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# EXISTENCE OF SELECTION MAP AND THE RELATED FIXED POINT RESULTS ON HYPERCONVEX PRODUCT SPACES

A. Herminau Jothy [a,*] P. S. Srinivasan [b] and R. Theivaraman [c]

Abstract. The main aim of this article is to present new fixed point results concerning existence of selection for a multivalued map on hyperconvex product space taking values on bounded, externally hyperconvex subsets under some appropriate hypothesis. Our results are significant extensions of some pioneering results in the literature, in particular M. A. Khamsi, W. A. Krik and Carlos Martinez Yanez, have proved the existence of single valued selection of a lipschitzian multi-valued map on hyperconvex space. Some suitable examples are also given to support and understand the applicability of our results.

## 1. Introduction

Since the early 19th century, many researchers have studied various extensions of Banach contraction principle ($BCP$) and have shown their existence and uniqueness of fixed points for many problems, followed by recent development in finding approximate fixed points (see [24] and [25]). But here we study the existence of selection for multivalued maps on hyperconvex product spaces without using the proof of $BCP$. Originally, in 1956, author Micheal [3] proposed the concept of selection maps. Further, this concept extended by Henry [6] in 1971 and it has more advantages for solving the generalized differential equations and played a major role in differential inclusions. Later, there were many articles which proposed and generalized selection theorem with reduced assumptions, see [9], [10] and [26]. In 2000, M. A. Khamsi et al. [11], proved selection theorem on hyperconvex spaces.

The notion of hyperconvexity was first introduced by N. Aronszajn, and P. Panitchpakdi in [17], they proved that a hyperconvex metric space can be seen as a

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*Corresponding author.

non-expansive retract of any metric space in which it is isometrically embedded. This became a major interests among researches then these facts are used to prove many fixed point results such as in [12], [16], [18] and [27]. In this regard, complete $\mathbb{R}$-trees hyperconvex hulls are uniquely determined by Marcin Borkowski et al. [13]. Also, they have proved hyperconvexity of subsets of normed spaces implies their convexity if and only if the space under consideration is strictly convex. In addition to that, they have proposed a Krein-Milman type theorem for $\mathbb{R}$-trees. Recently, Reny George et al. [20] presented further extensions of the best approximation theorem in hyper convex spaces obtained by Khamsi. Further, some theorems related to common fixed points of set-valued mappings in hyperconvex metric spaces was derived by M. Balaji et al. [14], (refer also [1], [4], [5]). In the similar way, Jack Markin et al. [8], proved the existence of common fixed points for commuting and non-expansive mappings on hyperconvex spaces. Many authors have also proved the existence of approximate fixed points in hyperconvex spaces and general spaces as in [22] and [23]. The authors M. A. Khamsi et al [11], proved the existence of fixed point for a non expansive mapping on a bounded hyperconvex space and have also proved the existence of selection for a non-expansive multivalued map taking values in bounded externally hyperconvex subsets. Inspired by all the above observations, in this paper, we extend the result and find a suitable conditions for the existence of fixed point. Furthermore, we prove the existence of selection for multivalued maps on product space taking values in bounded externally hyperconvex subsets.

This manuscript is structured as follows: Section 1 is, naturally, the introductory part. In section 2, we recall the notations, basic notions, essential definitions, and lemmas needed throughout the paper. In section 3, we prove the existence of selection maps and demonstrate the fixed-point results for some special kind of map on hyperconvex spaces. In section 4, we provide some application to support our findings. In section 5, we reach a conclusion.

## 2. Preliminaries

In this section, we revisit some preliminary definitions, lemmas, and results from previous literature related to our main findings. Additionally, we present specific terminology and concepts that will be employed throughout the remainder of this paper.

**Definition 2.1** ([18])**.** A metric space $M$ is said to be *hyperconvex* if $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$ for any collection of points $\{x_\alpha\}_{\alpha\in\Gamma}$ in $M$ and positive real number $\{r_\alpha\}_{\alpha\in\Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any $\alpha$ and $\beta$ in $\Gamma$.

**Definition 2.2** ([21])**.** A subset $N$ of a metric space $M$ is said to be *externally hyperconvex* (relative to $M$) if $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$ for any family of points $\{x_\alpha\}_{\alpha\in\Gamma}$ in $M$ and positive real number $\{r_\alpha\}_{\alpha\in\Gamma}$ satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \quad \text{and} \quad dist(x_\alpha, N) \leq r_\alpha$$

**Remark 2.3.** If $M$ is externally hyperconvex subset of $H$. Then for any given $x \in H$ there exists $m \in M$ such that

$$d(x, m) = dist(x, M),$$

where

$$dist(x, M) := \inf\{d(x, y) : y \in M\}.$$

**Proposition 2.4** ([15])**.** Any hyperconvex metric space is complete.

**Definition 2.5** ([21])**.** Let $Y$ be a metric space a subset $E$ of $Y$ is said to be *proximinal* with respect to $Y$ if $E \cap B(y, dist(y, E)) \neq \emptyset$, for all $y \in Y$.

