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On the Bayes risk of a sequential design for estimating a mean difference

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Abstract

The problem addressed is that of sequentially estimating the difference between the means of two populations with respect to the squared error loss, where each population distribution is a member of the one-parameter exponential family. A Bayesian approach is adopted in which the population means are estimated by the posterior means at each stage of the sampling process and the prior distributions are not specified but have twice continuously differentiable density functions. The main result determines an asymptotic second-order lower bound, as $t \to \infty$, for the Bayes risk of a sequential procedure that takes M observations from the first population and $t - M$ from the second population, where M is determined according to a sequential design, and t denotes the total number of observations sampled from both populations.

Keywords: Bayes risk, Fatou's lemma, the martingale convergence theorem, one-parameter exponential family, sequential design, squared error loss, uniform integrability

1. Introduction

Let Ω denote an open interval and let $F_\theta, \theta \in \Omega$, denote a one-parameter exponential family of probability distributions; that is, for each $\theta \in \Omega$,

$$dF_\theta(x) = \exp\{\theta x - \psi(\theta)\}\, d\lambda(x) \quad \text{for} \quad -\infty < x < \infty,$$

where ψ is a twice continuously differentiable function on Ω and λ is a non-degenerate sigma-finite measure on the Borel sets of $(-\infty, \infty)$. It is well known that if X is a random variable with a distribution F_θ, then the mean and variance of X are $\psi'(\theta)$ and $\psi''(\theta)$, respectively (Lehmann, 1959).

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote populations with independent distributions F_{θ_1} and F_{θ_2}, where $\theta_1, \theta_2 \in \Omega$ are unknown. A total of t observations are to be taken from the two populations, and the objective of the study is to estimate the mean difference $\psi'(\theta_1) - \psi'(\theta_2)$ with respect to the squared error loss, using a Bayesian approach.

Let $X_1, X_2, \ldots$ denote observations sampled from the first population, $\mathcal{P}_1$, and let $Y_1, Y_2, \ldots$ denote observations from the second population, $\mathcal{P}_2$. In the Bayesian framework, it is assumed that $X_1, X_2, \ldots$ are conditionally independent sharing a common distribution F_{θ_1}, given $\Theta_1 = \theta_1$. Similarly, $Y_1, Y_2, \ldots$ are presumed to be conditionally independent with a common distribution F_{θ_2}, given $\Theta_2 = \theta_2$. Additionally, $X_1, X_2, \ldots$ are conditionally independent of $Y_1, Y_2, \ldots$, given $\Theta_1 = \theta_1$ and $\Theta_2 = \theta_2$; and that Θ_1 and Θ_2 are independent random variables with respective prior density functions ξ_1 and ξ_2.

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For $m \geq 1$ and $n \geq 1$, let $\mathcal{F}_{m,n}$ indicate the sigma-algebra generated by $X_1, \ldots, X_m$, and $Y_1, \ldots, Y_n$. Then, $\mathcal{D}$ denotes a sequential design defined as a sequence of indicators $D_1, \ldots, D_t$, where $D_k = 0$ if the k^{th} value is sampled from $\mathcal{P}_2$ and $D_k = 1$ if the k^{th} value is from $\mathcal{P}_1$. The constants, D_1 and D_2, satisfy $D_1 + D_2 = 1$ where D_k is $\mathcal{F}_{m_k,n_k}$-measurable for $k = 3, \ldots, t$ with $m_k = D_1 + \cdots + D_k$ and $n_k = k - m_k$ for $k = 1, \ldots, t$. In the remainder of this paper, we denote m_t and n_t by M and N and $\mathcal{F}_{m_t,n_t}$ by $\mathcal{F}_t$.

A sequential procedure for estimating the difference between the two population means, $\mu(\theta_1, \theta_2) = \psi'(\theta_1) - \psi'(\theta_2)$, is the pair $(\mathcal{D}, \hat{\mu}_t)$, where $\mathcal{D}$ is the sequential design defined above and $\hat{\mu}_t = E\{\psi'(\Theta_1) - \psi'(\Theta_2)|\mathcal{F}_t\}$. The Bayes risk incurred by the sequential procedure $\mathcal{P} = (\mathcal{D}, \hat{\mu}_t)$ is defined as

$$R_t(\mathcal{P}) = E\left[(\hat{\mu}_t - \mu(\Theta_1, \Theta_2))^2\right]. \tag{1.1}$$

The problem considered is to find a sequential design for which the risk is minimal and to derive its optimality. To this end, Woodroofe and Hardwick (1990) devise a quasi-Bayesian approach to derive an asymptotic lower bound for the integrated risk and propose a three-stage procedure for two normal distributions with a unit variance. For non-linear estimation, Shapiro (1985) adopts an allocation strategy that has shown that the myopic rule is asymptotically optimal. In this context, Rekab (1989, 1992) derives a first-order sequential procedure and asymptotic lower bound for the Bayes risk, and Benkamra *et al.* (2015) further derive a nearly second-order asymptotically optimal three-stage design. Song and Rekab (2017) extend the former approaches with a three-stage design to obtain first-order efficiency.

In order to yield a minimal error in estimating the function of the parameters from two populations, obtaining a lower bound contributes to a more refined approximation. However, obtaining a closed-form expression of an exact lower bound for the Bayes risk, particularly in the absence of explicitly specified prior density functions, is notably challenging. Rekab (1990) derives the first-order Bayes risk lower bound for the difference between the means with the conjugate priors. Rekab and Tahir (2004) further extend to a second-order lower bound for the Bayes risk. However, their work specifies the priors as a conjugate, in which forms are provided by Diaconis and Ylvistaker (1979).

Now consider the difference between the two populations from the one-parameter exponential family with the Bayes risk. When the conjugate prior is known, the lower bound of the Bayes risk is as follows:

$$\mathcal{R}_t(\mathcal{P}) \geq \frac{E\left[(\sqrt{\psi''(\Theta_1)} + \sqrt{\psi''(\Theta_2)})^2\right]}{t} + o\left(\frac{1}{t}\right)$$

as $t \to \infty$.

