

On the second order property of elliptical multivariate regular variation

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Abstract

Multivariate regular variation is a popular framework of multivariate extreme value analysis. However, a suitable parametric model needs to be introduced for efficient estimation of its spectral measure. In such a view, elliptical distributions have been employed for deriving such models. On the other hand, the second order behavior of multivariate regular variation has to be specified for investigating the property of the estimator. This paper derives such a behavior by imposing a widely adopted second order regular variation condition on the representation of elliptical distributions. As result, the second order variation for the convergence to spectral measure is characterized by a signed measure with a regular varying index. Moreover, it leads to the asymptotic bias of the estimator. For demonstration, multivariate t-distribution is considered.

Keywords: multivariate regular variation, the second order behavior, spectral measure, elliptical distribution, asymptotic bias

1. Introduction

In diverse fields such as finance, meteorological hydorrisk management, and, internet traffic operation, it is a prominent issue to model multivariate extreme events. Multivariate regular variation is a popular framework for multivariate extreme value analysis (cf. Cai *et al.*, 2011; Weller and Cooly, 2014; Li and Hua, 2015; Einmahl *et al.*, 2020). It describes the behavior of multivariate extremes in terms of their radii and directions (cf. Resnick, 2008): The directional behavior is specified by a probability measure on a measurable space of unit vectors, which is called spectral measure. Meanwhile the radius obeys a univariate regular variation, and furthermore the radius and direction are asymptotically independent as the former gets large. However, for efficient estimation, it is crucially required to introduce a suitable model for the spectral measure which itself has no parameter of finite dimension. In such a view, elliptical distributions have received much attention since they not only constitute a subclass of multivariate regular variation but also provide a tractable parametric model for spectral measure and tail dependence (cf. Hult and Lindskog, 2002; Klüppelberg *et al.*, 2007; Joe and Li, 2019).

Kim (2021) derived a density function of the spectral measure of elliptical distributions and then proposed a maximum likelihood estimator based on it. However, for investigating the property of the resulting estimator, we need to specify how accurately the directional behavior at extreme level in

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finite sample is described by the spectral measure. This paper mainly focuses on the second order property of convergence to spectral measure for the class of elliptical distributions: Let $\{f_t(x) : t > 0\}$, and $f(x)$ be real valued functions defined on a space \mathcal{X} such that $f_t(x) \rightarrow f(x)$ as $t \rightarrow 0$ for each $x \in \mathcal{X}$. In this case, the second order property of the convergence indicates the behavior characterized by $\gamma > 0$ and a real valued function $g(x)$ such that

$$f_t(x) - f(x) = t^\gamma g(x) + t^\gamma o(1), \quad \text{as } t \rightarrow 0.$$

By imposing a widely adopted second order (univariate) regular variation condition on the representation of elliptical distributions (cf. Hall, 1982; Hult and Lindskog, 2002), this paper derives the second order behaviors with respect to radii and directions of the multivariate extremes.

The remainder of this paper is organized as follows: Section 2 presents some preliminary definitions and theorem, and briefly reviews the second order property of univariate cases; Section 3 deals with the second order property of elliptical multivariate regular variation.

2. Preliminary

2.1. Elliptical multivariate regular variation

This section provides the preliminary definitions and theorem. First, it is necessary to clarify the topological space on which the spectral measure is defined. Let

$$\mathbb{S}^{d-1} := \{s = (s_1, \dots, s_d)' \in \mathbb{R}^d : s_1^2 + \dots + s_d^2 = 1\}$$

denote the unit sphere in d -dimensional space for a positive integer $d \geq 2$. \mathbb{S}^{d-1} is equipped with the metric that takes $|s_1 - s_2|$ as the distance between s_1 and s_2 in \mathbb{S}^{d-1} , where $|\cdot|$ indicates the Euclidean norm. The induced topology generates a Borel σ -field on \mathbb{S}^{d-1} , which is denoted by \mathcal{S}^{d-1} hereafter.