**Proposition 2.6.** Let $(H_1, d_1)$ and $(H_2, d_2)$ be hyperconvex metric space and let $H = H_1 \times H_2$, then $(H, d_\infty)$ is also an hyperconvex metric space where $d_\infty$ is defined by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

*Proof.* Let $\{(x_\alpha, y_\alpha)\}$ of points in $H$ and a family $\{r_\alpha\}$ of positive real number with $d_\infty((x_\alpha, y_\alpha), (x_\beta, y_\beta)) \leq r_\alpha + r_\beta$. The proof follows from the fact that

$$B_{d_\infty}((x_\alpha, y_\alpha), r_\alpha) = B_{d_1}(x_\alpha, r_\alpha) \times B_{d_2}(y_\alpha, r_\alpha).$$

□

Let $M$ be a metric space and $A$ a subset of $M$ then the $\epsilon$ neighborhood of $A$ is given by

$$N_\epsilon(A) := \{x \in M : dist(x, A) \leq \epsilon\}$$

and let

$$L_\epsilon(A) := \bigcup_{a\in A} B((a, \epsilon)).$$

We use $\mathcal{P}(M)$ to denote the collection of all non-empty subsets of $M$ and $\mathcal{P}_{\mathcal{B},\mathcal{E}}(M)$ as the set of all subsets of $M$ which are non-empty, bounded and externally hyperconvex. That is,

$$\mathcal{P}(M) := \{A \subseteq M : A \neq \emptyset\}$$

and

$$\mathcal{P}_{\mathcal{B},\mathcal{E}}(M) = \{A \in \mathcal{P}(M) : A \text{ is bounded and externally hyperconvex}\}.$$

For any $A, B \in \mathcal{P}(M)$, let

$$\delta(A, B) := \sup\{dist(a, B) : a \in A\},$$

and

$$d_{\mathcal{H}}(A, B) := \max\{\delta(A, B), \delta(B, A)\},$$

where $d_{\mathcal{H}}$ is the Hausdorff distance.

A multivalued map $T : (M_1, d_1) \times (M_2, d_2) \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(M_1)$ is said to *satisfied the property* $\mathcal{N}$ if

$$d_{\mathcal{H}}(T(x_1, y), T(x_2, y)) \leq d_1(x_1, x_2),$$

for all $x_1, x_2 \in M_1$ and $y \in M_2$ and $T$ is said to satisfy property $\mathcal{K}$ if there exist $k \in (0, 1)$ such that for all $x_1, x_2 \in M_1$ and $y \in M_2$,

$$d_{\mathcal{H}}(T(x_1, y), T(x_2, y)) \leq kd_1(x_1, x_2).$$

Then, the fixed point set of $T$ is defined as

$$P_T(y) := \{x \in M_1 : x \in T(x, y)\}.$$

Therefore, $P_T$ can be viewed as a multivalued map from $M_2$ to $\mathcal{P}(M_1)$.

Let $(M, d)$ be a metric space, for a subset $A$ of $(M \times M, d_\infty)$ we define the following sets:

(i) $r_{(x,y)}(A) = \sup\{d_\infty((x, y), (a_1, a_2)) : (a_1, a_2) \in A\}$
(ii) $r_{M\times M}(A) = \inf\{r_{(x,y)}(A) : (x, y) \in M \times M\}$
(iii) $r(A) = \inf\{r_{(x,y)}(A) : (x, y) \in A\}$
(iv) $diam(A) = \sup\{d_\infty(x, y) : (x, y) \in A\}$
(v) $C_{M\times M}(A) = \{(x, y) \in M \times M : r_{(x,y)}(A) = r(A)\}$
(vi) $C(A) = \{(x, y) \in A : r_{(x,y)} = r(A)\}$
(vii) $cov(A) = \cap\{B_i : B_i \text{ is a closed ball with respect to } d_\infty \text{ and } A \subset B_i\}$

**Definition 2.7** ([21])**.** Let $A$ be a bounded subset of a metric space. $A$ is said to be *admissible* if $A = cov(A)$. The collection of all admissible subsets of $M \times M$ is denoted by $\mathcal{A}(M \times M)$.

