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A STUDY ON MILNE-TYPE INEQUALITIES FOR A SPECIFIC FRACTIONAL INTEGRAL OPERATOR WITH APPLICATIONS

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Abstract. Fractional integral operators have been studied extensively in the last few decades by various mathematicians, because it plays a vital role in the developments of new inequalities. The main goal of the current study is to establish some new Milne-type inequalities by using the special type of fractional integral operator i.e Caputo Fabrizio operator. Additionally, generalization of these developed Milne-type inequalities for s-convex function are also given. Furthermore, applications to some special means, quadrature formula, and q-digamma functions are presented.

1. Introduction

Recently fractional calculus became one of the most important field in applied research. The essential applications of fractional calculus can be seen in numerous fields, including numerical physical science [1], fluid mechanics [2], and biological modelling [3] etc. Inequalities play a vital role in different sciences specially in engineering. Several authors used different classes of functions to generalize various types of inequalities. Lipschitz functions were explored by Alomari [4] and compared to the generalized trapezoidal inequality . Dragomir [5] investigated bounded variation functions in relation to the trapezoid formula. Sarikaya and Aktan [6] discovered some novel inequalities of the trapezoid types. Sarikaya and Budak investigated fractional trapezoid-type inequalities [7]. For differentiable convex functions, Kirmaci et. al [8] established midpoint-type inequality. In [9], Dragomir described the findings for the functions of bounded variation. For twice differentiable functions, Sarikaya et al. [10] established numerous new inequalities. Simpson type inequality have received a lot of attention from authors for various classes of mappings. Additionally, a number of mathematicians developed Simpson-type inequalities for differentiable convex mappings [11], s-convex functions [12], extended (s, m)-convex mappings [13], bounded functions [14], and twice differentiable convex functions [15]. Numerous scholars have presented applications for fractional operators, we suggest studying the works [16,17]. For more information on fraction operator see these references [18]- [23]. We know

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that the most famous and well-known inequality, is Simpson type [24] which is followed as:

$$\left|\frac{1}{3}\left[\frac{\hbar(\mho)+\hbar(\eth)}{2}+2\hbar\left(\frac{\mho+\eth}{2}\right)-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar(\ell)\,d\ell\right]\right|\leq\frac{(\eth-\mho)^4}{2880}\left\|\hbar^4\right\|_\infty,$$

where $\hbar:[\mho,\eth]\to\mathbb{R}$ is 4-time differentiable function on $(\mho,\eth)$ and $\left\|\hbar^4\right\|_\infty=\sup_{\ell\in(\mho,\eth)}\left|\hbar^4(\ell)\right| < \infty$.

Milne's formula of open type is parallel to Simpson's formula which is of closed type, in terms of Newton-Cotes formulas, because they hold under the same conditions.

Theorem 1. *Suppose that* $\hbar:[\mho,\eth]\to\mathbb{R}$ *is 4-times continuously differentiable mapping on* $(\mho,\eth)$ *and let* $\left\|\hbar^4\right\|_\infty=\sup_{\ell\in(\mho,\eth)}\left|\hbar^4(\ell)\right|<\infty$, *then the inequality shown in [25] is given below:*

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar(\ell)\,d\ell\right|\leq\frac{7(\eth-\mho)^4}{23040}\left\|\hbar^4\right\|_\infty. \tag{1.1}$$

Definition 1. [26] Let I be a convex set on $\mathbb{R}$. If $\hbar:I\to\mathbb{R}$ is convex on I then:

$$\hbar(\omega\mho+(1-\omega)\eth)\leq\omega\hbar(\mho)+(1-\omega)\hbar(\eth),$$

for all $\mho,\eth\in I$ and $\omega\in[0,1]$.

In [27] Hudzik et. al introduced s-convex functions in the second senses.

Definition 2. Let $\hbar$ be a real valued function on $I=[0,\infty)$ and $s\in(0,1]$. Then $\hbar$ is called a s-convex in second sense if:

$$\hbar((1-\omega)\mho+\omega\eth)\leq(1-\omega)^s\hbar(\mho)+\omega^s\hbar(\eth),$$

for all $\mho,\eth\in I$ and $\omega\in[0,1]$.

The concepts of fractional operators have recently drawn the interest of several scholars. We recall the well-known fractional operators as follows.