Let \mathbf{X} be a d -dimensional random vector, and $R = |\mathbf{X}|$ and $\mathbf{S} = \mathbf{X}/R$ represent the radius and direction of \mathbf{X} . Then, R and \mathbf{S} are positive and \mathbb{S}^{d-1} -valued random variables, respectively. \mathbf{X} is said to have multivariate regularly varying tail if there exist $\alpha > 0$ and a probability Borel measure Λ on \mathbb{S}^{d-1} such that for $x > 0$ and $A \in \mathcal{S}^{d-1}$ with $\Lambda(\partial A) = 0$,

$$\lim_{t \rightarrow 0} \frac{\mathbb{P}(R > x/t, \mathbf{S} \in A)}{\mathbb{P}(R > 1/t)} = x^{-\alpha} \Lambda(A) \quad (2.1)$$

(cf. Resnick, 2008). Taking $A = \mathbb{S}^{d-1}$, the above equation reduces to

$$\lim_{t \rightarrow 0} \frac{\mathbb{P}(R > x/t)}{\mathbb{P}(R > 1/t)} = x^{-\alpha}, \quad \text{for } x > 0. \quad (2.2)$$

i.e., $x \mapsto \mathbb{P}(R > x)$ is (univariate) regularly varying with exponent $-\alpha$ as $x \rightarrow \infty$ (Bingham *et al.*, 1987). It means that the conditional distribution of $t|\mathbf{X}|$ given $t|\mathbf{X}| > 1$ converges to a Pareto distribution as $t \rightarrow 0$. Meanwhile, if $x = 1$ in (2.1), then it is

$$\lim_{t \rightarrow 0} \mathbb{P}(\mathbf{S} \in A \mid R > 1/t) = \Lambda(A), \quad (2.3)$$

viz, the conditional distribution of \mathbf{S} given $t|\mathbf{X}| > 1$ approximates to Λ as $t \rightarrow 0$. α is called the tail exponent or tail index, while Λ is done the spectral measure. Furthermore, it follows from (2.1)–(2.3) that asymptotically R and \mathbf{S} are of conditional independence given $t|\mathbf{X}| > 1$ as $t \rightarrow 0$. These features give rise to a simulation method of multivariate extremes (Kim, 2022).

This study concentrates on the tail behavior characterized by a non-singular heavy-tailed elliptical distribution. For a comprehensive overview of elliptical distributions, refer to Hult and Lindskog (2002). Let $\mathbf{Z} = (Z_1, \dots, Z_d)'$, where Z_1, \dots, Z_d are independent standard normal random variables. Then $\mathbf{W} := \mathbf{Z}/|\mathbf{Z}|$ is uniformly distributed on \mathbb{S}^{d-1} . In particular, if

$$\mathbf{X} = V\Sigma^{\frac{1}{2}}\mathbf{W}, \tag{2.4}$$

where Σ is a symmetric positive definite matrix, V is a positive random variable with a regularly varying tail of tail exponent α and independent of \mathbf{W} , then, according to Theorem 3.1 of Hult and Lindskog (2002), \mathbf{X} follows an elliptical distribution. Moreover, it exhibits a multivariate regularly varying tail, and its spectral measure can be readily proven to be

$$\Lambda(A) = \frac{E\{|\Sigma^{1/2}\mathbf{W}|^\alpha I(\Sigma^{1/2}\mathbf{W}/|\Sigma^{1/2}\mathbf{W}| \in A)\}}{E\{|\Sigma^{1/2}\mathbf{W}|^\alpha\}} \quad \text{for } A \in \mathcal{S}^{d-1}. \tag{2.5}$$

Define $\mu(A) := P(\mathbf{Z}/|\mathbf{Z}| \in A)$ for $A \in \mathcal{S}^{d-1}$. Then, Kim (2021) shows that Λ in (2.5) has a density function with respect to μ which of form is presented in the following theorem:

Theorem 1. *Let*

$$\lambda(s) = \frac{(s'\Sigma^{-1}s)^{-\frac{\alpha+d}{2}}}{C(\Sigma, \alpha)}, \quad s \in \mathbb{S}^{d-1}, \quad \text{with } C(\Sigma, \alpha) = \int_{s \in \mathbb{S}^{d-1}} (s'\Sigma^{-1}s)^{-\frac{\alpha+d}{2}} d\mu(s). \tag{2.6}$$

Then, we have

$$\Lambda(A) = \int_{s \in A} \lambda(s) d\mu(s), \quad \text{for } A \in \mathcal{S}^{d-1}. \tag{2.7}$$

This theorem is related to the limit Λ in (2.3), which can be readily applied to the estimation (cf. Kim, 2021). However, to investigate the asymptotic property of the resulting estimator, the second order behaviors of (2.2)–(2.3) have to be specified. The next section will deal with it.