**Proposition 2.8** ([15])**.** *If $(H, d)$ is bounded hyperconvex metric space and if $f : H \to H$ is non-expansive, then the fixed point set $Fix(f)$ of $f$ in $H$ is nonempty and hyperconvex.*

**Proposition 2.9** ([21])**.** *If $E$ is either an admissible, externally hyperconvex or weakly externally hyperconvex subset of a hyperconvex metric space $H$ , then $E$ is proximinal in $H$.*

**Lemma 2.10** ([15])**.** *Suppose $H$ is a hyperconvex metric space and let $A \in \mathcal{A}(H)$. Then for each $\epsilon > 0$ there is a non-expansive retraction $R$ of $N_\epsilon(A)$ onto $A$ which has the property $d(x, R(x)) \leq \epsilon$ for each $x \in N_\epsilon(A)$.*

**Theorem 2.11** ([11])**.** *Suppose $H$ is a hyperconvex metric space and suppose $T : H \to H$ is non expansive. Then for each $\epsilon > 0$, the set $F_\epsilon(T)$ is hyperconvex, where*

$$F_\epsilon(T) := \{x \in H : d(x, T(x)) \leq \epsilon\}.$$

## 3. Main Result

The main purpose of this section is to obtain the existence of a selection, corresponding to the given multivalued map which also fixes values in the first variable for those points that are in the fixed point set of the multivalued map.

**Lemma 3.1.** *Let $H$ be a bounded hyperconvex space and $F : H \times H \to H$ satisfies the property $\mathcal{N}$. Then for every $h \in H$ there exist $m_h \in H$ such that $F(m_h, h) = m_h$.*

*Proof.* Define $F_m : H \to H$ by $F_m(h) = F(h, m)$, then $F_m$ becomes a non expansive mapping on a bounded hyperconvex space. By Proposition 2.8, $Fix(F_m) \neq \emptyset$, which completes the claim. □

**Theorem 3.2.** *Let $H$ be a hyperconvex space and $E$ be any set and let $F : E \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$. Then there exist a map $f : E \times H \to H$ such that $f(x, y) \in F(x, y)$, for each $(x, y) \in E \times E$ and also $d(f(x_1, y), f(x_2, y)) \leq d_{\mathcal{H}}(F(x_1, y), F(x_2, y))$, for every $x_1, x_2, y$ in $E \times H$.*

*Proof.* Let $\mathcal{F}$ denote the family of all pairs $(S \times H, f)$, where $S \subset E$, $f : S \times H \to H$, $f(x,y) \in F(x,y)$, for all $(x,y) \in S \times H$ and $d(f(x_1,y), f(x_2,y)) \leq d_{\mathcal{H}}(F(x_1,y), F(x_2,y))$, for every $x_1, x_2, y$ in $S \times H$. Note that $\mathcal{F} \neq \emptyset$ as $(\{x_0\} \times H, \Pi) \in \mathcal{F}$ where $\Pi(x_0, y) = x_0$. This works for any choice of $x_0$.
(Reason: Since $F(x_0,y) \in \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ and $f(x_0,y) \in H$, there exists $m \in F(x_0,y)$ such that $d(x_0, m) = dist(x_0, F(x_0,y)))$.
Therefore, $B(x_0, r) \cap F(x_0,y) \neq \emptyset$ where $r = d(x_0, m)$ and observe that $x_0 \in B(x_0, r) \cap F(x_0, y)$.

Now, define an order relation on $\mathcal{F}$ by setting $(S_1 \times H, f_1) \preccurlyeq (S_2 \times H, f_2)$ iff $S_1 \subseteq S_2$, and $f_2|_{S_1 \times H} = f_1$.

Let $(S_\alpha \times H, f_\alpha)$ be an increasing chain in $(\mathcal{F}, \preccurlyeq)$. Then, it can be seen that $\big(\bigcup\limits_{\alpha} S_\alpha \times H, f\big) \in \mathcal{F}$ where $f|_{S_\alpha \times H} = f_\alpha$ and is the upper bound for the chain $(\mathcal{F}, \preccurlyeq)$. Hence, by Zorn's lemma, $(\mathcal{F}, \preccurlyeq)$ has a maximal element $(S \times H, f)$(say). We claim that $S \times D = E \times H$. suppose there exists $x_0 \in E$ such that $x_0 \notin S$. Then, the set $S_0 = S \cup \{x_0\}$. Now, consider the set

$$G_y = \bigcap_{x \in S} B\big(f(x,y); d_{\mathcal{H}}(F(x_0,y), F(x,y)\big) \cap F(x_0,y).$$

We show that $G_y \neq \emptyset$, $\forall y \in H$. Since $F(x,y) \in \mathcal{E}(H)$, $\forall (x,y) \in H \times H$, $G_y \neq \emptyset$ iff for each $x \in S$,

$$dist(f(x,y), F(x_0,y)) \leq d_{\mathcal{H}}(F(x_0,y), F(x,y)\big).$$

As we know that $F(x_0,y)$ is a proximal subset of $H$ the previous inequality is true iff

$$\forall x \in S, B(f(x,y); d_{\mathcal{H}}(F(x_0,y), F(x,y))\big) \cap F(x_0,y) \neq \emptyset.$$

