditions

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $S(x, y, z) = S(x, z, y) = S(z, y, x) = S(y, x, z) = S(z, x, y) = S(y, z, x)$;
(iii) $S(x, y, z) \leq S(x, u, u) + S(v, y, v) + S(w, w, z)$;
(v) for all $x, y, \in Y$, $S(x, x, y) = S(x, y, y)$.

Then (Y, S) is called a *generalized tripled metric space* (GTMS).

Definition 2.2. Let T, $f : Y \to Y$ and $\alpha : Y \times Y \times Y \to [0, \infty)$. The mapping T is f-α-admissible, if for all $x, y, z \in Y$, such that $\alpha(fx, fy, fz) \geq 1$, we have $\alpha(Tx, Ty, Tz) \geq 1$. If f is the identity mapping, then T is called *α-admissible.*

Definition 2.3. Let (Y, S) be a generalized tripled metric space and $\alpha : Y \times Y \times Y \to [0, \infty)$. Y is α-regular, if for every sequence $\{y_n\} \subset Y$ such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ and $S(x, x, x_n) \to 0$ or $S(x, x_n, x_n) \to 0$ as $n \to \infty$, then there exists a subsequence of $\{x_n\}$ such that $\alpha(x_{n_k}, x, x) \geq 1$ or $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ for all $n \in \mathbb{N}$.

Definition 2.4. Let (Y, S) be a GTMS, $\{y_n\}$ be a sequence in Y and $y \in Y$, then

(i) we define that $\{y_n\}$ is *GTMS convergent* to y, if and only if $S(y_n, y, y) \to 0$ or $S(y_n, y_n, y) \to 0$ as $n \to \infty$;
(ii) we define that $\{y_n\}$ is *GTMS Cauchy sequence* if and only if, for each $\varepsilon > 0$, there exists a natural number $n(\varepsilon)$ such that $S(y_n, y_m, y_m) < \varepsilon$, for all $n > m > n(\varepsilon)$;
(iii) (Y, S) is called *GTMS complete* if every GTMS-Cauchy sequence is GTMS convergent in Y.

Theorem 2.5. *Let (Y, S) be a GTMS and let T and f be self-mappings on Y such that $TY \subseteq fY$ and $\alpha : Y \times Y \times Y \to [0, \infty)$. Assume that (fY, S) is a complete GTMS and that the following condition holds*

$$\psi(\alpha(fx, fy, fz)S(Tx, Ty, Tz)) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right) \tag{2.1}$$

for all $x, y, z \in Y$*, where* $\psi \in \Psi$*,* $\varphi \in \Phi$ *and* $M(x, y, z) = \max\{S(fx, fy, fz)\},$

$$S(fx, Tx, Tx), S(fy, Ty, Ty), S(fz, Tz, Tz).$$

Let also that the following condition hold

(*i*) T *is* f*-*α*-admissible;*

(*ii*) *there exists* $x_0, x_1 \in Y$ *such that* $\alpha(fx_0, Tx_0, Tx_1) \geq 1$*;*

(*iii*) Y *is* α*-regular and for every sequence* $\{x_n\} \subseteq Y$ *such that* $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$*, we have* $\alpha(x_m, x_n, x_n) \geq 1$ *or* $\alpha(x_m, x_m, x_n) \geq 1$ *for all* $m, n \in \mathbb{N}$*;*

(*iv*) *either* $\alpha(fu, fv, fw) \geq 1$ *or* $\alpha(fv, fu, fw) \geq 1$ *or* $\alpha(fw, fv, fu) \geq 1$ *or* $\alpha(fu, fw, fv) \geq 1$ *whenever* $fu = Tu$*,* $fv = Tv$ *and* $fw = Tw$*.*

Then T *and* f *have a unique point of coincidence in* Y*. Moreover, if* T *and* f *are weakly compatible, then* T *and* f *have a unique common fixed point.*

Proof. Suppose $x_0 \in Y$ such that $\alpha(fx_0, Tx_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ and $\{y_n\}$ in Y by $y_n = fx_{n+1} = Tx_n$, $(n \in \mathbb{N} \cup \{0\})$. Moreover, we assume that if $y_n = Tx_n = Tx_{n+p} = y_{n+p}$, then we choose $x_{n+p+1} = x_{n+1}$. Since $TY \subseteq fY$. In particular, if $y_n = y_{n+1}$, then y_{n+1} is a point of coincidence of T and f, consequently, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. By condition (*ii*) we have $\alpha(fx_0, Tx_0, Tx_1) \geq 1$, thus $\alpha(fx_0, fx_1, fx_2) \geq 1$. Since by hypotheses T is f-α-admissible, we obtain $\alpha(Tx_0, Tx_1, Tx_2) = \alpha(fx_1, fx_2, fx_3) \geq 1$ and $\alpha(Tx_2, Tx_3, Tx_4) = \alpha(fx_3, fx_4, fx_5) \geq 1$. By induction, we get $\alpha(fx_n, fx_{n+1}, fx_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now by (2.1), we have