2.2. The second order behavior of univariate regular variation

Before dealing with a second order behavior of multivariate regular variation, this subsection briefly reviews the univariate case (2.2) with respect to α -estimation. Let R_1, \dots, R_n be a random sample of R . For estimating α ,

$$H_n = \frac{1}{k} \sum_{i=1}^n (\log R_i - \log R_{(k+1)})_+$$

is popularly employed, where $k < n$ is a positive integer and $R_{(k+1)}$ indicates the $(k + 1)^{th}$ largest value in R_1, \dots, R_n (cf. Hill, 1975; Hsing, 1991). Assuming that $k := k_n$ satisfies $k \rightarrow \infty$ and $k/n \rightarrow 0$ as $n \rightarrow \infty$, H_n converges in probability to $1/\alpha$ as $n \rightarrow \infty$. For establishing a further asymptotic property, we need to impose a more stringent condition on (2.2) which is called second order regular variation: For instance, letting $L(x) := x^\alpha P(R > x)$, $x > 0$, we assume that there exist $\gamma > 0$, $c_1 > 0$, and, $c_2 \in \mathbb{R}$ such that

$$L(x) = c_1 \{1 + c_2 x^{-\gamma} + x^{-\gamma} o(1)\} \quad \text{as } x \rightarrow \infty \tag{2.8}$$

(cf. Hall, 1982). If $\sqrt{k}(k/n)^{\gamma/\alpha} \rightarrow M$ as $n \rightarrow \infty$ for some $M \geq 0$, we then have

$$\sqrt{k} \left(H_n - \frac{1}{\alpha} \right) \Rightarrow \frac{Z}{\alpha} + \frac{\gamma c_1^{-\gamma/\alpha} c_2 M}{\alpha(\alpha + \gamma)} \quad \text{as } n \rightarrow \infty,$$

where Z is a standard normal variable and \Rightarrow denotes ‘convergence in distribution’ (cf. Hsing, 1991). (2.8) is a special and prominent case of

$$\frac{L(x/t)}{L(1/t)} - 1 = t^\gamma \gamma c_2 \frac{x^{-\gamma} - 1}{\gamma} + t^\gamma o(1) \quad \text{as } t \rightarrow 0, \quad \text{for } x > 0,$$

which is the second order property of $\lim_{t \rightarrow 0} L(x/t)/L(1/t) = 1$ for each $x > 0$.

3. Main result

For investigating the asymptotic property of the estimator of Σ in (2.4), we need to explore the second order behavior of (2.3) (see Remark 1). Let $\mathbf{X} = V\Sigma^{1/2}\mathbf{W}$ be the random vector in (2.4). Moreover, it is assumed that there exist $\alpha > 0$, $\gamma > 0$, $a_1 > 0$, and $a_2 \in \mathbb{R}$, such that

$$\bar{F}_V(x) := P(V > x) = a_1 x^{-\alpha} \{1 + a_2 x^{-\gamma} + o(x^{-\gamma})\} \quad \text{as } x \rightarrow \infty, \tag{3.1}$$

(cf. (2.8)). Let

$$\Lambda_t(\cdot) := P(\mathbf{S} \in \cdot \mid R > 1/t), \quad \text{for } t > 0.$$