By the definition of Hausdorff distance, for each $\epsilon > 0$,

$$F(x,y) \subset N_{d_H(F(x,y),F(x_0,y))+\epsilon}(F(x_0,y)).$$

By our assumption $f(x,y) \in F(x,y)$. Hence, for each $\epsilon > 0$,

$$\bigcap_{x \in S} B\big(f(x,y); d_{\mathcal{H}}(F(x_0,y), F(x,y)) + \epsilon\big) \cap F(x_0,y) \neq \emptyset.$$

Since $F(x_0,y)$ is proximinal in $H$ we have,

$$\bigcap_{x \in S} B\big(f(x,y); d_{\mathcal{H}}(F(x_0,y), F(x,y))\big) \cap F(x_0,y) \neq \emptyset.$$

Therefore, for every $y \in H$, $G_y \neq \emptyset$. For each $y \in Y$ Choose $a_y \in G_y$.

Define $f_0 : S_0 \times H \to H$ by $f_0(x,y) = \begin{cases} f(x,y), & \text{if} \quad x \neq x_0 \\ a_y, & \text{if}\, x = x_0 \end{cases}$

By the choice of $a_y$, $f_0(x_0,y) \in F(x_0,y)$ and

$$d(f_0(x_0,y), f_0(x_1,y)) = d(a_y, f(x_1,y)) \leq d_{\mathcal{H}}(F(x_0,y), F(x_1,y)).$$

Therefore, $(S_0 \times H, f_0) \in \mathcal{F}$, which contradicts the maximality of $(S \times H, f)$. Hence $S = H$. □

**Corollary 3.3.** *Let $H$ be a hyperconvex space and let $(G,\rho)$ be a metic space Suppose $F : (G \times H, d_\infty) \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ satisfying property $\mathcal{N}$. Then, there exist $f : G \times H \to H$ with $d(f(x_1,y), f(x_2,y)) \leq \rho(x_1,x_2)$, for each $x_1, x_2 \in G$ , $y \in H$ and $f(x,y) \in F(x,y)$, for all $(x,y) \in G \times H$.*

*Proof.* The existence of the selection $f$ is guaranteed by previous result (Theorem 3.2) and since $F$ satisfies property $\mathcal{N}$, it follows that $d(f(x_1,y), f(x_2,y)) \leq \rho(x_1,x_2)$ is true for all $x_1, x_2 \in G$ and $y \in H$. □

**Corollary 3.4.** *Let $H$ be a bounded hyperconvex space and $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ satisfying property $\mathcal{N}$. Then, for each $y \in H$, $P_F(y) \neq \emptyset$.*

*Proof.* The proof is the direct application of Lemma 3.1 and Theorem 3.2. □

**Corollary 3.5.** *Let $H$ be a hyperconvex space and let $(G,\rho)$ be a metic space. Suppose $F : (G \times H, d_\infty) \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ satisfying property $\mathcal{K}$. Then, there exist $f : G \times H \to H$ with $d(f(x_1,y), f(x_2,y)) \leq k\rho(x_1,x_2)$, for each $x_1, x_2 \in G$ , $y \in H$ and $f(x,y) \in F(x,y)$, for all $(x,y) \in G \times H$.*

**Corollary 3.6.** *Let $H$ be a hyperconvex metric space and $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ satisfying property $\mathcal{K}$. Then, for each $y \in H$, $P_F(y) \neq \emptyset$ and $P_F(y)$ is singleton.*

*Proof.* By Corollary 3.5, there exist $f : H \times H \to H$ satisfying $d(f(x_1,y), f(x_2,y)) \leq k(x_1,x_2)$, for all $x_1, x_2, y \in H$. As defined in lemma, $f_m$ becomes a contractive type mapping. Proof follows from the famous $BCP$. □

**Remark 3.7.** Note that the boundedness is not required for the existence of fixed point in the above case.

**Theorem 3.8.** *Let $H$ be a hyperconvex space and $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ be a multivalued map satisfying property $\mathcal{N}$. Assume that $\bigcap_{y \in H} P_F(y) \neq \emptyset$. Then, there exist $f : H \times H \to H$ with $f(x,y) = x$, for every $x \in P_F(y)$ such that $f(x,y) \in$*

*$F(x,y)$, for all $(x,y) \in H \times H$ and satisfying $d(f(x_1,y), f(x_2,y)) \leq d(x_1,x_2)$, for all $x_1, x_2, y \in H$.*

*Proof.* Let $\mathcal{F}$ denote the family of all pairs $(B \times H, f)$, where $f : B \times H \to H$ Satisfying the following conditions:

(1) $f(x,y) \in F(x,y)$, for all $(x,y) \in B \times H$.
(2) $d(f(x_1,y), f(x_2,y)) \leq d(x_1,x_2)$, for all $x_1, x_2 \in B$ and $y \in H$.
(3) $f(x,y) = x$, for all $x \in P_F(y) \cap B$.