$$\begin{aligned} \psi\left(S(Tx_n, Tx_{n+1}, Tx_{n+1})\right) &\leq \psi\left(\alpha(fx_n, fx_{n+1}, fx_{n+1}) S(Tx_n, Tx_{n+1}, Tx_{n+1})\right) \\ &\leq \psi\left(M(x_n, x_{n+1}, x_{n+1})\right) - \varphi\left(M(x_n, x_{n+1}, x_{n+1})\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= \max\big\{S(fx_n, fx_{n+1}, fx_{n+1}), S(fx_n, Tx_n, Tx_n), \\ &\quad S(x_{n+1}, Tx_{n+1}, Tx_{n+1}), S(fx_{n+1}, Tx_{n+1}, Tx_{n+1})\big\} \\ &= \max\big\{S(y_{n-1}, y_n, y_n), S(y_{n-1}, y_n, y_n), \\ &\quad S(y_n, y_{n+1}, y_{n+1}), S(y_n, y_{n+1}, y_{n+1})\big\} \\ &= \max\big\{S(y_{n-1}, y_n, y_n), S(y_n, y_{n+1}, y_{n+1})\big\}. \end{aligned}$$

From (2.2) we get

$$\psi\left(S(y_n, y_{n+1}, y_{n+1})\right) \leq \psi\left(M(x_n, x_{n+1}, x_{n+1})\right) - \varphi\left(M(x_n, x_{n+1}, x_{n+1})\right). \tag{2.3}$$

If $\max\{S(y_{n-1}, y_n, y_n), S(y_n, y_{n+1}, y_{n+1})\} = S(y_n, y_{n+1}, y_{n+1})$, from (2.3), we have

$$\psi\left(S(y_n, y_{n+1}, y_{n+1})\right) \leq \psi\left(S(y_{n+1}, y_{n+1}, y_n)\right) - \varphi\left(S(y_{n+1}, y_{n+1}, y_n)\right) \tag{2.4}$$

and hence $S(y_{n+1}, y_{n+1}, y_n) = 0$ which is a contradiction. Thus

$$M(x_n, x_{n+1}, x_{n+1}) = S(y_{n-1}, y_n, y_n) > 0.$$

From (2.4) we get

$$\psi\left(S(y_n, y_{n+1}, y_{n+1})\right) \leq \psi\left(S(y_{n-1}, y_n, y_n)\right) - \varphi\left(S(y_{n-1}, y_n, y_n)\right) < \psi\left(S(y_{n-1}, y_n, y_n)\right).$$

Because ψ is nondecreasing, then

$$S(y_n, y_{n+1}, y_{n+1}) < S(y_{n-1}, y_n, y_n)$$

for all $n \in \mathbb{N}$. That is the sequence of nonnegative numbers $\{S(y_n, y_{n+1}, y_{n+1})\}$ is decreasing, Hence, it converges to a nonnegative number, say $t \geq 0$. If $t > 0$, then letting $n \to \infty$ in (2.4), we obtain $\psi(t) \leq \psi(t) - \varphi(t)$ which implies $t = 0$, that is $\lim_{n\to\infty} S(y_n, y_{n+1}, y_{n+1}) = 0$. Suppose that $y_n \neq y_m$ for all $m \neq n$ and prove that $\{y_n\}$ is a GTMS Cauchy sequence. First, we show that the sequence $\{S(y_n, y_{n+2}, y_{n+2})\}$ is bounded. Since $S(y_n, y_{n+1}, y_{n+1}) \to 0$ as $n \to \infty$, there exists $L > 0$ such that $S(y_n, y_{n+1}, y_{n+1}) \leq L$, for all $n \in \mathbb{N}$. If $S(y_n, y_{n+2}, y_{n+2}) > L$, for all $n \in \mathbb{N}$, from

$$\begin{aligned} M(x_n, x_{n+2}, x_{n+2}) &= \max\big\{S(fx_n, fx_{n+2}, fx_{n+2}), S(fx_n, Tx_n, Tx_n), \\ &\qquad S(fx_{n+2}, Tx_{n+2}, Tx_{n+2}), S(fx_{n+2}, Tx_{n+2}, Tx_{n+2})\big\} \\ &= \max\big\{S(y_{n-1}, y_{n+1}, y_{n+1}), S(y_{n-1}, y_n, y_n), S(y_{n+1}, y_{n+2}, y_{n+2})\big\}. \end{aligned}$$

Because

$$S(y_{n+1}, y_{n+2}, y_{n+2}) < S(y_n, y_{n+1}, y_{n+1}), \text{ and } S(y_n, y_{n+1}, y_{n+1}) < S(y_{n-1}, y_n, y_n),$$

thus we have

$$M(x_n, x_{n+2}, x_{n+2}) = \max\big\{S(y_{n-1}, y_n, y_n), S(y_{n-1}, y_{n+1}, y_{n+1})\big\}$$

and by (2.1) we conclude that

(2.5)

$$\begin{aligned} \psi\left(S(y_n, y_{n+2}, y_{n+2})\right) &= \psi\left(S(Tx_n, Tx_{n+2}, Tx_{n+2})\right) \\ &\leq \psi\left(\alpha(fx_n, fx_{n+2}, fx_{n+2})\right) - \varphi(S(Tx_n, Tx_{n+2}, Tx_{n+2})) \\ &\leq \psi\left(M(x_n, x_{n+2}, x_{n+2})\right) - \varphi\left(M(x_n, x_{n+2}, x_{n+2})\right) \\ &< \psi\left(M(x_n, x_{n+2}, x_{n+2})\right). \end{aligned}$$