Under (3.1), we are going to derive the behavior of $\Lambda_t(\cdot) - \Lambda(\cdot)$ as $t \rightarrow 0$. Let

$$\begin{aligned} \rho &= E\{|\Sigma^{1/2}\mathbf{W}|^{\alpha+\gamma}\} / E\{|\Sigma^{1/2}\mathbf{W}|^\alpha\}, \quad c_1 = a_1 E\{|\Sigma^{1/2}\mathbf{W}|^\alpha\}, \quad c_2 = a_2 \rho, \\ \Lambda_{(\gamma)}(A) &:= \frac{E\{|\Sigma^{1/2}\mathbf{W}|^{\alpha+\gamma} \mathbf{I}_A\}}{E\{|\Sigma^{1/2}\mathbf{W}|^{\alpha+\gamma}\}} \quad \text{with } \mathbf{I}_A = \mathbf{I}(\Sigma^{\frac{1}{2}}\mathbf{W} / |\Sigma^{\frac{1}{2}}\mathbf{W}| \in A) \text{ for } A \in \mathcal{S}^{d-1}, \end{aligned}$$

and

$$\Gamma(A) := a_2 \rho \{ \Lambda_{(\gamma)}(A) - \Lambda(A) \} \quad \text{for } A \in \mathcal{S}^{d-1}.$$

Theorem 2. Assume that Σ is a symmetric positive definite matrix and (3.1) holds. Then,

- (i) $P(|\mathbf{X}| > 1/t) = c_1 t^\alpha (1 + c_2 t^\gamma + o(t^\gamma))$ as $t \rightarrow 0$, i.e., (2.8) holds;
- (ii) for each $A \in \mathcal{S}^{d-1}$, $\Lambda_t(A) - \Lambda(A) = t^\gamma \Gamma(A) + t^\gamma o(1)$ as $t \rightarrow 0$, where the $o(1)$ -term is uniformly negligible in $A \in \mathcal{S}^{d-1}$.

Proof: We first verify (i). Since V is independent of \mathbf{W} and we have $|\Sigma^{1/2}\mathbf{W}| > 0$ with probability 1 due to positive definiteness of Σ , (2.4) implies that for $A \in \mathcal{S}^{d-1}$,

$$\begin{aligned} P(|\mathbf{X}| > 1/t, \mathbf{X}/|\mathbf{X}| \in A) &= P(|\Sigma^{1/2}\mathbf{W}|V > 1/t, \Sigma^{1/2}\mathbf{W}/|\Sigma^{1/2}\mathbf{W}| \in A) \\ &= E\left(\mathbf{I}(|\Sigma^{1/2}\mathbf{W}|V > 1/t) \mathbf{I}_A\right) = E\left\{E\left(\mathbf{I}(|\Sigma^{1/2}\mathbf{W}|V > 1/t) \mathbf{I}_A \mid \mathbf{W}\right)\right\} \\ &= E\left\{P\left(V > \frac{1}{t|\Sigma^{1/2}\mathbf{W}|} \mid \mathbf{W}\right) \mathbf{I}_A\right\} = E\left\{\bar{F}_V\left(\frac{1}{t|\Sigma^{1/2}\mathbf{W}|}\right) \mathbf{I}_A\right\}. \end{aligned}$$

Moreover, since there exists $c > 1$ such that almost surely $1/c < |\Sigma^{1/2}\mathbf{W}| < c$ due to positive definiteness of Σ , (3.1) implies that

$$\begin{aligned} \frac{\bar{F}_V(1/(t|\Sigma^{1/2}\mathbf{W}|))}{\bar{F}_V(1/t)} &= |\Sigma^{1/2}\mathbf{W}|^\alpha \frac{1 + a_2 t^\gamma |\Sigma^{1/2}\mathbf{W}|^\gamma + t^\gamma o(1)}{1 + a_2 t^\gamma + t^\gamma o(1)} \\ &= |\Sigma^{1/2}\mathbf{W}|^\alpha \left[1 + a_2 t^\gamma \left(|\Sigma^{1/2}\mathbf{W}|^\gamma - 1 \right) + t^\gamma o(1) \right] \quad \text{as } t \rightarrow 0, \end{aligned}$$

where the $o(1)$ -terms are all bounded by a constant for sufficiently small $t > 0$. Thus, applying bounded convergence theorem we have