The objective of this study is, without an explicit assumption on the conjugate priors as such provided by Diaconis and Ylvisaker (1979), to establish an asymptotic second-order lower bound for the Bayes risk. This result extends the lower bound result of Woodroofe and Hardwick (1990) and further generalizes Rekab (1990), obtained for the difference between the means of two normal populations with unit variance, using the difference of the sample means instead of the Bayes estimator.

The paper is organized as follows. In Section 2, we further describe the preliminary notations and present the main result with an example. In Section 3, the proof of the main result is provided with lemmas and remarks. In Section 4, we illustrate the implementation of the main result with a numerical simulation showcasing the performance of the Bayes risk lower bound. Section 5 concludes with some remarks on the main result and the future direction.

2. An asymptotic second-order lower bound

Let $R_t(\mathcal{P})$ be as in (1.1). Then, it follows that

$$R_t(\mathcal{P}) = E\left[\mathrm{Var}\{\psi'(\Theta_1) \mid X_1,\ldots,X_M\} + \mathrm{Var}\{\psi'(\Theta_2) \mid Y_1,\ldots,Y_N\}\right].$$

Furthermore, Lemma A.2 (see Appendix) shows that

$$\begin{aligned}\mathrm{Var}\{\psi'(\Theta_1) \mid X_1,\ldots,X_M\} = {} & E\{(\psi'(\Theta_1) - \bar{X}_M)^2 \mid X_1,\ldots,X_M\} \\ & - \frac{1}{M^2}\left(E\{\alpha_1(\Theta_1) \mid X_1,\ldots,X_M\}\right)^2,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}\{\psi'(\Theta_2) \mid Y_1,\ldots,Y_N\} = {} & E\{(\psi'(\Theta_2) - \bar{Y}_N)^2 \mid Y_1,\ldots,Y_N\} \\ & - \frac{1}{N^2}(E\{\alpha_2(\Theta_2) \mid Y_1,\ldots,Y_N\})^2,\end{aligned}$$

where for $i = 1, 2$,

$$\alpha_i(\theta) = \begin{cases} \frac{\xi_i'(\theta)}{\xi_i(\theta),} & \text{if } \theta \in \{\theta \in \Omega : \xi_i(\theta) > 0\} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover, if ξ_1 and ξ_2 have compact supports in Ω, it follows from Woodroofe (1985) that

$$\begin{aligned}E\{(\psi'(\Theta_1) - \bar{X}_M)^2 \mid X_1,\ldots,X_M\} = {} & \frac{1}{M}E\{\psi''(\Theta_1) \mid X_1,\ldots,X_M\} \\ & + \frac{1}{M^2}E\{\beta_1(\Theta_1) \mid X_1,\ldots,X_M\},\end{aligned}$$

and

$$\begin{aligned}E\{(\psi'(\Theta_2) - \bar{Y}_N)^2 \mid Y_1,\ldots,Y_N\} = {} & \frac{1}{N}E\{\psi''(\Theta_2) \mid Y_1,\ldots,Y_N\} \\ & + \frac{1}{N^2}E\{\beta_2(\Theta_2) \mid Y_1,\ldots,Y_N\},\end{aligned}$$

where for $i = 1, 2$,

$$\beta_i(\theta) = \begin{cases} \frac{\xi_i''(\theta)}{\xi_i(\theta),} & \text{if } \theta \in \{\theta \in \Omega : \xi_i(\theta) > 0\} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Hence, the Bayes risk becomes the following.

$$\mathcal{R}_t(\mathcal{P}) = E\left[\frac{U_M}{M} + \frac{V_N}{N}\right] + E\left[\frac{A_M}{M^2}\right] + E\left[\frac{B_N}{N^2}\right], \tag{2.3}$$

where

$$U_M = E\{\psi''(\Theta_1) \mid X_1, \ldots, X_M\}, \quad V_N = E\{\psi''(\Theta_2) \mid Y_1, \ldots, Y_N\},$$
$$A_M = E\{\beta_1(\Theta_1) \mid X_1, \ldots, X_M\} - [E\{\alpha_1(\Theta_1)| X_1, \ldots, X_M\}]^2,$$
$$B_N = E\{\beta_2(\Theta_2)| Y_1, \ldots, Y_N\} - [E\{\alpha_2(\Theta_2)| Y_1, \ldots, Y_N\}]^2.$$

Next,

$$E\left[\frac{U_M}{M} + \frac{V_N}{N}\right] = \frac{1}{t}E\left[\left(\sqrt{U_M} + \sqrt{V_N}\right)^2\right] + \frac{1}{t}E\left[\frac{(N\sqrt{U_M} - M\sqrt{V_N})^2}{MN}\right].$$

Thus, (2.3) becomes

$$\mathcal{R}_t(\mathcal{P}) = \frac{1}{t}E\left[U_M + V_N + 2\sqrt{U_M V_N}\right] + \frac{1}{t}E\left[\frac{(N\sqrt{U_M} - M\sqrt{V_N})^2}{MN}\right] + E\left[\frac{A_M}{M^2}\right] + E\left[\frac{B_N}{N^2}\right]. \tag{2.4}$$

In the remainder of this paper,

$$C(\theta_1, \theta_2) = \frac{\sqrt{\psi''(\theta_1)}}{\sqrt{\psi''(\theta_1)} + \sqrt{\psi''(\theta_2)}}.$$

For the following main result, the simple regularity conditions, as in Woodroofe (1985), are assumed.