If $M(x_n, x_{n+2}, x_{n+2}) = S(y_{n-1}, y_{n+1}, y_{n+1})$ from (2.5), we have

$$\psi\left(S(y_n, y_{n+2}, y_{n+2})\right) < \psi\left(S(y_{n-1}, y_{n+1}, y_{n+1})\right).$$

Therefore the sequence $\{S(y_n, y_{n+2}, y_{n+2})\}$ is decreasing and hence is bounded. If

$$M(x_n, x_{n+2}, x_{n+2}) = S(y_{n-1}, y_n, y_n)$$

from (2.5), we have $\psi\left(S(y_n, y_{n+2}, y_{n+2})\right) < \psi\left(S(y_{n-1}, y_n, y_n)\right)$. Since ψ is nondecreasing, thus we have

$$S(y_n, y_{n+2}, y_{n+2}) < S(y_{n-1}, y_n, y_n) < \cdots < S(y_1, y_2, y_2).$$

and the sequence $\{S(y_n, y_{n+2}, y_{n+2})\}$ is bounded. If for some $n \in \mathbb{N}$, we have $S(y_{n-1}, y_{n+1}, y_{n+1}) \leq L$ and $S(y_n, y_{n+2}, y_{n+2}) > L$, then

$$\begin{aligned}\psi\left(S(y_n, y_{n+2}, y_{n+2})\right) &= \psi\left(S(Tx_n, Tx_{n+2}, Tx_{n+2})\right)\\ &\leq \psi\left(\alpha(fx_n, fx_{n+2}, fx_{n+2})\right) S(Tx_n, Tx_{n+2}, Tx_{n+2})\\ &\leq \psi\left(M(x_n, x_{n+2}, x_{n+2})\right) - \varphi\left(M(x_n, x_{n+2}, x_{n+2})\right)\\ &< \psi\left(M(x_n, x_{n+2}, x_{n+2})\right).\end{aligned}$$

Now, if

$$M(x_n, x_{n+2}, x_{n+2}) = S(y_{n-1}, y_{n+1}, y_{n+1}) \leq L,$$

we obtain $S(y_n, y_{n+2}, y_{n+2}) < L$, a contradiction. If

$$M(x_n, x_{n+2}, x_{n+2}) = S(y_{n-1}, y_n, y_n) \leq L,$$

we obtain $\psi\left(S(y_n, y_{n+2}, y_{n+2})\right) < \psi(L)$ and $S(y_n, y_{n+2}, y_{n+2}) < L$, a contradiction. Then $S(y_n, y_{n+2}, y_{n+2}) > L$ or $S(y_n, y_{n+2}, y_{n+2}) \leq L$ for all $n \in \mathbb{N}$ and in both cases the sequence $\{S(y_n, y_{n+2}, y_{n+2})\}$ is bounded. Now, if

$$\lim_{n\to\infty} S(y_n, y_{n+2}, y_{n+2}) = 0 \tag{2.6}$$

does not satisfied, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$S(y_{n_k}, y_{n_{k+2}}, y_{n_{k+2}}) \to t > 0 \text{ as } k \to \infty.$$

From

$$\begin{aligned}S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}) &\le S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) + S(y_{n_{k+1}}, y_{n_{k+2}}, y_{n_{k+2}}) \\ &\quad + S(y_{n_{k+1}}, y_{n_{k+3}}, y_{n_{k+3}}) \\ &\le S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) + S(y_{n_{k+1}}, y_{n_{k+2}}, y_{n_{k+2}}) \\ &\quad + S(y_{n_{k+1}}, y_{n_{k-1}}, y_{n_{k-1}}) + S(y_{n_{k+3}}, y_{n_{k+2}}, y_{n_{k+2}}) \\ &\quad + S(y_{n_{k+3}}, y_{n_{k+4}}, y_{n_{k+4}}).\end{aligned}$$

If $k \to \infty$ we have

$$\begin{aligned}\lim_{k\to\infty} S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}) &\le 0 + 0 + \lim_{k\to\infty} S(y_{n_{k+1}}, y_{n_{k+3}}, y_{n_{k+3}}) \\ &\le 0 + 0 + \lim_{k\to\infty} S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}) + 0 + 0 \\ &= \lim_{k\to\infty} S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}).\end{aligned}$$

We obtain that $\lim_{k\to\infty} S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}) = t$. By (2.1) with $x = x_{n_k}$ and $y = x_{n_{k+2}}$, we have

(2.7)
$$\begin{aligned}\psi\left(S(Tx_{n_k}, Tx_{n_{k+2}}, Tx_{n_{k+2}})\right) &\le \psi\left(\alpha(fx_{n_k}, fx_{n_{k+2}}, fx_{n_{k+2}})\right) S(Tx_{n_k}, Tx_{n_{k+2}}, Tx_{n_{k+2}}) \\ &\le \psi\left(M(x_{n_k}, x_{n_{k+2}}, x_{n_{k+2}})\right) - \varphi\left(M(x_{n_k}, x_{n_{k+2}}, x_{n_{k+2}})\right)\end{aligned}$$

where

$$\begin{aligned}M(x_{n_k}, x_{n_{k+2}}, x_{n_{k+2}}) &= \max\big\{S(fx_{n_k}, fx_{n_{k+2}}, fx_{n_{k+2}}), \\ &\quad S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}), S(fx_{n_{k+2}}, Tx_{n_{k+2}}, Tx_{n_{k+2}})\big\} \\ &= \max\big\{S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}), S(y_{n_{k-1}}, y_{n_k}, y_{n_k}), \\ &\quad S(y_{n_{k+1}}, y_{n_{k+2}}, y_{n_{k+2}})\big\}.\end{aligned}$$