$$\begin{aligned} \frac{\mathbb{P}(|\mathbf{X}| > 1/t, \mathbf{X}/|\mathbf{X}| \in A)}{\mathbb{P}(V > 1/t)} &= \frac{\mathbb{E} \left\{ \frac{\bar{F}_V(1/(t|\Sigma^{1/2}\mathbf{W}|))}{\bar{F}_V(1/t)} \mathbf{I}_A \right\}}{\bar{F}_V(1/t)} = \mathbb{E} \left\{ \frac{\bar{F}_V(1/(t|\Sigma^{1/2}\mathbf{W}|))}{\bar{F}_V(1/t)} \mathbf{I}_A \right\} \\ &= \mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \left[1 + a_2 t^\gamma \left(|\Sigma^{1/2}\mathbf{W}|^\gamma - 1 \right) + t^\gamma o(1) \right] \mathbf{I}_A \right\} \\ &= \mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A \right\} + t^\gamma \mathbb{E} \left\{ a_2 \left(|\Sigma^{1/2}\mathbf{W}|^{\alpha+\gamma} - |\Sigma^{1/2}\mathbf{W}|^\alpha \right) \mathbf{I}_A \right\} + t^\gamma o(1) \\ &= \mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A \right\} + t^\gamma h(A) + t^\gamma o(1) \quad \text{as } t \rightarrow 0, \end{aligned} \quad (3.2)$$

where $h(A) = a_2 \mathbb{E} \{ (|\Sigma^{1/2}\mathbf{W}|^{\alpha+\gamma} - |\Sigma^{1/2}\mathbf{W}|^\alpha) \mathbf{I}_A \}$ is a signed Borel measure on \mathbb{S}^{d-1} and the $o(1)$ -terms are uniformly negligible for $A \in \mathcal{S}^{d-1}$. Taking $A = \mathbb{S}^{d-1}$, we have that as $t \rightarrow 0$,

$$\begin{aligned} \mathbb{P}(|\mathbf{X}| > 1/t) &= \mathbb{P}(V > 1/t) \left[\mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \right\} + t^\gamma h(\mathbb{S}^{d-1}) + t^\gamma o(1) \right] \\ &= a_1 \mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \right\} t^\alpha (1 + a_2 t^\gamma + t^\gamma o(1)) (1 + t^\gamma a_2 (\rho - 1) + t^\gamma o(1)) \\ &= a_1 \mathbb{E} \left\{ |\Sigma^{1/2}\mathbf{W}|^\alpha \right\} t^\alpha (1 + t^\gamma a_2 \rho + t^\gamma o(1)) = c_1 t^\alpha (1 + c_2 t^\gamma + t^\gamma o(1)), \end{aligned}$$

which asserts (i).

Next, we verify (ii). From (3.2), it follows that as $t \rightarrow 0$,

$$\begin{aligned} \Lambda_t(A) &= \mathbb{P}(\mathbf{X}/|\mathbf{X}| \in A \mid |\mathbf{X}| > 1/t) = \frac{\mathbb{P}(|\Sigma^{1/2}\mathbf{W}|V > 1/t, \Sigma^{1/2}\mathbf{W}/|\Sigma^{1/2}\mathbf{W}| \in A)}{\mathbb{P}(|\Sigma^{1/2}\mathbf{W}|V > 1/t)} \\ &= \frac{\mathbb{P}(|\Sigma^{1/2}\mathbf{W}|V > 1/t, \Sigma^{1/2}\mathbf{W}/|\Sigma^{1/2}\mathbf{W}| \in A)/\mathbb{P}(V > 1/t)}{\mathbb{P}(|\Sigma^{1/2}\mathbf{W}|V > 1/t)/\mathbb{P}(V > 1/t)} \\ &= \frac{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A \} + t^\gamma h(A) + t^\gamma o(1)}{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \} + t^\gamma h(\mathbb{S}^{d-1}) + t^\gamma o(1)}, \end{aligned}$$

where $o(1)$ -terms are uniformly negligible in $A \in \mathcal{S}^{d-1}$. Moreover, by applying mean value theorem we have that as $t \rightarrow 0$

$$\frac{1}{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \} + t^\gamma h(\mathbb{S}^{d-1}) + t^\gamma o(1)} = \frac{1}{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \}} - t^\gamma \frac{h(\mathbb{S}^{d-1})}{(\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \})^2} + t^\gamma o(1).$$