Theorem 1. *If Θ_1 and Θ_2 have compact supports in Ω, then for any sequential procedure $\mathcal{P} = (\mathcal{D}, \hat{\mu}_t)$ such that*

$$\frac{M}{t} \to C(\Theta_1, \Theta_2) \quad \textit{w.p.1 as } t \to \infty \tag{2.5}$$

then, the Bayes risk of $\mathcal{P}$ satisfies the following asymptotic lower bound:

$$\liminf_{t\to\infty}\left(t^2\mathcal{R}_t(\mathcal{P}) - tE\left[\left(\sqrt{\psi''(\Theta_1)} + \sqrt{\psi''(\Theta_2)}\right)^2\right]\right) \geq$$
$$E\left[\frac{[\gamma'(\psi'(\Theta_1))]^2\psi''(\Theta_1)\psi''(\Theta_2)}{C(\Theta_1, \Theta_2)}\right] + E\left[\frac{[\gamma'(\psi'(\Theta_2))]^2\psi''(\Theta_1)\psi''(\Theta_2)}{C(\Theta_2, \Theta_1)}\right] +$$
$$E\left[\frac{\beta_1(\Theta_1) - [\alpha_1(\Theta_1)]^2}{[C(\Theta_1, \Theta_2)]^2}\right] + E\left[\frac{\beta_2(\Theta_2) - [\alpha_2(\Theta_2)]^2}{[C(\Theta_2, \Theta_1)]^2}\right],$$

where $\gamma(\eta) = \sqrt{\psi''(g(\eta))}$ with g being the inverse of ψ' and $\alpha_1(\theta)$, $\alpha_2(\theta)$, $\beta_1(\theta)$ and $\beta_2(\theta)$ are defined by (2.1) *and* (2.2)*.*

The proof of Theorem 1 hinges on lemmas provided in Section 3.

Example 1. Suppose that F_θ is the exponential distribution with mean $|\theta|^{-1}$, where $\theta \in \Omega = (-\infty, 0)$ and that Θ_i has p.d.f.

$$\xi_i(\theta) = \frac{s_i^{r_i}}{\Gamma(r_i)}|\theta|^{r_i - 1}e^{-s_i|\theta|} \quad \text{for } -\infty < \theta < 0,$$

where r_i and s_i are given positive real numbers. Then,

$$\psi(\theta) = -ln(-\theta), \quad \psi'(\theta) = -\frac{1}{\theta}, \quad \psi''(\theta) = \frac{1}{\theta^2}, \quad \gamma(\eta) = \eta$$
$$\alpha_i(\theta) = s_i - \frac{r_i - 1}{|\theta|} \quad \text{and} \quad \beta_i(\theta) = s_i^2 - \frac{2s_i(r_i-1)}{|\theta|} + \frac{r_i^2 - 3r_i + 2}{\theta^2}.$$

Using the fact that, for $i = 1, 2$,

$$E\left[[\psi''(\Theta_i)]^p\right] = \frac{s_i^{2p}\Gamma(r_i - 2p)}{\Gamma(r_i)}$$

for any $p > 0$, yields

$$\mathcal{R}_t(\mathcal{P}) \ \geq \ \frac{L_1}{t} + \frac{L_2 + L_3}{t^2}$$

for sufficiently large t, provided that $r_1 > 3$ and $r_2 > 3$, where

$$L_1 = \frac{s_1^2}{(r_1-1)(r_1-2)} + \frac{s_2^2}{(r_2-1)(r_2-2)} + \frac{2r_1r_2}{(r_1-1)(r_2-2)}$$
$$L_2 = \frac{s_1 s_2^2}{(r_1-1)(r_2-2)(r_2-3)} + \frac{s_2^3}{(r_1-1)(r_2-2)(r_2-3)}$$
$$+ \frac{s_1^2 s_2^2}{(r_1-1)(r_2-2)(r_2-1)^2} + \frac{s_1^3 s_2}{(r_1-1)(r_1-2)(r_1-3)(r_2-1)}$$
$$L_3 = 1 - \frac{3r_1 - 5}{r_1 - 2} + \frac{r_2 s_1^2(1 - s_2^2)}{(r_1-1)(r_1-2)s_2} - \frac{(r_2-1)s_1^2}{(r_1-1)(r_2-2)}.$$

3. Proof of Theorem 1

The following lemmas are needed for the proof of the theorem.

Lemma 1. *Let $\gamma(\eta)$ be as in the statement of Theorem 1. Then,*

$$Var\left\{\sqrt{\psi''(\Theta_1)} \mid X_1, \ldots, X_M\right\} \leq \frac{1}{M} E\left\{[\gamma'(\psi'(\Theta_1))]^2 \psi''(\Theta_1) \mid X_1, \ldots, X_M\right\}$$
$$+ \frac{1}{M} E\left\{\frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M}\alpha_1(\Theta_1) \mid X_1, \ldots, X_M\right\},$$

and

$$Var\left\{\sqrt{\psi''(\Theta_2)} \mid Y_1, \ldots, Y_N\right\} \leq \frac{1}{N} E\left\{[\gamma'(\psi'(\Theta_2))]^2 \psi''(\Theta_2) \mid Y_1, \ldots, Y_N\right\}$$
$$+ \frac{1}{N} E\left\{\frac{[\gamma(\psi'(\Theta_2)) - \gamma(\bar{Y}_N)]^2}{\psi'(\Theta_2) - \bar{Y}_N}\alpha_2(\Theta_2) \mid Y_1, \ldots, Y_N\right\}$$

w.p.1.