This implies $\lim_{k\to\infty} M(x_{n_k}, x_{n_{k+2}}, x_{n_{k+2}}) = t$. From (2.7) as $k \to \infty$, we get $\psi(t) \le \psi(t) - \varphi(t)$, which implies $t = 0$. Now, if possible, let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\varepsilon > 0$ fro which we can find subsequence $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k > k$ such that

$$S(y_{m_k}, y_{n_k}, y_{n_k}) \ge \varepsilon, \tag{2.8}$$

where corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k - m_k \ge 3$ and satisfying (2.8). Then

$$S(y_{m_k}, y_{m_{k-1}}, y_{m_{k-1}}) < \varepsilon. \tag{2.9}$$

By using (2.8), (2.9) and the rectangular inequality, we get

$$
\begin{aligned}
\varepsilon &\leq S(y_{n_k}, y_{n_k}, y_{m_k}) \\
&\leq S(y_{n_k}, y_{n_{k-2}}, y_{n_{k-2}}) + S(y_{n_k}, y_{n_{k-1}}, y_{n_{k-1}}) + S(y_{m_k}, y_{n_{k-3}}, y_{n_{k-3}}) \\
&< S(y_{n_k}, y_{n_{k-2}}, y_{n_{k-2}}) + S(y_{m_k}, y_{n_{k-3}}, y_{n_{k-3}}) + \varepsilon \\
&< S(y_{n_k}, y_{n_{k-2}}, y_{n_{k-2}}) + \varepsilon + S(y_{m_k}, y_{n_k}, y_{n_k}) \\
&\quad + S(y_{n_{k-3}}, y_{n_{k-2}}, y_{n_{k-2}}) + S(y_{n_{k-3}}, y_{n_{k-1}}, y_{n_{k-1}}) \\
&< \varepsilon + \varepsilon + S(y_{m_k}, y_{n_k}, y_{n_k}) + \varepsilon + S(y_{n_{k-3}}, y_{n_{k-1}}, y_{n_{k-1}}) \\
&< \varepsilon + \varepsilon + S(y_{m_k}, y_{n_k}, y_{n_k}) + \varepsilon + S(y_{n_{k-3}}, y_{n_{k-2}}, y_{n_{k-2}}) \\
&\quad + S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) + S(y_{n_{k-1}}, y_{n_{k+1}}, y_{n_{k+1}}).
\end{aligned} \tag{2.10}
$$

Letting $k \to \infty$ in the above inequality, using (2.4) and (2.5), we obtain

$$
S(y_{m_k}, y_{n_k}, y_{n_k}) \to \varepsilon^+. \tag{2.11}
$$

From

$$
\begin{aligned}
S(y_{m_k}, y_{n_k}, y_{n_k}) &- S(y_{m_{k-1}}, y_{m_k}, y_{m_k}) - S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) \\
&\leq S(y_{n_{k-1}}, y_{m_{k-1}}, y_{m_{k-1}}) \\
&\leq S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) + S(y_{m_{k-1}}, y_{m_k}, y_{m_k}) + S(y_{m_{k-1}}, y_{m_{k+1}}, y_{m_{k+1}}).
\end{aligned}
$$

Letting $k \to \infty$, we get

$$
S(y_{n_{k-1}}, y_{m_{k-1}}, y_{m_{k-1}}) \to \varepsilon^+. \tag{2.12}
$$

From (2.1) with $x = x_{n_k}$ and $y = x_{n_k}$m we obtain

$$
\begin{aligned}
\psi\left(S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k})\right) &\leq \psi\left(\alpha(fx_{m_k}, fx_{n_k}, fx_{n_k})\right) S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) \\
&\leq \psi\left(M(fx_{m_k}, fx_{n_k}, fx_{n_k})\right) - \varphi\left(M(fx_{m_k}, fx_{n_k}, fx_{n_k})\right).
\end{aligned}
$$

So

$$
\begin{aligned}
&M(fx_{m_k}, fx_{n_k}, fx_{n_k}) \\
&= \max\left\{S(fx_{m_k}, fx_{n_k}, fx_{n_k}), S(fx_{m_k}, Tx_{m_k}, Tx_{m_k}), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k})\right\} \\
&= \max\left\{S(y_{m_{k-1}}, y_{n_{k-1}}, y_{n_{k-1}}), S(y_{m_{k-1}}, y_{m_k}, y_{m_k}), S(y_{n_{k-1}}, y_{n_k}, y_{n_k})\right\} \\
&= \max\left\{S(y_{m_{k-1}}, y_{n_{k-1}}, y_{n_{k-1}}), S(y_{n_{k-1}}, y_{n_k}, y_{n_k}), S(y_{m_{k-1}}, y_{m_k}, y_{m_k})\right\}.
\end{aligned}
$$