Definition 3. [28] Let $\hbar\in L[\mho,\eth]$, the left and right-sides Riemann-Liouville fractional integrals of order $\alpha>0$ defined by:

$$\begin{aligned}I^{\alpha}_{\mho+}\hbar(\omega) &= \frac{1}{\Gamma\alpha}\int_{\mho}^{\ell}(\ell-\omega)^{\alpha-1}\hbar(\omega)\,d\omega,\ \ell>\mho\\ I^{\alpha}_{\eth-}\hbar(\omega) &= \frac{1}{\Gamma\alpha}\int_{\ell}^{\eth}(\omega-\ell)^{\alpha-1}\hbar(\omega)\,d\omega,\ \ell<\eth,\end{aligned}$$

where $\Gamma(.)$ is the gamma function and $I^{0}_{\mho+}\hbar(\omega)=I^{0}_{\eth-}\hbar(\omega)=\hbar(\omega)$.

Theorem 2. [29] *Let* $\hbar:[\mho,\eth]\to\mathbb{R}$ *be a differentiable function on* I^o, $\mho,\eth\in I^o$ *with* $\mho<\eth$, *where* $\hbar'\in L[\mho,\eth]$. *If* $|\hbar'|$ *is convex function on* $[\mho,\eth]$, *then we get:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\hbar\left(\mho\right)-\hbar\left(\frac{\mho+\eth}{2}\right)+2\hbar\left(\eth\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\eth-\mho\right)^{\alpha}}\left[J_{\mho+}^{\alpha}\hbar\left(\frac{\mho+\eth}{2}\right)+J_{\eth-}^{\alpha}\hbar\left(\frac{\mho+\eth}{2}\right)\right]\right|\\\leq\ &\frac{\eth-\mho}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(\left|\hbar'\left(\mho\right)\right|+\left|\hbar'\left(\eth\right)\right|\right).\end{aligned}$$

PROPOSITION 1. *If we take $\alpha=1$ in Theorem 2, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\hbar\left(\mho\right)-\hbar\left(\frac{\mho+\eth}{2}\right)+2\hbar\left(\eth\right)\right]-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar\left(u\right)du\right|\\\leq\ &\frac{5\left(\eth-\mho\right)}{24}\left(\left|\hbar'\left(\mho\right)\right|+\left|\hbar'\left(\eth\right)\right|\right).\end{aligned}$$

THEOREM 3. [29] *Let $\hbar:[\mho,\eth]\rightarrow\mathbb{R}$ be a differentiable function on I^o, $\mho,\eth\in I^o$ with $\mho<\eth$, where $\hbar'\in L\left[\mho,\eth\right]$. If $|\hbar'|$ is an L-Lipschitz function on $[\mho,\eth]$, then we get:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\hbar\left(\mho\right)-\hbar\left(\frac{\mho+\eth}{2}\right)+2\hbar\left(\eth\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\eth-\mho\right)^{\alpha}}\left[J_{\mho+}^{\alpha}\hbar\left(\frac{\mho+\eth}{2}\right)+J_{\eth-}^{\alpha}\hbar\left(\frac{\mho+\eth}{2}\right)\right]\right|\\=\ &\frac{\left(\eth-\mho\right)^{2}}{24}\left(\frac{\alpha+8}{\alpha+2}\right)L.\end{aligned}$$

PROPOSITION 2. *If we take $\alpha=1$, in Theorem 3, we have*

$$\left|\frac{1}{3}\left[2\hbar\left(\mho\right)-\hbar\left(\frac{\mho+\eth}{2}\right)+2\hbar\left(\eth\right)\right]-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar\left(u\right)du\right|\leq\frac{\left(\eth-\mho\right)^{2}}{8}L.$$

DEFINITION 4. [30] Let $\hbar\in H^1\left(\mho,\eth\right)$, $\mho<\eth$, for all $\alpha\in[0,1]$, then the left and right fractional integrals are defined by:

$$\begin{aligned}\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)\left(\ell\right)&=\frac{1-\alpha}{B\left(\alpha\right)}\hbar\left(\ell\right)+\frac{\alpha}{B\left(\alpha\right)}\int_{\mho}^{\ell}\hbar\left(\ell\right)d\ell,\\\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)\left(\ell\right)&=\frac{1-\alpha}{B\left(\alpha\right)}\hbar\left(\ell\right)+\frac{\alpha}{B\left(\alpha\right)}\int_{\ell}^{\eth}\hbar\left(\ell\right)d\ell,\end{aligned}$$

where $B\left(\alpha\right)>0$ is a normalizer satisfying $B\left(0\right)=B\left(1\right)=1$.