Proof: Let $m \geq 1$ be a value of M, and let $\mathbf{x} = (x_1, \ldots, x_m)$, where x_i is a value of X_i. Also, let $\overline{x}_m = (x_1 + \cdots + x_m)/m$ denote the value of $\overline{X}_M$. Then,

$$\begin{aligned}\mathrm{Var}\left\{ \sqrt{\psi''(\Theta_1)} \mid M = m, \mathbf{X} = \mathbf{x} \right\} &= \mathrm{Var}\left\{\gamma(\psi'(\Theta_1)) \mid M = m, \mathbf{X} = \mathbf{x}\right\} \\ &\leq \frac{1}{c_m} \int \left[\gamma(\psi'(\theta_1)) - \gamma(\bar{x}_m)\right]^2 L_m(\theta_1)\xi_1(\theta_1)d\theta_1 \\ &= -\frac{1}{mc_m} \int \frac{[\gamma(\psi'(\theta_1)) - \gamma(\bar{x}_m)]^2}{\psi'(\theta_1) - \bar{x}_m} L'_m(\theta_1)\xi_1(\theta_1)d\theta_1,\end{aligned}$$

where

$$L_m(\theta_1) = \exp\left\{m\theta_1\bar{x}_m - m\psi(\theta_1)\right\} \quad \text{and} \quad c_m = \int L_m(\theta_1)\xi_1(\theta_1)d\theta_1.$$

Let $\eta = \psi'(\theta_1)$. Then, $\theta_1 = g(\eta)$ and $d\theta_1 = g'(\eta)d\eta$; so that

$$\begin{aligned}\mathrm{Var}\left\{ \sqrt{\psi''(\Theta_1)} \mid M = m, \mathbf{X} = \mathbf{x} \right\} &\leq \frac{-1}{mc_m} \int \frac{[\gamma(\eta) - \gamma(\bar{x}_m)]^2}{\eta - \bar{x}_m} L'_m(g(\eta))\xi_1(g(\eta))g'(\eta)d\eta, \\ &= \frac{1}{mc_m} \int \frac{d}{d\eta}\left[\frac{[\gamma(\eta) - \gamma(\bar{x}_m)]^2}{\eta - \bar{x}_m} \xi_1(g(\eta)) \right] L_m(g(\eta))d\eta\end{aligned}$$

by performing integration by parts. Next,

$$\begin{aligned}\frac{d}{d\eta}\left[\frac{[\gamma(\eta) - \gamma(\bar{x}_m)]^2}{\eta - \bar{x}_m} \right] &= 2\gamma'(\eta)\frac{\gamma(\eta) - \gamma(\bar{x}_m)}{\eta - \bar{x}_m} - \left[\frac{\gamma(\eta) - \gamma(\bar{x}_m)}{\eta - \bar{x}_m} \right]^2 \\ &= -\left[\gamma'(\eta) - \frac{\gamma(\eta) - \gamma(\bar{x}_m)}{\eta - \bar{x}_m} \right]^2 + [\gamma'(\eta)]^2 \\ &\leq [\gamma'(\eta)]^2 .\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Var}\left\{ \sqrt{\psi''(\Theta_1)} \mid M = m, \mathbf{X} = \mathbf{x} \right\} &\leq \frac{1}{mc_m} \int [\gamma'(\eta)]^2 L_m(g(\eta))\xi_1(g(\eta))d\eta \\ &\quad + \frac{1}{mc_m} \int \frac{[\gamma(\eta) - \gamma(\bar{x}_m)]^2}{\eta - \bar{x}_m} \xi'_1(g(\eta))g'(\eta)L_m(g(\eta))d\eta \\ &= \frac{1}{mc_m} \int [\gamma'(\psi'(\theta_1))]^2 L_m(\theta_1)\xi_1(\theta_1)\psi''(\theta_1)d\theta_1 \\ &\quad + \frac{1}{mc_m} \int \frac{[\gamma(\psi'(\theta_1)) - \gamma(\bar{x}_m)]^2}{\psi'(\theta_1) - \bar{x}_m} \xi'_1(\theta_1)L_m(\theta_1)d\theta_1 \\ &= \frac{1}{m} E\left\{ [\gamma'(\psi'(\Theta_1))]^2 \psi''(\Theta_1) \mid M = m, \mathbf{X} = \mathbf{x} \right\} \\ &\quad + \frac{1}{m} E\left\{ \frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_m)]^2}{\psi'(\Theta_1) - \bar{X}_m} \alpha_1(\Theta_1) \middle| M = m, \mathbf{X} = \mathbf{x} \right\}.\end{aligned}$$

The first assertion of the lemma follows. A parallel argument leads to the subsequent assertion. □

Lemma 2. *Let $\gamma(\eta)$ be as in the statement of Theorem 1.*

$$Var\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)} \mid \mathcal{F}_t\} \le$$
$$\frac{1}{M}E\left\{[\gamma'(\psi'(\Theta_1))]^2\psi''(\Theta_1)\psi''(\Theta_2) + \frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M}\alpha_1(\Theta_1)\psi''(\Theta_2) \mid \mathcal{F}_t\right\}$$
$$+ \frac{1}{N}E\left\{[\gamma'(\psi'(\Theta_2))]^2\psi''(\Theta_1)\psi''(\Theta_2) + \frac{[\gamma(\psi'(\Theta_2)) - \gamma(\bar{Y}_N)]^2}{\psi'(\Theta_2) - \bar{Y}_N}\alpha_2(\Theta_2)\psi''(\Theta_1) \mid \mathcal{F}_t\right\}$$

w.p.1.