By using the continuity of ψ and the lower semi-continuous of φ as $k \to \infty$, we obtain $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon)$ which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Hence $\{y_n\}$ is a GTMS Cauchy sequence. Since (fY, S) is GTMS complete, there exists $z \in fY$ such that $y_n \to z$. Let $y \in Y$ be such that $fy = z$. Since Y is

α-regular there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_{k-1}}, fy, fy) \geq 1$ for all $k \in \mathbb{N}$. If $fy \neq Ty$, applying (2.1) with $x = x_{n_k}$, we get

$$\begin{aligned}\psi\left(S(Tx_{n_k}, Ty, Ty)\right) &\leq \psi\left(\alpha(fx_{n_k}, fy, fy)S(Tx_{n_k}, Ty, Ty)\right)\\ &\leq \psi\left(M(fx_{n_k}, fy, fy)\right) - \varphi\left(M(fx_{n_k}, fy, fy)\right),\end{aligned}$$

where

$$\begin{aligned}M(fx_{n_k}, f_y, f_y) &= \max\left\{S(fx_{n_k}, fy, fy), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}), S(fy, Ty, Ty)\right\}\\ &= \max\left\{S(y_{n_{k-1}}, fy, fy), S(y_{n_{k-1}}, y_{n_k}, y_{n_k}), S(fy, Ty, Ty)\right\}.\end{aligned}$$

From $S(y_{n_{k-1}}, fy, fy)$, $S(y_{n_{k-1}}, y_{n_k}, y_{n_k}) \to 0$ as $k \to \infty$ for k great enough, we deduce $M(fx_{n_k}, fy, fy) = S(fy, Ty, Ty)$. On the other hand

$$\begin{aligned}S(Ty, fy, fy) &= S(fy, Ty, Ty)\\ &\leq S(fy, y_{n_{k-1}}, y_{n_{k-1}}) + S(Ty, y_{n_k}, y_{n_k}) + S(Ty, y_{n_{k+1}}, y_{n_{k+1}})\end{aligned}$$

implies

$$S(fy, Ty, Ty) \leq \liminf_{k\to\infty} S(Ty, Tx_{n_{k+1}}, Tx_{n_{k+1}}).$$

Because ψ is continuous and non-decreasing, for k great enough, we get

$$\begin{aligned}\psi\left(S(Ty, fy, fy)\right) &\leq \liminf_{k\to\infty} \psi\left(S(Ty, Tx_{n_{k+1}}, Tx_{n_{k+1}})\right)\\ &\leq \psi\left(S(fy, fy, Ty)\right) - \varphi\left(S(fy, fy, Ty)\right)\end{aligned}$$

which implies $S(fy, fy, Ty) = 0$, that is $fy = Ty = z$ and so z is a point of coincidence for T and f. Suppose that there exist $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$, We prove that $p = 1$, then $fx_{n+1} = Tx_n = Tx_{n+1} = y_{n+1}$ and so y_{n+1} is a point of coincidence of T and f. Let $p > 1$, this implies that $S(y_{n+p-1}, y_{n+p}, y_{n+p}) > 0$. using (2.3), we obtain

$$\begin{aligned}\psi\left(S(y_n, y_{n+1}, y_{n+1})\right) &= \psi\left(S(y_{n+p}, y_{n+p+1}, y_{n+p+1})\right)\\ &\leq \psi\left(S(y_{n+p-1}, y_{n+p}, y_{n+p})\right) - \varphi\left(S(y_{n+p-1}, y_{n+p}, y_{n+p})\right)\\ &< \psi\left(S(y_{n+p-1}, y_{n+p}, y_{n+p})\right).\end{aligned}$$

Because the sequence $\{S(y_n, y_{n+1}, y_{n+1})\}$ is decreasing, we deduce

$$\psi\left(d(y_n, y_{n+1}, y_{n+1})\right) < \psi\left(S(y_n, y_{n+1}, y_{n+1})\right),$$

a contradiction and hence $p = 1$. We deduce that T and f have a point of coincidence. The uniqueness of the point coincidence is a consequence of conditions (2.1), (iv) and so we omit the details. Now, if z is the point of coincidence of T and f as

T and f are weakly compatible. We deduce that $fz = Tz$ and so $z = fz = Tz$. Consequently, z is the unique common fixed point of T and f. □

If we choose $f = I_Y$ the identity mapping on Y, we deduce the following corollary.

Corollary 2.6. *Let (Y, S) be a complete GTMS. Let T be a self-mapping on Y, and $\alpha : Y^3 \to [0, \infty)$. Assume that the following condition holds:*

$$\psi\left(\alpha(x, y, z)S(Tx, Ty, Tz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right)$$

for all $x, y, z \in Y$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y, z) = \max\left\{S(x, y, z), S(x, Tx, Tx), S(y, Ty, Ty), S(z, Tz, Tz)\right\}.$$

Assume also that the following condition hold:

- *(i) T is α-admissible;*
- *(ii) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$;*
- *(iii) Y is α-regular, for every sequence $\{x_n\} \subset Y$ such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$, we have $\alpha(x_m, x_n, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*
- *(iv) either $\alpha(u, v, w) \geq 1$ or $\alpha(w, v, u) \geq 1$ or $\alpha(u, w, v) \geq 1$ or $\alpha(v, u, w) \geq 1$ whenever $u = Tu$, $v = Tv$ and $w = Tw$. Then T has a unique fixed point.*

From Theorem 1.3, if the function $\alpha : Y \times Y \times Y \to [0, \infty)$ is such that $\alpha(x, y, z) = 1$ for all $x, y, z \in Y$, we deduce the following theorem.