The main goal in this paper is to establish a new integral identity by using the Caputo-Fabrizio fractional integral operator, and generalize some novel Milne type inequality (1.1) for s-convex functions on the base of this identity. We also include the applications to special means, quadrature formula, and, q-digamma functions of these results taking many special cases of the main findings.

2. Milne-type inequalities for differentiable functions

Before proceeding toward our main theorems regarding generalization of Milne-type inequalities using Caputo-Fabrizio fractional operator, we begin with the following lemma.

Lemma 1. *Let* $\hbar : I \to \mathbb{R}$ *be a differentiable function on* I^o, $\mho, \eth \in I^o$ *with* $\mho < \eth$, *and* $\hbar' \in L[\mho, \eth]$, *then we get*

$$\begin{aligned}
&\left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth)\right] \\
&-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k) + \left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \\
= &\frac{(\eth-\mho)}{4}\left[\int_0^1\left(\omega - \frac{4}{3}\right)\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right. \\
&\left.+\int_0^1\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right],
\end{aligned}$$

$B(\alpha)$ is a normalization function.

Proof. Let

$$I_1 = \int_0^1\left(\omega - \frac{4}{3}\right)\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega$$

and

$$I_2 = \int_0^1\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega.$$

Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1\left(\omega - \frac{4}{3}\right)\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega \\
&= \frac{2}{(\eth-\mho)}\left(\omega-\frac{4}{3}\right)\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)\Bigg|_0^1 \\
&\quad -\frac{2}{(\eth-\mho)}\int_0^1\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega \\
&= \frac{-2}{3(\eth-\mho)}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{8}{3(\eth-\mho)}\hbar(\mho) - \frac{4}{(\eth-\mho)^2}\int_{\mho}^{\frac{\mho+\eth}{2}}\hbar(u)\,du
\end{aligned}$$

$$\begin{aligned}
&\frac{\alpha(\eth-\mho)^2}{4B(\alpha)}\int_0^1\left(\omega-\frac{4}{3}\right)\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega \\
(2.1)\quad &= -\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\mho) - \frac{\alpha}{B(\alpha)}\int_{\mho}^{\frac{\mho+\eth}{2}}\hbar(u)\,du.
\end{aligned}$$

Analogously, we get

$$\begin{aligned} I_2 &= \int_0^1 \left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega \\ &= \frac{2}{(\eth-\mho)}\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)\Big|_0^1 \\ &\quad -\frac{2}{(\eth-\mho)}\int_0^1 \hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega \\ &= \frac{-2}{3(\eth-\mho)}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{8}{3(\eth-\eth)}\hbar(\eth)-\frac{4}{(\eth-\mho)^2}\int_{\frac{\mho+\eth}{2}}^{\eth}\hbar(u)\,du \end{aligned}$$

$$\begin{aligned} &\frac{\alpha(\eth-\mho)^2}{4B(\alpha)}\int_0^1\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega \\ &= -\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\eth)-\frac{\alpha}{B(\alpha)}\int_{\frac{\mho+\eth}{2}}^{\eth}\hbar(u)\,du. \end{aligned} \tag{2.2}$$

Summing (2.1), (2.2) and subtracting $\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)$ both sides, we get