Proof:

$$\begin{aligned}\mathrm{Var}\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)} \mid \mathcal{F}_t\} &= E\{\psi''(\Theta_1)\psi''(\Theta_2) \mid \mathcal{F}_t\} - [E\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)} \mid \mathcal{F}_t\}]^2\\ &= (E\{\psi''(\Theta_1) \mid \mathcal{F}_t\} - [E\{\sqrt{\psi''(\Theta_1)}|\mathcal{F}_t\}]^2)E\{\psi''(\Theta_2)|\mathcal{F}_t\}\\ &\quad + (E\{\psi''(\Theta_2)|\mathcal{F}_t\} - [E\{\sqrt{\psi''(\Theta_2)}|\mathcal{F}_t\}]^2)[E\{\sqrt{\psi''(\Theta_1)}|\mathcal{F}_t\}]^2\\ &\le \mathrm{Var}\{\sqrt{\psi''(\Theta_1)} \mid X_1,\dots,X_M\}E\{\psi''(\Theta_2) \mid Y_1,\dots,Y_N\}\\ &\quad + \mathrm{Var}\{\sqrt{\psi''(\Theta_2)} \mid Y_1,\dots,Y_N\}E\{\psi''(\Theta_1) \mid X_1,\dots,X_M\}.\end{aligned}$$

Now, use Lemma 1 to complete the proof. □

Lemma 3. *For any sequential procedure $\mathcal{P} = (\mathcal{D}, \hat{\mu}_t)$ that satisfies Condition* (2.5)

$$tVar\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)}|\mathcal{F}_t\} \to [C(\Theta_1,\Theta_2)]^{-1}[\gamma'(\psi'(\Theta_1))]^2\psi''(\Theta_1)\psi''(\Theta_2) + [C(\Theta_2,\Theta_1)]^{-1}[\gamma'(\psi'(\Theta_2))]^2\psi''(\Theta_1)\psi''(\Theta_2)$$

w.p.l. as $t \to \infty$.

Proof: Since

$$\frac{1}{M}E\left\{\frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M}\alpha_1(\Theta_1) \,\middle|\, X_1,\dots,X_M\right\} \to 0,$$

and

$$\frac{1}{N}E\left\{\frac{[\gamma(\psi'(\Theta_2)) - \gamma(\bar{Y}_N)]^2}{\psi'(\Theta_2) - \bar{Y}_N}\alpha_2(\Theta_2) \,\middle|\, Y_1,\dots,Y_N\right\} \to 0$$

w.p.l as $t \to \infty$, by Lemma A.3, it follows from Lemma 2 that

$$\begin{aligned}&\limsup_{t\to\infty} \mathrm{tVar}\left\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)}\,|\mathcal{F}_t\right\}\\ &\le \limsup_{t\to\infty}\frac{t}{M}E\left\{[\gamma'(\psi'(\Theta_1))]^2\,\psi''(\Theta_1)\,\middle|\, X_1,\dots,X_M\right\}E\{\psi''(\Theta_2) \mid Y_1,\dots,Y_N\}\\ &+ \limsup_{t\to\infty}\frac{t}{N}E\left\{[\gamma'(\psi'(\Theta_2))]^2\,\psi''(\Theta_2)\,\middle|\, Y_1,\dots,Y_N\right\}E\{\psi''(\Theta_1) \mid X_1,\dots,X_M\}.\end{aligned}$$

Moreover,

$$\frac{t}{M}E\left\{[\gamma'(\psi'(\Theta_1))]^2\,\psi''(\Theta_1)\mid X_1,\ldots,X_M\right\}\to\frac{1}{C(\Theta_1,\Theta_2)}\,[\gamma'(\psi'(\Theta_1))]^2\,\psi''(\Theta_1),$$

and

$$\frac{t}{N}E\left\{[\gamma'(\psi'(\Theta_2))]^2\,\psi''(\Theta_2)\mid Y_1,\ldots,Y_N\right\}\to\frac{1}{C(\Theta_2,\Theta_1)}\,[\gamma'(\psi'(\Theta_2))]^2\,\psi''(\Theta_2)$$

w.p.l. as $t\to\infty$, by Condition (2.5) of the theorem and Lemma A.4. Also,

$$E\{\psi''(\Theta_1)\mid X_1,\ldots,X_M\}\to\psi''(\Theta_1)\quad\text{and}\quad E\{\psi''(\Theta_2)\mid Y_1,\ldots,Y_N\}\to\psi''(\Theta_2)$$

w.p.l. as $t\to\infty$, by Lemma A.4. Combining these results yield

$$\limsup_{t\to\infty}\mathrm{tVar}\left\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)}\mid\mathcal{F}_t\right\}\leq[C(\Theta_1,\Theta_2)]^{-1}\,[\gamma'(\psi'(\Theta_1))]^2\,\psi''(\Theta_1)\psi''(\Theta_2)+$$
$$[C(\Theta_2,\Theta_1)]^{-1}\,[\gamma'(\psi'(\Theta_2))]^2\,\psi''(\Theta_1)\psi''(\Theta_2).$$

To establish the reverse inequality, first, write

$$\begin{aligned}\mathrm{tVar}\left\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)}|\mathcal{F}_t\right\}=&\frac{t}{M}\mathrm{Var}\left\{\sqrt{M\psi''(\Theta_1)}\mid X_1,\ldots,X_M\right\}(E\{\sqrt{\psi''(\Theta_2)}\mid Y_1,\ldots,Y_N\})^2\\&+\frac{t}{N}\mathrm{Var}\{\sqrt{N\psi''(\Theta_2)}|\,Y_1,\ldots,Y_N\}(E\{\sqrt{\psi''(\Theta_1)}|\,X_1,\ldots,X_M\})^2,\end{aligned}\tag{3.1}$$

as in the proof of Lemma 2. Next, use Taylor's expansion for $\gamma\circ\psi'$ at $\hat{\Theta}_M=E\{\Theta_1|X_1,\ldots,X_M\}$ to obtain

$$\sqrt{\psi''(\Theta_1)}=\gamma\circ\psi'(\Theta_1)=\gamma(\psi'(\hat{\Theta}_M))+\gamma'(\psi'(\Theta_M^*))\psi''(\Theta_M^*)(\Theta_1-\hat{\Theta}_M),$$

where Θ_M^* is an intermediate variable between Θ_1 and $\hat{\Theta}_M$. It follows that

$$\begin{aligned}&\frac{t}{M}\mathrm{Var}\left\{\sqrt{M}\,\sqrt{\psi''(\Theta_1)}\mid X_1,\ldots,X_M\right\}=\\&\frac{t}{M}\frac{1}{\psi''(\hat{\Theta}_M)}\mathrm{Var}\left\{\gamma'(\psi'(\Theta_M^*))\psi''(\Theta_M^*)\sqrt{M\psi''(\hat{\Theta}_M)}(\Theta_1-\hat{\Theta}_M)\middle|\,X_1,\ldots,X_M\right\}\end{aligned}$$