Theorem 2.7. *Let (Y, S) be a GTMS and let T and f be self-mapping on X, such that $TY \subseteq fY$. Assume that (fY, S) is a complete GTMS and that the following condition holds:*

$$\psi\left(S(Tx, Ty, Tz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right)$$

for all $x, y, z \in Y$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y, z) = \max\left\{S(fx, fy, fz), S(fx, Tx, Tx), S(fy, Ty, Ty), S(fz, Tz, Tz)\right\}.$$

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

From Theorem 2.5 in the setting of partially ordered GTMS spaces, we get the follow theorem.

Theorem 2.8. *Let $(Y, S, \preceq)$ be a partially ordered GTMS and let T and f be a self-mappings on Y such that $TY \subseteq fY$. Assume that (fY, S) is a complete GTMS and that the following condition holds*

$$\psi\left(S(Tx, Ty, Tz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right),$$

for all $x, y, z \in Y$ such that $fx \preceq fy \preceq fz$, where $\psi \in \Psi$ and $\varphi \in \Phi$ with $\psi(t) - \varphi(t) \geq 0$, for all $t \geq 0$, and

$$M(x, y, z) = \max\left\{S(fx, fy, fz), S(fx, Tx, Tx), S(fy, Ty, Ty), S(fz, Tz, Tz)\right\}.$$

Assume also that the following conditions hold.

(i) *T is a f-nondecreasing;*
(ii) *there exists $x_0 \in Y$ such that $fx_0 \preceq Tx_0$;*
(iii) *if $\{x_n\} \subset Y$ is such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*
(iv) *for all $u, v \in Y$ such that $fu = Tu$ and $fv = Tv$ then fu and fv are comparable.*

Then T and f have a unique point of coincidence in Y. Moreover, if T and f are weakly comparable then T and f have a unique common fixed point.

Proof. Define the mapping $\alpha : Y \times Y \times Y \to [0, \infty)$ by

$$\alpha(x, y, z) = \begin{cases} 1, & x, y, z \in fY \text{ and } x \preceq y \preceq z, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Then we can verify easily that T is an f-α-admissible mapping. Let $\{x_n\}$ be a sequence in Y such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$, for all $n \in \mathbb{N}$ and $x_n \to x \in Y$ as $n \to \infty$. By the definition of α, we have $x_n, x_{n+1}, x_{n+2} \in fY$ and $x_n \preceq x_{n+1} \preceq x_{n+2}$ for all $n \in \mathbb{N}$. Since fY is complete, we deduce that $x \in fY$. By (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$, for all $k \in \mathbb{N}$, and so $\alpha(x_{n_k}, x, x) \geq 1$ and $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$ and so Y is α-regular. Moreover, $\alpha(x_m, x_n, x_n) \geq 1$ for all $m, n \in \mathbb{N}$, with $m < n$. Hence, (iii) of Theorem 2.5 holds. The same considerations show that (ii) and (iv) of this Theorem imply (ii) and (iv) of Theorem 2.5. Thus the hypothesis (i) - (iv) of Theorem 2.5 are satisfied. Also the contractive condition (2.2) is satisfied, since $\alpha(fx, fy, fz) = 1$ for all $x, y, z \in Y$ such that $fx \preceq fy \preceq fz$. Otherwise, $\psi\left(\alpha(fx, fy, fz)S(Tx, Ty, Tz)\right) = 0$ and so condition (2.1) holds. From Theorem 2.5, T and f have a unique common fixed point. □

From Theorem 2.5, we can derive many interesting fixed point results in GTMS. Denote by Γ the set of functions $\mu : [0,\infty) \to [0,\infty)$ lebesque integrable on each compact subset of $[0,\infty)$ such that, for rvery $\varepsilon > 0$, we have $\int_0^\varepsilon \mu(s)\,ds > 0$. As the function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \int_0^t \mu(s)\,ds$ belongs to Ψ, we obtain the following theorem.

Theorem 2.9. *Let (Y,S) be a GTMS and let T and f be self-mappings on Y such that $TY \subseteq fY$ and $\alpha : Y \times Y \times Y \to [0,\infty)$. Assume that (fY,S) is a complete GTMS and that the following condition holds*

$$\int_0^{\alpha(fx,fy,fz)S(Tx,Ty,Tz)} \mu(s)\,ds \leq \int_0^{M(x,y,z)} \mu(s)\,ds - \int_0^{M(x,y,z)} \mu(s)\,ds,$$

for all $x,y,z \in Y$, where $\lambda, \delta \in \Gamma$ and

$$M(x,y,z) = \max\left\{S(fx,fy,fz), S(fx,Tx,Tx), S(fy,Ty,Ty), S(fz,Tz,Tz)\right\}.$$

Assume also that the following conditions hold.