$$\begin{aligned} &\frac{\alpha(\eth-\mho)^2}{4B(\alpha)}\left[\int_0^1\left(\omega-\frac{4}{3}\right)\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right. \\ &\left.+\int_0^1\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right]-\frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \\ =\ & -\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\mho)-\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right) \\ &+\frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\eth)-\frac{\alpha}{B(\alpha)}\left(\int_{\mho}^{\frac{\mho+\eth}{2}}\hbar(u)\,du+\int_{\frac{\mho+\eth}{2}}^{\eth}\hbar(u)\,du\right)-\frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \\ =\ & -\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\mho)-\frac{\alpha(\eth-\mho)}{6B(\alpha)}\hbar\left(\frac{\mho+\eth}{2}\right) \\ &+\frac{2}{3}\frac{\alpha(\eth-\mho)}{B(\alpha)}\hbar(\eth)-\left(\frac{\alpha}{B(\alpha)}\int_{\mho}^{\eth}\hbar(u)\,du\right)-\frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \\ =\ & \frac{2}{3}\left(\hbar(\mho)+\hbar(\eth)\right)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) \\ &-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left(\frac{\alpha}{B(\alpha)}\int_{\mho}^{k}\hbar(u)\,du-\frac{(1-\alpha)}{B(\alpha)}\hbar(k)\right. \\ &\qquad\qquad\left.+\frac{\alpha}{B(\alpha)}\int_{k}^{\eth}\hbar(u)\,du-\frac{(1-\alpha)}{B(\alpha)}\hbar(k)\right) \\ =\ & \frac{2}{3}\left(\hbar(\mho)+\hbar(\eth)\right)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]. \end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\frac{(\eth-\mho)}{4}\left[\int_0^1\left(\omega-\frac{4}{3}\right)\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right.\\
&\left.+\int_0^1\left(\omega+\frac{1}{3}\right)\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right]\\
=\;&\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\\
&-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k).
\end{aligned}
$$

Hence proved Lemma 1. □

THEOREM 4. *Let $\hbar:[\mho,\eth]\to\mathbb{R}$ be a differentiable function on I^o, $\mho,\eth\in I^o$ with $\mho<\eth$, where $\hbar'\in L[\mho,\eth]$. If $|\hbar'|$ is s-convex function on $[\mho,\eth]$, then*

$$
\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|\\
\le\;&\frac{\eth-\mho}{4}\left(\frac{1}{3(2+3s+s^2)}\right)\left[5+4s\left(|\hbar'(\mho)|+|\hbar'(\eth)|\right)+2(5+s)\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|\right].
\end{aligned}
$$

Proof. By taking Mod of Lemma 1, and using the s-convexity of $|\hbar'|$, we have

$$
\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|\\
\le\;&\frac{(\eth-\mho)}{4}\left[\int_0^1\left|\omega-\frac{4}{3}\right|\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right|\right.\\
&\left.+\int_0^1\left|\omega+\frac{1}{3}\right|\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right|\right]\\
\le\;&\frac{(\eth-\mho)}{4}\left[\int_0^1\left|\omega-\frac{4}{3}\right|\left((1-\omega)^s|\hbar'(\mho)|+\omega^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|\right)\right.\\
&\left.+\int_0^1\left|\omega+\frac{1}{3}\right|\left((1-\omega)^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|+\omega^s|\hbar'(\eth)|\right)\right]\\
\le\;&\frac{\eth-\mho}{4}\left(\frac{1}{3(2+3s+s^2)}\right)\left[5+4s\left(|\hbar'(\mho)|+|\hbar'(\eth)|\right)+2(5+s)\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|\right].
\end{aligned}
$$

Hence proved. □

Corollary 1. *In Theorem 4, we use the further convexity of* $|\hbar'|$, *we have*

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.$$
$$\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|$$
$$\leq \frac{\eth-\mho}{4}\left[\frac{2^{-s}\left(5\left(2+2^{s}\right)+\left(2+2^{2+s}\right)s\right)}{3\left(2+3s+s^{2}\right)}\right]\left(|\hbar'(\mho)|+|\hbar'(\eth)|\right).$$

Corollary 2. *Choosing* $s=1$ *in corollary 1, then we have*

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.$$
$$\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|$$
$$\leq \frac{5(\eth-\mho)}{24}\left(|\hbar'(\mho)|+|\hbar'(\eth)|\right).$$

Remark 1. If we choose $\alpha=1$ and $B(0)=B(1)=1$, in corollary 2 then we have

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar(u)\,du\right|$$
$$\leq \frac{5(\eth-\mho)}{24}\left(|\hbar'(\mho)|+|\hbar'(\eth)|\right).$$

Which is obtained by Budak et al. in [29, Remark 1].