Thus, $\Lambda_t(A)$ is equal to

$$\frac{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A \}}{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \}} + t^\gamma \left[\frac{h(A)}{\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \}} - \frac{h(\mathbb{S}^{d-1}) \mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A \}}{(\mathbb{E} \{ |\Sigma^{1/2}\mathbf{W}|^\alpha \})^2} \right] + t^\gamma o(1), \quad (3.3)$$

as $t \rightarrow 0$, where $o(1)$ -term is uniformly negligible in $A \in \mathbb{S}^{d-1}$. Moreover, we have that for every $A \in \mathbb{S}^{d-1}$, the leading term in (3.3) is $\Lambda(A)$ by (2.5) and

$$\begin{aligned} \frac{h(A)}{\mathbb{E}\{|\Sigma^{1/2}\mathbf{W}|^\alpha\}} - \frac{h(\mathbb{S}^{d-1})\mathbb{E}\{|\Sigma^{1/2}\mathbf{W}|^\alpha \mathbf{I}_A\}}{(\mathbb{E}\{|\Sigma^{1/2}\mathbf{W}|^\alpha\})^2} &= a_2 \left\{ \rho \Lambda_{(\gamma)}(A) - \Lambda(A) - (\rho - 1)\Lambda(A) \right\} \\ &= a_2 \rho \left\{ \Lambda_{(\gamma)}(A) - \Lambda(A) \right\} = \Gamma(A). \end{aligned}$$

This proves (ii). \square

Theorem 2 is a strong result. Actually, we are satisfied with the second behavior of $\Lambda_t(A) - \Lambda(A)$ as $t \rightarrow 0$ while A runs over a subclass of \mathbb{S}^{d-1} rather than the entire. For bounded measurable function $f(s)$ defined on \mathbb{S}^{d-1} with $\int_{s \in \mathbb{S}^{d-1}} f(s) d\Lambda(s) = 0$, let

$$A_f(y) = \{s \in \mathbb{S}^{d-1} : f(s) > y\} \quad \text{and} \quad G_f(y) = \Gamma(A_f(y)) \quad \text{for } y \in \mathbb{R}.$$

The following corollary is readily established.

Corollary 1. *Under the same conditions of Theorem 2, we have the followings:*

(i) *For bounded measurable function $f(s)$ defined on \mathbb{S}^{d-1} ,*

$$\Lambda_t(A_f(y)) - \Lambda(A_f(y)) = t^\gamma G_f(y) + t^\gamma o(1) \quad \text{as } t \rightarrow 0,$$

where the $o(1)$ -term is uniformly negligible in $y \in \mathbb{R}$;

(ii) *Moreover, if $\int_{s \in \mathbb{S}^{d-1}} f(s) d\Lambda(s) = 0$, then, $\int_{-\infty}^{\infty} G_f(y) dy = a_2 \rho \int_{s \in \mathbb{S}^{d-1}} f(s) d\Lambda_{(\gamma)}(s)$.*

Proof: Since (i) is readily established by (ii) of Theorem 2, it suffices to verify (ii). Note that $G_f(y)$ is integrable since it is bounded and $G_f(y)$ vanishes outside a compact interval due to the boundedness of $f(s)$. Thus, we have

$$\int_{-\infty}^{\infty} G_f(y) dy = a_2 \rho \int_{-\infty}^{\infty} \left\{ \Lambda_{(\gamma)}(A_f(y)) - \Lambda(A_f(y)) \right\} dy.$$