(i) *T is a f-α-nondecreasing;*

(ii) *there exists $x_0 \in Y$ such that $\alpha(fx_0,Tx_0,Tx_0) \geq 1$;*

(iii) *Y is α-rgular and for every sequence $\{x_n\} \subset Y$, $\alpha(x_n,x_{n+1},x_{n+2}) \geq 1$, we have $\alpha(x_m,x_n,x_n) \geq 1$ for all $m,n \in \mathbb{N}$ with $m < n$;*

(iv) *either $\alpha(fu,fv,fw) \geq 1$ or $\alpha(fv,fu,fw) \geq 1$, or $\alpha(fw,fv,fu) \geq 1$, or $\alpha(fu,fw,fv) \geq 1$, whenever $fu = Tu$, $fv = Tv$, and $fw = Tw$.*

Then T and f have a unique point of coincidence in Y. Moreover, if T and f are weakly compatible then T and f have a unique common fixed point.

Taking $\delta(s) = (1-k)\lambda(s)$ for $k \in [0,1)$ in Theorem 2.9, we obtain the following result.

Theorem 2.10. *Let (Y,S) be a GTMS and let T and f be self-mappings on Y such that $TY \subseteq fY$ and $\alpha : Y \times Y \times Y \to [0,\infty)$. Assume that (fY,S) is a complete GTMS and that the following condition hold.*

$$\int_0^{\alpha(fx,fy,fz)S(Tx,Ty,Tz)} \mu(s)\,ds \leq k \int_0^{M(x,y,z)} \mu(s)\,ds,$$

for all $x,y,z \in Y$, where $k \in [0,1)$. Let also thet the following condiotons hold.

(i) *T is a f-α-admissibble;*

(ii) *there exists $x_0 \in Y$ such that $\alpha(fx_0,Tx_0,Tx_0) \geq 1$;*

(iii) *Y is α-regular and for every sequence $\{x_n\} \subset Y$ such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$, we have $\alpha(x_m, x_n, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*

(iv) *either $\alpha(fu, fv, fw) \geq 1$ or $\alpha(fv, fu, fw) \geq 1$, or $\alpha(fw, fv, fu) \geq 1$, or $\alpha(fu, fw, fv) \geq 1$, whenever $fu = Tu$, $fv = Tv$, and $fw = Tw$.*

Then T and f have a unique point of coincidence in Y. Moreover, if T and f are weakly compatible then T and f have a unique common fixed point.

Example 2.11. Let $Y = [0, 1]$ and $B = \{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$. We define the GTMS, S on Y as follows.

$$S\left(\frac{1}{4}, \frac{1}{7}, \frac{1}{6}\right) = \frac{3}{7}, \qquad S\left(\frac{1}{4}, \frac{1}{7}, \frac{1}{7}\right) = \frac{8}{7},$$
$$S\left(\frac{1}{7}, \frac{1}{8}, \frac{1}{11}\right) = \frac{1}{11}, \qquad S\left(\frac{1}{7}, \frac{1}{8}, \frac{1}{11}\right) = \frac{1}{7},$$
$$S\left(\frac{1}{7}, \frac{1}{9}, \frac{1}{9}\right) = \frac{8}{7}, \qquad S\left(\frac{1}{8}, \frac{1}{10}, \frac{1}{10}\right) = \frac{3}{7},$$
$$S\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{7}\right) = \frac{8}{7}, \qquad S\left(\frac{1}{10}, \frac{1}{8}, \frac{1}{8}\right) = \frac{3}{7},$$
$$S\left(\frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) = \frac{4}{7}, \qquad S\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{9}\right) = \frac{8}{7},$$
$$S\left(\frac{1}{6}, \frac{1}{10}, \frac{1}{10}\right) = \frac{3}{7}, \qquad S\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{10}\right) = \frac{3}{7},$$
$$S\left(\frac{1}{11}, \frac{1}{5}, \frac{1}{5}\right) = \frac{3}{7}, \qquad S\left(\frac{1}{5}, \frac{1}{7}, \frac{1}{7}\right) = \frac{4}{7},$$
$$S\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{7}\right) = \frac{4}{7}, \qquad S\left(\frac{1}{11}, \frac{1}{10}, \frac{1}{10}\right) = \frac{2}{7},$$
$$S\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right) = \frac{4}{7}.$$

We have $S(x, y, z) = |x - y| + |y - z|$. (Y, S) is a complete GTMS. Let $T : Y \to Y$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ be defined by $T(x) = \frac{1}{6}$, if $x \in B$ else $T(x) = 1 - x$, $\varphi(t) = \frac{t}{7}$ and $\psi(t) = t$. Finally, $\alpha : Y \times Y \times Y \to [0, \infty)$ given by $\alpha(x, y, z) = 1$ when $x, y, z \in B$ or $x = y = z$, otherwise $\alpha(x, y, z) = 0$. We have

$$\psi\left(\alpha(x, y, z)S(Tx, Ty, Tz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right),$$

where

$$M(x, y, z) = \max\left\{S(x, y, z), S(x, Tx, Tx), S(y, Ty, Ty), S(z, Tz, Tz)\right\}.$$

We give a few cases.
CASE I. If $x = \frac{1}{4}$, $y = \frac{1}{5}$ and $z = \frac{1}{6}$. Then $\alpha(x,y,z) = 1$, $\psi(1) = 1$, $S\left(T\frac{1}{4}, T\frac{1}{5}, T\frac{1}{6}\right) = 0$.
CASE II. If $x = \frac{1}{4}$, $y = \frac{1}{7}$ and $z = \frac{1}{10}$, we have

$$S\left(\frac{1}{6}, \frac{1}{6}, \frac{9}{10}\right) = \left|\frac{1}{6} - \frac{9}{10}\right| + \left|\frac{1}{6} - \frac{1}{6}\right| = \frac{11}{15},$$

$$M\left(\frac{1}{4}, \frac{1}{7}, \frac{1}{10}\right) = \max\left\{\frac{11}{15}, \frac{6}{7}, \frac{6}{7}, \frac{4}{5}\right\} = \frac{6}{7},$$