Thus,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{t}{M}\mathrm{Var}\{\sqrt{M\psi''(\Theta_1)}\mid X_1,\ldots,X_M\}\quad E\{\psi''(\Theta_2)\mid Y_1,\ldots,Y_N\}\geq\\&\frac{1}{C(\Theta_1,\Theta_2)}\psi''(\Theta_1)[\gamma'(\psi'(\Theta_1))]^2\psi''(\Theta_2)\end{aligned}\tag{3.2}$$

w.p.1, by first using Fatou's lemma, then Condition (2.5) of the theorem, the fact that $\psi''(\hat{\Theta}_M)\to\psi''(\Theta_1)$ w.p.1, the fact that

$$\mathrm{Var}\left\{\gamma'(\psi'(\Theta_M^*))\psi''(\Theta_M^*)\sqrt{M\psi''(\hat{\Theta}_M)}(\Theta_1-\hat{\Theta}_M)\middle|X_1,\ldots,X_M\right\}\;\to\;[\gamma'(\psi'(\Theta_1))]^2[\psi''(\Theta_1)]^2$$

w.p.1 as $t \to \infty$ since the posterior distribution of $\sqrt{M\psi''(\hat{\Theta}_M)}(\Theta_1 - \hat{\Theta}_M)$, given $X_1, \ldots, X_M$, is asymptotically normal with mean 0 and variance 1(see Bickel and Yahav, 1969) and the fact that $E\{\psi''(\Theta_2) \mid Y_1, \ldots, Y_N\} \to \psi''(\Theta_2)$ by Lemma A.4. A similar argument will yield

$$\liminf_{t\to\infty} \frac{t}{N} \mathrm{Var}\left\{ \sqrt{N\psi''(\Theta_2)} \mid Y_1, \ldots, Y_N \right\} \quad \left[E\left\{ \sqrt{\psi''(\Theta_1)} \mid X_1, \ldots, X_M \right\} \right]^2 \geq \frac{1}{C(\Theta_2, \Theta_1)} \psi''(\Theta_2) \left[\gamma'(\psi'(\Theta_2))\right]^2 \psi''(\Theta_1) \tag{3.3}$$

w.p.l. Now take the liminf in (3.1) and use (3.2) and (3.3) to complete the proof. □

Proof of Theorem 1: It follows from (2.4) that

$$\mathcal{R}_t(\mathcal{P}) \geq \frac{1}{t} E\left[U_M + V_N + 2\sqrt{U_M V_N}\right] + E\left[\frac{A_M}{M^2}\right] + E\left[\frac{B_N}{N^2}\right]. \tag{3.4}$$

Next, let $W_t = E\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)} | \mathcal{F}_t\}$ and $Z_t = Var\{\sqrt{\psi''(\Theta_1)\psi''(\Theta_2)} | \mathcal{F}_t\}$. Then, $Z_t = U_M V_N - W_t{}^2$, which implies that

$$\sqrt{U_M V_N} = W_t + \frac{Z_t}{\sqrt{U_M V_N} + W_t}.$$

Thus,

$$\begin{aligned} E\left[U_M + V_N + 2\sqrt{U_M V_N}\right] &= E[U_M] + E[V_N] + 2E[W_t] + E\left[\frac{2Z_t}{\sqrt{U_M V_N} + W_t}\right] \\ &= E\left[\left(\sqrt{\psi''(\Theta_1)} + \sqrt{\psi''(\Theta_2)}\right)^2\right] + E\left[\frac{2Z_t}{\sqrt{U_M V_N} + W_t}\right]. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$t^2\mathcal{R}_t(\mathcal{P}) - tE\left[(\sqrt{\psi''(\Theta_1)} + \sqrt{\psi''(\Theta_2)})^2\right] \geq E\left[\frac{2tZ_t}{\sqrt{U_M V_N} + W_t}\right] + E\left[\frac{t^2}{M^2} A_M\right] + E\left[\frac{t^2}{N^2} B_N\right]. \tag{3.6}$$

Furthermore, by Lemma 3 and the martingale convergence theorem,

$$\frac{2tZ_t}{\sqrt{U_M V_N} + W_t} \;\to\; \frac{[\gamma'(\psi'(\Theta_1))]^2 \sqrt{\psi''(\Theta_1)}\psi''(\Theta_2) + [\gamma'(\psi'(\Theta_2))]^2 \sqrt{\psi''(\Theta_2)}[\psi''(\Theta_1)]^{3/2}}{C(\Theta_1, \Theta_2)}$$

w.p.l. as $t \to \infty$; so that

$$\liminf_{t\to\infty} E\left\{\frac{2tZ_t}{\sqrt{U_M V_N} + W_t}\right\} \geq E\left[\frac{[\gamma'(\psi'(\Theta_1))]^2 \sqrt{\psi''(\Theta_1)}\psi''(\Theta_2) + [\gamma'(\psi'(\Theta_2))]^2[\sqrt{\psi''(\Theta_1)}]^3 \sqrt{\psi''(\Theta_2)}}{C(\Theta_1, \Theta_2)}\right], \tag{3.7}$$

Table 1: Second-order optimality of sequential design with uniform priors

t	$\mathcal{R}_t(P)$	$\mathcal{R}_t(O)$	EF
10	0.0471716	0.0458398	0.13317
20	0.0273274	0.0267399	0.23499
40	0.0148670	0.0145854	0.45066
50	0.0120676	0.0118844	0.45818
100	0.0062212	0.0061707	0.50473
200	0.0031577	0.0031458	0.47633
300	0.0021133	0.0021110	0.21545