Theorem 5. *Let* $\hbar: I \to \mathbb{R}$ *be a differentiable function on* I^{o}, $\mho,\eth \in I^{o}$ *with* $\mho<\eth$, *where* $\hbar' \in L[\mho,\eth]$. *If* $|\hbar'|^{q}$, $q>1$ *is s-convex function on* $[\mho,\eth]$, *then*

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.$$
$$\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|$$
$$\leq \frac{(\eth-\mho)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(|\hbar'(\mho)|^{q}+\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^{q}+|\hbar'(\eth)|^{q}\right)^{\frac{1}{q}}\right].$$

Proof. Again by taking Mod of Lemma 1, we have

$$\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.$$
$$\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|$$
$$\leq \frac{(\eth-\mho)}{4}\left[\int_{0}^{1}\left|\omega-\frac{4}{3}\right|\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right|\right.$$
$$\left.+\int_{0}^{1}\left|\omega+\frac{1}{3}\right|\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right|\right]. \tag{2.3}$$

By utilizing the Hölder's inequality in (2.3) and s-convexity of $|\hbar'|^q$, we have

$$\begin{aligned}
&\le \frac{(\eth-\mho)}{4}\left[\left(\int_0^1\left|\omega-\frac{4}{3}\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left|\omega+\frac{1}{3}\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]\\
&\le \frac{(\eth-\mho)}{4}\left[\left(\int_0^1\left|\omega-\frac{4}{3}\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\omega)^s|\hbar'(\mho)|^q+\omega^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q\right)d\omega\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left|\omega+\frac{1}{3}\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\omega)^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q+\omega^s|\hbar'(\eth)|^q\right)d\omega\right)^{\frac{1}{q}}\right]\\
&\le \frac{(\eth-\mho)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(|\hbar'(\mho)|^q+\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q+|\hbar'(\eth)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence proved. □

REMARK 2. If we choose $\alpha = s = 1$ and $B(0) = B(1) = 1$ in Theorem 5, we have

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]-\frac{1}{\eth-\mho}\int_\mho^\eth\hbar(u)\,du\right|\\
&\le \frac{(\eth-\mho)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\hbar'(\mho)|^q+|\hbar'(\eth)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\hbar'(\mho)|^q+3|\hbar'(\eth)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Which is proved by Budak et al. in [29].

THEOREM 6. *Let $\hbar : I \to \mathbb{R}$ be a differentiable function on I^o, and $\mho, \eth \in I^o$ with $\mho < \eth$, where $\hbar' \in L[\mho,\eth]$. If $|\hbar'|^q$, $q \ge 1$ is s-convex function on $[\mho,\eth]$, then*

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^\alpha\hbar\right)(k)+\left({}^{CF}I^\alpha_{\eth}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|\\
&\le \frac{(\eth-\mho)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{3(2+3s+s^2)}\right)^{\frac{1}{q}}\\
&\quad\times\left[(5+4s)\left(|\hbar'(\mho)|^q+|\hbar'(\eth)|^q\right)+2(5+s)\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1, properties of modulus, the power-mean and s-convexity of $|\hbar'|^q$, we have

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|\\
\leq\;& \frac{(\eth-\mho)}{4}\left[\int_0^1\left|\omega-\frac{4}{3}\right|\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right|\right.\\
&\left.+\int_0^1\left|\omega+\frac{1}{3}\right|\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right|\right]\\
\leq\;& \frac{(\eth-\mho)}{4}\left[\left(\int_0^1\left|\omega-\frac{4}{3}\right|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\omega-\frac{4}{3}\right|\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left|\omega+\frac{1}{3}\right|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\omega+\frac{1}{3}\right|\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]\\
\leq\;& \frac{(\eth-\mho)}{4}\left[\left(\int_0^1\left|\omega-\frac{4}{3}\right|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\omega-\frac{4}{3}\right|\left((1-\omega)^s\left|\hbar'(\mho)\right|^q+\omega^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q\right)d\omega\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left|\omega+\frac{1}{3}\right|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\omega+\frac{1}{3}\right|\left((1-\omega)^s\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q+\omega^s\left|\hbar'(\eth)\right|^q\right)d\omega\right)^{\frac{1}{q}}\right]\\
\leq\;& \frac{(\eth-\mho)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{3(2+3s+s^2)}\right)^{\frac{1}{q}}\\
&\times\left[(5+4s)\left(\left|\hbar'(\mho)\right|^q+\left|\hbar'(\eth)\right|^q\right)+2(5+s)\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Hence proved. □

COROLLARY 3. *Choosing $s=1$ in Theorem 6, we have*

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|
\end{aligned}$$

$$\leq\frac{5(\eth-\mho)}{24}\left[\left(\frac{4\left|\hbar'(\mho)\right|^q+\left|\hbar'(\eth)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{\left|\hbar'(\mho)\right|^q+4\left|\hbar'(\eth)\right|^q}{5}\right)^{\frac{1}{q}}\right]. \tag{2.4}$$