Moreover, letting $A_f^c(y) = \mathbb{S}^{d-1} \setminus A_f(y) = \{s \in \mathbb{S}^{d-1} : f(s) \leq y\}$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\{ \Lambda_{(\gamma)}(A_f(y)) - \Lambda(A_f(y)) \right\} dy \\ &= \int_0^{\infty} \left\{ \Lambda_{(\gamma)}(A_f(y)) - \Lambda(A_f(y)) \right\} dy + \int_{-\infty}^0 \left\{ \Lambda_{(\gamma)}(A_f(y)) - \Lambda(A_f(y)) \right\} dy \\ &= \int_0^{\infty} \left\{ \Lambda_{(\gamma)}(A_f(y)) - \Lambda(A_f(y)) \right\} dy + \int_{-\infty}^0 \left\{ \Lambda(A_f^c(y)) - \Lambda_{(\gamma)}(A_f^c(y)) \right\} dy \\ &= \int_0^{\infty} \Lambda_{(\gamma)}(A_f(y)) dy - \int_{-\infty}^0 \Lambda_{(\gamma)}(A_f^c(y)) dy - \left[\int_0^{\infty} \Lambda(A_f(y)) dy - \int_{-\infty}^0 \Lambda(A_f^c(y)) dy \right] \\ &= \int_{s \in \mathbb{S}^{d-1}} f(s) d\Lambda_{(\gamma)}(s), \end{aligned}$$

where the last equation holds due to the fact that $\int_{s \in \mathbb{S}^{d-1}} f(s) d\Lambda(s) = 0$. This completes the proof. \square

Remark 1. The typical choice of f in Corollary 1 is the partial derivatives of $\log \lambda(\mathbf{s})$ with respect to parameters representing Σ and α . Investigating the asymptotic property of Σ -estimator, we encounter the integral $\int_{\mathbf{s} \in \mathbb{S}^{d-1}} f(\mathbf{s}) d\Lambda_t(\mathbf{s})$ for $t > 0$, which represents the expected value of score function. Furthermore,

$$\lim_{t \rightarrow 0} t^{-\gamma} \left(\int_{\mathbf{s} \in \mathbb{S}^{d-1}} f(\mathbf{s}) d\Lambda_t(\mathbf{s}) - \int_{\mathbf{s} \in \mathbb{S}^{d-1}} f(\mathbf{s}) d\Lambda(\mathbf{s}) \right)$$

is related to the bias. According to Corollary 1, the limit is $a_2 \rho \int_{\mathbf{s} \in \mathbb{S}^{d-1}} f(\mathbf{s}) d\Lambda_{(\gamma)}(\mathbf{s})$ under (2.4) and (3.1).

Remark 2. An example related to Theorem 2 is multivariate t -distribution: let $\mathbf{Z} = (Z_1, \dots, Z_d)'$, where Z_1, \dots, Z_d are independent standard normal random variables, Σ be a positive definite and symmetric $d \times d$ matrix, and U follow χ^2 -distribution with degree of freedom $\nu > 0$. Then, if \mathbf{Z} and U are independent, then, $\mathbf{Z}/|\mathbf{Z}|$ is independent of $|\mathbf{Z}|/\sqrt{U/\nu}$ and thus

$$\frac{\Sigma^{1/2} \mathbf{Z}}{\sqrt{U/\nu}} = \frac{|\mathbf{Z}|}{\sqrt{U/\nu}} \Sigma^{1/2} \frac{\mathbf{Z}}{|\mathbf{Z}|},$$

satisfies (2.4) with $V = |\mathbf{Z}|/\sqrt{U/\nu}$. Moreover, V^2 follows an F -distribution with 1 and ν degrees of freedom whose a probability density function is

$$\begin{aligned} x &\mapsto \frac{x^{-1/2}(1+x/\nu)^{-(1+\nu)/2}}{\nu^{1/2} \text{Beta}(1/2, \nu/2)} \quad (x > 0) \\ &= \frac{1}{\nu^{(2+\nu)/2} \text{Beta}(1/2, \nu/2)} x^{-(2+\nu)/2} \left\{ 1 - \frac{\nu(1+\nu)}{2} \frac{1}{x} + \frac{o(1)}{x} \right\} \quad \text{as } x \rightarrow \infty, \end{aligned}$$

where Beta indicates the beta function. By integrating the above function we can check that (3.1) holds with

$$\alpha = \nu, \quad a_1 = \frac{2\nu^{-(4+\nu)/2}}{\text{Beta}(1/2, \nu/2)}, \quad a_2 = \frac{\nu^2(1+\nu)}{2(2+\nu)}, \quad \gamma = 2.$$

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