Note that $\mathcal{F} \neq \emptyset$, since the function $\Pi : \bigcap_{y \in H} P_F(y) \times H \to H$ defined by $\Pi(x,y) = x$ belongs to $\mathcal{F}$. Define an order relation on $\mathcal{F}$ by setting

$$(B_1 \times H, f_1) \preccurlyeq (B_2 \times H, f_2) \text{ iff } B_1 \subseteq B_2, \text{ and } f_2|_{(B_1 \times H)} = f_1.$$

Let $(B_\alpha \times H, f_\alpha)$ be an increasing chain in $(\mathcal{F}, \preccurlyeq)$. Then, it can be seen that $\big(\bigcup_\alpha (B_\alpha \times H), f\big) \in \mathcal{F}$ where $f|_{(B_\alpha \times H)} = f_\alpha$ and is the upper bound for the chain $(\mathcal{F}, \preccurlyeq)$. Hence by Zorn's lemma,$(\mathcal{F}, \preccurlyeq)$ has a maximal element $(B \times H, f)$(say). We claim that $B = H$. suppose there exists $x_0 \in H$ such that $x_0 \notin B$. Then set $B_0 = B \cup \{x_0\}$. Now, consider the set

$$G_y = \bigcap_{x \in B} B\big(f(x,y); d(x_0,x)\big) \cap F(x_0,y).$$

We show that for each $y \in H$, $G_y \neq \emptyset$. Since $F(x,y) \in \mathcal{E}(H)$ for all $(x,y) \in H \times H$, $G_y \neq \emptyset$ iff for each $x \in B$, $dist(f(x,y), F(x_0,y)) \leq d(x,x_0)$. As we know that $F(x_0,y)$ is a proximal subset of $H$ the previous inequality is true iff for each $x \in B$,

$$B(f(x,y); d(x,x_0)) \cap F(x_0,y) \neq \emptyset$$

By the definition of the Hausdorff distance and $F$ satisfies property $\mathcal{N}$, one can observe that for each $\epsilon > 0$,

$$F(x,y) \subset N_{d_H(F(x,y),F(x_0,y))+\epsilon}(F(x_0,y)) \subset N_{d(x,x_0)+\epsilon}(F(x_0,y)).$$

By our assumption $f(x,y) \in F(x,y)$. Therefore, for every $y \in H$, $G_y \neq \emptyset$. For each $y \in H$ choose $a_y \in G_y$.

Define $f_0 : B_0 \times H \to H$ by $f_0(x,y) = \begin{cases} f(x,y), & \text{if } \ x \neq x_0 \\ a_y, & \text{if } \ x_0 \notin P_F(y) \\ x_0 & \text{otherwise} \end{cases}$

By the choice of $a_y$, $f_0(x_0,y) \in F(x_0,y)$ and we have

$$d(f_0(x_0,y), f_0(x_1,y)) = d(a_y, f_0(x_1,y)) \leq d(x_0, a_y).$$

Also for $x_0 \in P_F(y)$, $f_0(x_0, y) = x_0 \in F(x_0, y)$. Hence, $(B_0 \times H, f_0) \in \mathcal{F}$, this contradicts the maximality of $(B \times H, f)$. Therefore $B = H$. ☐

**Example 3.9.** Let $F : l^\infty \times l^\infty \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(l^\infty)$ defined by $F(x, y) = \{a \in l^\infty : \|a\|_\infty \leq \|x - y\|_\infty\}$, that is $F(x, y) = B(0, r_{xy})$ where $r_{xy} = \|x - y\|_\infty$. Then $\bigcap_{y \in l^\infty} P_F(y) \neq \emptyset$, since $0 \in \bigcap_{y \in l^\infty} P_F(y)$. Also, for $x_1, x_2, y \in l^\infty$, we have $d_{\mathcal{H}}(F(x_1, y), F(x_2, y)) = d_{\mathcal{H}}(B(0, r_{x_1y}), B(0, r_{x_2y}))$. Observe that either $r_{x_1y} \leq r_{x_2y}$ or $r_{x_2y} \leq r_{x_1y}$. Therefore either $d_{\mathcal{H}}(B(0, r_{x_1y}), B(0, r_{x_2y})) = r_{x_1y} - r_{x_2y}$ or $d_{\mathcal{H}}(B(0, r_{x_1y}), B(0, r_{x_2y})) = r_{x_2y} - r_{x_1y}$. That is either $r_{x_1y} - r_{x_2y} = \|x_1 - y\|_\infty - \|x_2 - y\|_\infty \leq \|x_1 - x_2\|_\infty$ or $r_{x_2y} - r_{x_1y} = \|x_2 - y\|_\infty - \|x_1 - y\|_\infty \leq \|x_1 - x_2\|_\infty$. Therefore in either case $d_{\mathcal{H}}(F(x_1, y), F(x_2, y)) \leq d(x_1, x_2)$. Hence by Theorem 3.8 there exists a selection $f : l^\infty \times l^\infty \to l^\infty$ with $f(x, y) = x$, for every $x \in P_F(y)$ and such that $f(x, y) \in F(x, y)$, for all $(x, y) \in l^\infty \times l^\infty$ and satisfying $d(f(x_1, y), f(x_2, y)) \leq d(x_1, x_2)$, for all $x_1, x_2, y \in l^\infty$.