Note. $\mathcal{R}_t(P)$ represents the Bayes risk incurred by the sequential design.
$\mathcal{R}_t(O)$ represents the Bayes risk incurred by the optimal design.

by Fatou's lemma. Finally,

$$\lim_{t\to\infty} E\left[\frac{t^2}{M^2}A_M\right] = E\left[\frac{\beta_1(\Theta_1) - [\alpha_1(\Theta_1)]^2}{[C(\Theta_1,\Theta_2)]^2}\right], \tag{3.8}$$

$$\lim_{t\to\infty} E\left[\frac{t^2}{N^2}B_N\right] = E\left[\frac{\beta_2(\Theta_2) - [\alpha_2(\Theta_2)]^2)}{[C(\Theta_2,\Theta_1)]^2}\right], \tag{3.9}$$

by Lemma A.4 (see Appendix) since $t^2/M^2 \to [C(\Theta_1,\Theta_2)]^{-2}$ w.p.1 and $t^2/N^2 \to [C(\Theta_2,\Theta_1)]^{-2}$ w.p.1 by Condition (2.5) of the theorem, $A_M \to \beta_1(\Theta_1) - [\alpha_1(\Theta_1)]^2$ w.p.1, $B_N \to \beta_2(\Theta_2) - [\alpha_2(\Theta_2)]^2$ w.p.1 by Lemma A.4, $A_M, t > 0$, and $B_N, t > 0$, are both uniformly integrable martingales, and t/M and t/N are bounded by Condition (2.5) of the theorem.

Then, the theorem follows by taking the liminf in (3.6) and using eqs. (3.7) to (3.9).

4. Numerical illustrations

In this section, we specialize the outcomes from Section 2 to Bernoulli trials, that is X_i takes values in 0, 1 for $i = 1, 2, \ldots$ with probabilities $1 - \theta_1$ and θ_1, respectively. Similarly, Y_i takes values in 0, 1 for $i = 1, 2, \ldots$ with probabilities $1 - \theta_2$ and θ_2, respectively. Here, θ_1 and θ_2 are independent random variables constrained within $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. The prior density functions of θ_1 and θ_2 are denoted as ξ_1 and ξ_2, respectively. Both ξ_1 and ξ_2 are standard uniform distributions, defined as $\xi_1(\theta_1) = 1$ for $0 \leq \theta_1 \leq 1$ and 0 otherwise, and $\xi_2(\theta_2) = 1$ for $0 \leq \theta_2 \leq 1$ and 0 otherwise.

Table 1 displays two columns of Bayes risks where $\mathcal{R}_t(P)$ and $\mathcal{R}_t(O)$ each correspond to the outcomes of the fully sequential procedure and the optimal sampling scheme as described in Rekab (1990). In Table 1, we denote EF as the rate of convergence or the excess of second-order Bayes risk as follows:

$$\text{EF} = t^2 \left\{\mathcal{R}_t(\mathcal{P}) - \mathcal{R}_t(O)\right\}.$$

The results indicate that the Bayes risks of the fully sequential procedure and optimal sampling scheme decrease with increasing t. The absolute excess of the Bayes risk comparing the fully sequential procedure to the optimal sampling scheme diminishes. Overall, the Bayes risk of the fully sequential procedure performs closely to the optimal sampling scheme in terms of the second-order excess or rate of convergence. Up to $t = 100$, the variability gain drives up the rate of convergence; however, it diminishes again as t increases. The numerical simulation confirms that the second-order excess behaves with increasing t under a non-conjugate prior for a density of one-parameter exponential families.

5. Concluding remarks

In this study, we address the problem of estimating the mean difference between two populations, $\mathcal{P}_1$ and $\mathcal{P}_2$, modeled by one-parameter exponential families of probability distributions. The objective was to estimate the difference $\psi'(\theta_1) - \psi'(\theta_2)$ using a Bayesian approach, with a focus on minimizing the Bayes risk through sequential designs. The central contribution of this work lies in establishing an asymptotic second-order lower bound for the Bayes risk without explicit assumptions on conjugate priors. By extending the results of previous works, such as Woodroofe and Hardwick (1990) and Rekab (1990), we generalized the framework beyond normal distributions with unit variance, offering a more comprehensive approach to Bayesian estimation in the context of one-parameter exponential families. The main result underscores the complexities inherent in Bayesian estimation, particularly in the absence of explicit prior specifications.

Application of the main result to the exponential distribution with nonstandard gamma prior, as well as a numerical illustration with Bernoulli distribution with a uniform prior, are given. The numerical simulation illustrates the second-order lower bound given by the full sequential design with the best optimal design for several values of the sequence size. While the study presents a fully sequential design with Bayes risk, and it is not specified for a stage-wise procedure, there are other designs that are of interest, including the two-stage design and the myopic design (see Terbeche, 2000). Furthermore, in order to refine the approximation of the Bayes risk further, it may be desirable to attain higher-order optimality (see Martinsek, 1983). Last but not least, this study can further benefit from examining the tightness of the lower bound without specifying the conjugate prior.

Appendix

Lemma A.1: *For $i = 1, 2$, let $\hat{\mu}_{in} = E\{\psi'(\Theta_i)|X_{i1}, \ldots, X_{in}\}$, where $X_{1j} = X_j$ and $X_{2j} = Y_j$. If Θ_i has a a compact support on Ω, then*

$$\hat{\mu}_{in} = \overline{X}_{in} + \frac{1}{n}\alpha_{in},$$

where $\alpha_{in} = E\{\alpha_i(\Theta_i)|X_{i1}, \ldots, X_{in}\}$.