$\psi(\frac{6}{7}) = \frac{6}{7}$ and $\varphi(\frac{6}{7}) = \frac{6}{49}$.
CASE III. If $x = \frac{1}{6}$, $y = \frac{1}{7}$ and $z = \frac{1}{8}$, we have

$$S\left(T\frac{1}{6}, T\frac{1}{7}, T\frac{1}{8}\right) = S\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) = 0.$$

CASE IV. If $x = \frac{1}{7}$, $y = \frac{1}{7}$ and $z = \frac{1}{7}$, we have

$$S\left(T\frac{1}{y}, T\frac{1}{7}, T\frac{1}{y}\right) = S\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) = 0.$$

CASE V. If $x = \frac{1}{10}$, $y = \frac{1}{11}$ and $z = \frac{1}{12}$, we have

$$S\left(T\frac{1}{10}, T\frac{1}{11}, T\frac{1}{12}\right) = S\left(\frac{9}{10}, \frac{10}{11}, \frac{11}{12}\right) = \frac{242}{14520},$$

and

$$\begin{aligned} M\left(\frac{1}{10}, \frac{1}{11}, \frac{1}{12}\right) &= \max\left\{S\left(\frac{1}{10}, \frac{1}{11}, \frac{1}{12}\right), S\left(\frac{1}{10}, \frac{9}{10}, \frac{9}{10}\right),\right. \\ &\qquad \left. S\left(\frac{1}{11}, \frac{10}{11}, \frac{10}{11}\right), S\left(\frac{1}{12}, \frac{11}{12}, \frac{11}{12}\right)\right\} \\ &= \left\{\frac{242}{14520}, \frac{4}{5}, \frac{9}{11}, \frac{5}{6}\right\} = \frac{5}{6}. \end{aligned}$$

Then T and α satisfy all the condition of Corollary 2.6 and hence T has a unique fixed point on Y that is $x = \frac{1}{6}$.

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Data sharing is not applicable to this article as obviously no data sets were generated or analyzed during the current study.

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References

1. A. Branciari: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Publ. Math. Debrecen, 57:3137, 2000. http//dx.doi.org/10.12988/imf.2014.4227
2. J. Ahmad, M. Arshad & C. Vetro: On a theorem of Khan in a generalized metric space. International Journal of Analysis, 2013, Article ID 852727, 6 pp. 2013. https//doi.org/10.1155/2013/852727
3. A. Azam & M. Arshad: Kannan fixed point theorem on generalized metric spaces. J. Nonlinear Sci. Appl., 1:4548, 2008. http//dx.doi.org/10.22436/jnsa.001.01.07
4. P. Das: A fixed point theorem in a generalized metric space. Soochow J. Math. 33:3339, 2007. https//doi.org/10.1515/dema-2013-0432
5. P. Das & B.K. Lahiri: Fixed point of contractive mappings in generalized metric spaces. Math. Slovaca, 59:499504, 2009. https//doi.org/10.2478/s12175-009-0143-2
6. A. Fora, A. Bellour & A. Al-Bsoul: Some results in fixed point theory concerning generalized metric spaces. Mat. Vesnik, 61:203208, 2009. http//doi.org/10.3934/math.2021106
7. B.K. Lahiri & P. Das: Fixed point of a Ljubomir C irics quasi-contraction mapping in a generalized metric space. Publ. Math. Debrecen, 61:589594, 2002. http//doi.org/10.5486/pmd.2002.2677
8. D. Mihet: On Kannan fixed point principle in generalized metric spaces. J. Nonlinear Sci. Appl., 2:9296, 2009. http//dx.doi.org/10.22436/jnsa.002.02.03
9. B. Samet: A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type. Int. J. Math. Anal. 3:12651271, 2009. http//doi.org/10.1155/s0161171202007524
10. B. Samet: Discussion on: A fixed point theorem of Banach Caccioppoli-type on a class of generalized metric spaces by A. Branciari, Publ. Math. Debrecen, 76:493494, 2010. http//doi.org/10.5486/pmd.2010.4595
11. I.R. Sarma, J.M. Rao & S.S. Rao: Contractions over generalized metric spaces. J. Nonlinear Sci. Appl., 2:180182, 2009. http//doi.org/10.22436/jnsa.002.03.06
12. H. Lakzian & B. Samet: Fixed point for (ψ, φ)-weakly contractive mappings in generalized metric spaces. Appl. Math. Lett. 25:902906, 2012. http//doi.org/10.1016/j.aml.2011.10.047
13. C. Di Bari & P. Vetro: Common fixed points in generalized metric spaces. Appl. Math. Comput. 218:73227325, 2012. http//doi.org/10.16/j.amc.2012.01.010
14. B. Samet, C. Vetro & P. Vetro: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., 75:21542165, 2012. http//doi.org/10.1016/j.na.2011.10.014

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