**Corollary 3.10.** *Let $H$ be a hyperconvex space and $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ satisfies property $\mathcal{N}$. If $\bigcap_{y \in H} P_F(y) \neq \emptyset$ then for each $y \in H$, $P_F(y)$ is hyperconvex.*

*Proof.* From Theorem 3.8, there exists $f : H \times H \to H$ with $f(x, y) = x$ whenever $x \in P_F(y)$ and $d(f(x_1, y), (x_2, y)) \leq d(x_1, x_2)$. Therefore $P_F(y) = \{x \in H : f(x, y) = x\} = \{x \in H : f_y(x) = x\}$ where $f_y : H \to H$ defined by $f_y(x) = f(x, y)$. Hence by Proposition 2.8, $P_F(y)$ is hyperconvex ☐

**Theorem 3.11.** *Let $H$ be a hyperconvex normed space and $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$ be a multivalued map satisfying property $\mathcal{K}$. Assume that $\bigcap_{y \in H} P_F(y) \neq \emptyset$. Then there exist $f : H \times H \to H$ with $f(x, y) = x$ for every $x \in P_F(y)$ such that $f(x, y) \in F(x, y)$ for all $(x, y) \in H \times H$ and satisfying $d(f(x_1, y), f(x_2, y)) \leq kd(x_1, x_2)$ for all $x_1, x_2, y \in H$.*

*Proof.* The proof is the exact replica of Theorem 3.8. The only non trivial part is showing the $\mathcal{F}$ is non empty. Let $G = \bigcap_{y \in H} P_F(y)$, observe that the map $\Pi : kG \times H \to H$ defined by $\Pi(x, y) = x$ belongs to $\mathcal{F}$. ☐

**Example 3.12.** Let $F : [0, 1] \times [0, 1] \to \mathcal{P}([0, 1])$. For each $y \in [0, 1]$ define
$F(x, y) = \left[\frac{x+1}{2}, 1\right]$. Then, $d_{\mathcal{H}}(F(x_1, y), F(x_2, y)) \leq \frac{1}{2} d(x_1, x_2) \leq d(x_1, x_2)$, also $\bigcap_{y \in H} P_F(y) \neq \emptyset$ as $F(1, y) = 1$ for all $y \in [0, 1]$. Hence, F satisfies Theorem 3.8 and

Theorem 3.11 with a constant $k = \frac{1}{2}$. Therefore, $F$ has a selection $f : [0,1] \times [0,1] \to [0,1]$ with $f(x,y) = x$ whenever $x \in P_F(y)$ and satisfying $d(f(x_1,y), f(x_2,y) \leq d(x_1, x_@)$.

**Definition 3.13.** For a subset $A$ of a metric space define

$$\mathrm{N}_\epsilon(A) = \bigcup_{(a_1,a_2)\in A} B((a_1,a_2),\epsilon).$$

**Lemma 3.14.** *Let $H$ be a hyperconvex space metric space and let $A \in \mathcal{A}(H \times H)$, say $A = \bigcap\limits_{\alpha\in\Gamma} B((x_\alpha, y_\alpha), r_\alpha)$. Then for each $\epsilon > 0$,*

$$\mathrm{N}_\epsilon(A) = \bigcap_{\alpha\in\Gamma} B((x_\alpha, y_\alpha), r_\alpha + \epsilon).$$

*Proof.* Let $(z_1, z_2) \in \mathrm{N}_\epsilon(A)$. Then $d((z_1,z_2),(a_1,a_2)) \leq \epsilon$ for some $(a_1,a_2) \in A$. Observe that for each $i \in \Gamma$, $d((z_1,z_2),(x_i,y_i) \leq d((a_1,a_2),(x_i,y_i)) + d((a_1,a_2),(y_1,y_2))$ $\leq r_i + \epsilon$, hence $\mathrm{N}_\epsilon(A) \subseteq \bigcap\limits_{\alpha\in\Gamma} B((x_\alpha,y_\alpha), r_\alpha + \epsilon)$. Now let $(z_1,z_2) \in \bigcap\limits_{\alpha\in\Gamma} B((x_\alpha,y_\alpha), r_\alpha + \epsilon)$. For each $i \in \Gamma$, we have $d((z_1,z_2),(x_i,y_i)) \leq r_i + \epsilon$. Since $A$ is non empty, for each $i,j \in \Gamma$, $d((x_i,y_i),(x_j,y_j)) \leq d((x_i,y_i),(a_1,a_2)) + d((x_j,y_j),(a_1,a_2)) \leq r_i + r_j$. By hyperconvexity of $H$, $\left(\bigcap\limits_{\alpha\in\Gamma} B((x_\alpha,y_\alpha),r_\alpha)\right) \cap B((z_1,z_2),\epsilon) \neq \emptyset$, But $A \cap B((z_1,z_2),\epsilon) = \left(\bigcap\limits_{\alpha\in\Gamma} B((x_\alpha,y_\alpha),r_\alpha)\right) \cap B(y,\epsilon)$. Therefore, there exists $(w_1,w_2) \in A$ such that $(w_1,w_2) \in B(y,\epsilon)$. This implies $(z_1,z_2) \in \mathrm{N}_\epsilon(A)$. □