Proof: For simplicity, the subscript "i" is omitted in the proof. Let

$$L_n(\theta) = \exp\{n\theta\bar{x}_n - n\psi(\theta)\} \quad \text{and} \quad c_n = \int L_n(\theta)\xi(\theta)d\theta,$$

where $x_1, \ldots, x_n$ are the observed values of $X_1, \ldots, X_n$. Then,

$$\begin{aligned}
\hat{\mu}_n &= E\{\psi'(\Theta)|X_1 = x_1, \ldots, X_n = x_n\} = \frac{1}{c_n}\int_\Omega \psi'(\theta)L_n(\theta)\xi(\theta)d\theta \\
&= -\frac{1}{nc_n}\int_\Omega L_n'(\theta)\xi(\theta)d\theta + \overline{x}_n = \frac{1}{nc_n}\int_\Omega L_n(\theta)\xi'(\theta)d\theta + \overline{x}_n \\
&= \frac{1}{nc_n}\int_\Omega L_n(\theta)\alpha(\theta)\xi(\theta)d\theta + \overline{x}_n = \frac{1}{n}E\{\alpha(\Theta)|X_1 = x_1, \ldots, X_n = x_n\} + \overline{x}_n
\end{aligned}$$

by using integration by parts. The lemma follows. □

Lemma A.2: *If Θ_i has a compact support, then*

$$
\begin{aligned}
Var\{\psi'(\Theta_i) \mid X_{i1}, \ldots, X_{in}\} = E\{[\psi'(\Theta_i) - \overline{X}_{in}]^2 \mid X_{i1}, \ldots, X_{in}\} \\
- \frac{1}{n^2}[E\{\alpha_i(\Theta_i)|X_{i1}, \ldots, X_{in}\}]^2,
\end{aligned}
$$

where $X_{1j} = X_j$ and $X_{2j} = Y_j$.

Proof: For simplicity, the subscript "i" is omitted in the proof. Lemma A.1 yields

$$
\psi'(\Theta) - \hat{\mu}_n = \psi'(\Theta) - \overline{X}_n - \frac{1}{n}\alpha_n.
$$

Thus,

$$
\begin{aligned}
\mathrm{Var}\{\psi'(\Theta) \mid X_1, \ldots, X_n\} &= E\{[\psi'(\Theta) - \hat{\mu}_n]^2 \mid X_1, \ldots, X_n\} \\
&= E\{[\psi'(\Theta) - \overline{X}_n]^2 \mid X_1, \ldots, X_n\} \\
&\quad - \frac{2}{n}\alpha_n[E\{\psi'(\Theta) \mid X_1, \ldots, X_n\} - \overline{X}_n] + \frac{1}{n^2}\alpha_n^2 \\
&= E\{[\psi'(\Theta) - \overline{X}_n]^2 \mid X_1, \ldots, X_n\} - \frac{2}{n}\alpha_n\left(\frac{1}{n}\alpha_n\right) + \frac{1}{n^2}\alpha_n^2 \\
&= E\{[\psi'(\Theta) - \overline{X}_n]^2 \mid X_1, \ldots, X_n\} - \frac{1}{n^2}\alpha_n^2.
\end{aligned}
$$

□

Lemma A.3: *Let M and N be as in Theorem 1. Then,*

$$
\frac{t}{M}E\left\{\frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M}\alpha_1(\Theta_1) \,\middle|\, X_1, \ldots, X_M\right\} \;\to\; 0,
$$

and

$$
\frac{t}{N}E\left\{\frac{[\gamma(\psi'(\Theta_2)) - \gamma(\bar{Y}_N)]^2}{\psi'(\Theta_2) - \bar{Y}_N}\alpha_2(\Theta_2) \,\middle|\, Y_1, \ldots, Y_N\right\} \;\to\; 0
$$

w.p.l as $t \to \infty$.

Proof: A simple expansion yields

$$
\gamma\left(\psi'(\Theta_1)\right) = \gamma(\overline{X}_M) + \gamma'(U_M^*)\left[\psi'(\Theta_1) - \overline{X}_M\right],
$$

where U_M^* is a random variable between $\psi'(\Theta_1)$ and $\bar{X}_M$. Combining this observation and Lemma A.1. yields

$$
\frac{t}{M}E\left\{\frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M}\alpha_1(\Theta_1) \,\middle|\, X_1, \ldots, X_M\right\} = \frac{t}{M^2}\left[\gamma'(U_M^*)\right]^2 E\left\{[\alpha_1(\Theta_1)]^2 \mid X_1, \ldots, X_M\right\}.
$$

Next, there exist positive numbers a and b such that $|\gamma'(U_M^*)| \leq a$ w.p.1 and $|\alpha_1(\Theta_1)| \leq b$ w.p.1. since γ is continuously differentiable on $\psi'(\Omega_1)$ and α_1 is continuously differentiable on Ω_1, the compact support of Θ_1. It follows from this observation that

$$\left| \frac{t}{M} E\left\{ \frac{[\gamma(\psi'(\Theta_1)) - \gamma(\bar{X}_M)]^2}{\psi'(\Theta_1) - \bar{X}_M} \alpha_1(\Theta_1) \,\middle|\, X_1, \ldots, X_M \right\} \right| \leq \frac{a^2 b^2}{t\delta^2} \to 0$$

w.p.1 as $t \to \infty$. The second assertion can be established similarly. □

Lemma A.4: *Let $h_1(\Theta_1)$ and $h_2(\Theta_2)$ be continuous functions of Θ_1 and Θ_2, respectively. Then, $E\{h_1(\Theta_1)|X_1, \ldots, X_M\}$ and $E\{h_2(\Theta_2)|Y_1, \ldots, Y_N\}$ are uniformly integrable martingales and*

$$E\{h_1(\Theta_1) \mid X_1, \ldots, X_M\} \;\to\; h_1(\Theta_1)$$
$$E\{h_2(\Theta_2) \mid Y_1, \ldots, Y_N\} \;\to\; h_2(\Theta_2)$$

w.p.1 as $t \to \infty$.

Proof. See Theorem 6.6.2 of Ash and Doleans-Dade (2000).

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