**Theorem 3.15.** *Let $H$ be hyperconvex space and let $T : H \times H \to H$ be such that for all $x_1, x_2, y \in H$, $d(T(x_1,y), T(x_2,y)) \leq d(x_1,x_2)$. Then, for each $\epsilon > 0$ and for each $h \in H$, the set $F_\epsilon(T)$ is hyperconvex where for each $h \in H$, $F_\epsilon(T)$ is defined as $F_\epsilon(T) := \{x \in H : d(x, T(x,h)) \leq \epsilon\}$. This $F_\epsilon(T)$ denotes the approximate fixed point set of $T$.*

*Proof.* Note that $F_\epsilon(T) = \{x \in H : d(x, T_h(x)) \leq \epsilon\}$, where $T_h(x) := T(x,h)$ also $T_h$ is non expansive for each $h \in H$. Let $x_\alpha \in F_\epsilon(T)$, and let $R_\alpha \geq 0$ satisfying $d(x_\alpha, y_\beta) \leq r_\alpha + r_\beta$. Since $H$ is hyperconvex $J = \bigcap\limits_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$. Also for $x \in J$ and for each $\alpha \in \Gamma$ we have,

$$d(x_\alpha, T_h(x)) \leq d(x_\alpha, T_h(x_\alpha)) + d(T_h(x_\alpha), T_h(x)) \leq \epsilon + d(x_\alpha, x) \leq r_\alpha + \epsilon$$

which shows that $T_h(x) \in N_\epsilon(J)$. By using the results in chapter 4 of [15], there exists a retraction $R$ of $N_\epsilon(J)$ onto $J$ with $d(R(x), x) \leq \epsilon$, for every $x \in N_\epsilon(J)$.

Additionally, the map $R \circ T_h : J \to J$ becomes a non expansive mapping. Therefore by Proposition 2.8, there exists a $z \in J$ such that $R \circ T_h(z) = z$. Hence $d(z, T_h(z)) = d(R \circ T_h(z), T_h(z)) \leq \epsilon$. Which implies $F_\epsilon(T) = \{a \in H : d(x, T_h(x)) \leq \epsilon\} \neq \emptyset$ and this is true for every $h \in H$. □

**Theorem 3.16.** *Let $H$ be a hyperconvex metric space and suppose $F : H \times H \to \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$. Then the family $\mathcal{S}(F)$, consisting of all mappings $f : H \times H \to H$ with $f(x, y) \in F(x, y)$ and $d(f(x_1, y), f(x_2, y)) \leq d_{\mathcal{H}}(F(x_1, y), F(x_2, y)$ for all $x_1, x_2, y \in H$, is hyperconvex.*

*Proof.* Let $\{f_\alpha\}$ in $\mathcal{S}(T)$ and $\{r_\alpha\}$ be such that $d(f_\alpha, f_\beta) \leq r_\alpha, r_\beta$. Then, for each $(x, y) \in H \times H$, $J(x, y) := \bigcap_{\alpha\in\Gamma} (B(f_\alpha(x, y), r_\alpha)) \cap F(x, y) \neq \emptyset$, since $F(x, y) \in \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$. Also, $J(x, y) \in \mathcal{P}_{\mathcal{B},\mathcal{E}}(H)$. Hence by Theorem 3.2, the mapping $(x, y) \to J(x, y)$ has a selection $f$ satisfying $d(f(x_1, y), f(x_2, y)) \leq d_{\mathcal{H}}(F(x_1, y), F(x_2, y))$ for all $x_1, x_2, y \in H$, thus $f \in \bigcap_{\alpha\in\Gamma} (B(f_\alpha, r_\alpha)) \cap \mathcal{S}(F)$. Therefore, $\mathcal{S}(F)$ is hyperconvex. □

## 4. Applications

Here we apply our result to prove a fixed edge theorem for an edge preserving map on an non cyclic, connected graph which contains no infinite path.

Let $V$ be any set and $E$ be a relation on $V$. A graph is the ordered pair $(V, E)$, where $V$ is the set of vertices and the relation $E$ are called the edges. We consider undirected graphs with no multiple edges and has a loop at every vertex. That is, $\forall x \in V$, $(x, x) \in E$. This kind of graphs are called reflexive.

**Definition 4.1.** Let $G$ be a graph associated with the vertex set $V$ and the relation $E$. A path is sequence $(x_n)$ where for each $x_i \in V$ and $(x_{i-1}, x_i) \in E$ for each $i \in \{0, 1, 2, ...\}$. A path is finite if $i$ varies in a finite index. A cycle is a finite path $(x_i)_{i=0}^k$, where $(x_k, x_0) \in E$. Let $a, b \in V$ a path form $a$ to $b$ is denoted by $[a, b]$. A graph is said to be *connected* if for any pair of vertices there is finite path joining them. A connected graph with no cycles is called *tree.*

**Definition 4.2.** For a graph $G = (V, E)$, a map $h : G \to G$ is called *edge-preserving* if $(x, y) \in E$ then $(h(x), h(y)) \in E$

**Theorem 4.3.** *Let $G = (V, E)$ be a reflexive graph that is connected, contains no cycles, and contains no infinite paths. Then given any edge preserving map from $G$ to itself for every pair of vertices, either the map fixes an edge or vertex.*

*Proof.* Let $h : G \to G$ be an edge preserving map. $G$ is tree since it is connected graph with no cycle. We can construct a $\mathbb{R}$-tree $T$ form the graph $G$ by identifying each non-trivial edge with unit interval of real line . Note that in $T$ the metric is defined to be the length of the shortest path between the vertices. This metric makes $T$ complete [19]. Thus by [7] $T$ is hyperconvex.

Now define $F : T \times T \to \mathcal{P}(T)$ by $F(a, b) = [a, b]$, since $G$ contains no infinite path, for every $(a, b) \in T \times T$, $[a, b]$ is bounded and externally hyperconvex. Also $\bigcap_{y \in T} P_F(y) \neq \emptyset$, in fact it is precisely equal to the vertex set of G. Let $a_1, a_2, b \in V$, then $F((a_1, b)) = [a_1, b]$ and $F((a_2, b)) = [a_2, b]$. Since $T$ is a metric tree $[a_1, b] \cap [a_2, b] = [b, w]$ for some $w \in V$, therefore $d_{\mathcal{H}}(F(a_1, b), F(a_2, b)) \leq d(a_1, a_2)$. As $d_{\mathcal{H}}(F(a_1, b), F(a_2, b)) = \max\{d(a_1, w), d(a_2, w)\} \leq d(a_1, a_2)$. By Theorem 3.8 there exists a $f : T \times T \to T$ by $f(x, y) = x$ for $x \in P_F(y)$ and $f(x, y) \in F(x, y)$ for all $(x, y) \in T \times T$ Therefore it is true for $f(h(a), h(b)) = h(a) \in F((h(a), h(b))) = [h(a), h(b)]$. Which either fixes an edge or the vertex. □

**Example 4.4.** Let $F : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by $F(x, y) = [x, y]$. Then by Theorem 3.2 in [7] $\mathbb{R}$ with usual metric becomes a metric tree. Observe that $[x, y]$ is a convex set. A. G. Aksoy and M. A. Khamsi proved that any non-empty convex subset of a metric tree is externally hyperconvex, see [2].

$$d_{\mathcal{H}}(F(x_1, y), F(x_2, y) = \begin{cases} \max\{d(x_1, y), d(x_2, y)\}, & \text{or} \\ d(x_2, x_1). \end{cases}$$

In either cases $d_{\mathcal{H}}(F(x_1, y), F(x_2, y) \leq d(x_1, x_2)$ also $\bigcap_{y \in H} P_F(y) \neq \emptyset$. Therefore by the above theorem $F$ has a selection $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $f(x, y) = x$ for every $x \in P_F(y)$ such that $f(x, y) \in F(x, y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$ and satisfying $d(f(x_1, y), f(x_2, y)) \leq d(x_1, x_2)$ for all $x_1, x_2, y \in \mathbb{R}$.

## 5. Conclusion

Although a large number of researchers have paid attention to finding the existence of fixed points on hyperconvex spaces and have obtained abundant achievements, there are still some interesting and open topics worth considering. Under this circumstance, we prove some results related to existence of selection map on hyperconvex spaces and existence of fixed point for such selection. As various future results can be demonstrated in a smaller setting or even generalization to ensure the existence of selection and its fixed point.

## 6. Conflict of Interest

All the authors declare that they have no conflict of interest.

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[a]Research Scholar: Department of Mathematics, Bharathidasan University, Trichy-620024, India
*Email address*: herminaujothy.a@bdu.ac.in

[b]Associate Professor: Department of Mathematics, Bharathidasan University, Trichy-620024, India
*Email address*: pssrini@bdu.ac.in

[c]Guest Lecturer: Department of Mathematics, Bharathidasan University, Trichy-620024, India
*Email address*: deivaraman@gmail.com