REMARK 3. By choosing $\alpha=1$ and $B(0)=B(1)=1$, in corollary 3, we have

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]-\frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar(u)\,du\right|\\
\leq\;& \frac{5(\eth-\mho)}{24}\left[\left(\frac{4\left|\hbar'(\mho)\right|^q+\left|\hbar'(\eth)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{\left|\hbar'(\mho)\right|^q+4\left|\hbar'(\eth)\right|^q}{5}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Which is obtained by Budak et al. in [29, Remark 2].

Theorem 7. *Let $\hbar : I \to \mathbb{R}$ be a differentiable function on I^o, $\mho, \eth \in I^o$ with $\mho < \eth$, where $\hbar' \in L[\mho, \eth]$. If $|\hbar'|^q$, is s-convex function on $[\mho, \eth]$, for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}
&\left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth) \right] \right. \\
&\left. - \frac{B(\alpha)}{\alpha(\eth-\mho)} \left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar \right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar \right)(k) \right) \right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \right| \\
\leq\ & \frac{\eth-\mho}{4} \left[\left(\frac{1}{p} \left(\frac{(4^{p+1}-1)\,3^{-1-p}}{1+p} \right) \right) + \left(\frac{1}{q} \left(\frac{|\hbar'(\mho)|^q + \left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q}{s+1} \right) \right. \right. \\
&\left. \left. + \left(\frac{\left|\hbar'\left(\frac{\eth+\eth}{2}\right)\right|^q + |\hbar'(\eth)|^q}{s+1} \right) \right) \right].
\end{aligned}$$

Proof. From Lemma 1, we get

$$\begin{aligned}
&\left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth) \right] \right. \\
&\left. - \frac{B(\alpha)}{\alpha(\eth-\mho)} \left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar \right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar \right)(k) \right) \right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \right| \\
\leq\ & \frac{(\eth-\mho)}{4} \left[\int_0^1 \left|\omega - \frac{4}{3}\right| \left| \hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right) \right) d\omega \right| \right. \\
&\left. + \int_0^1 \left|\omega + \frac{1}{3}\right| \left| \hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right) + \omega\eth \right) d\omega \right| \right].
\end{aligned}$$

By the Young's inequality, we have

$$\mho\eth \leq \frac{1}{p}\mho^q + \frac{1}{q}\eth^q,$$

$$\begin{aligned}
&\left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth) \right] \right. \\
&\left. - \frac{B(\alpha)}{\alpha(\eth-\mho)} \left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar \right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar \right)(k) \right) \right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \right| \\
\leq\ & \frac{(\eth-\mho)}{4} \left[\left(\frac{1}{p}\int_0^1 \left|\omega - \frac{4}{3}\right|^p d\omega \right) + \frac{1}{q}\left(\int_0^1 \left| \hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right) \right) \right| d\omega \right)^q \right. \\
&\left. + \left(\frac{1}{p}\int_0^1 \left|\omega + \frac{1}{3}\right|^p d\omega \right) + \frac{1}{q}\left(\int_0^1 \left| \hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right) + \omega\eth \right) \right| d\omega \right) \right] \\
\leq\ & \frac{(\eth-\mho)}{4} \left[\left(\frac{1}{p}\int_0^1 \left|\omega - \frac{4}{3}\right|^p d\omega \right) + \frac{1}{q}\left(\int_0^1 \left((1-\omega)^s |\hbar'(\mho)|^q + \omega^s \left| \hbar'\left(\frac{\mho+\eth}{2}\right) \right|^q \right) \right) d\omega \right. \\
&\left. + \left(\frac{1}{p}\int_0^1 \left|\omega + \frac{1}{3}\right|^p d\omega \right) + \frac{1}{q}\left(\int_0^1 \left((1-\omega)^s \left| \hbar'\left(\frac{\mho+\eth}{2}\right) \right|^q + \omega^s |\hbar'(\eth)|^q \right) d\omega \right) \right] \\
\leq\ & \frac{\eth-\mho}{4} \left[\left(\frac{1}{p} \left(\frac{(4^{p+1}-1)\,3^{-1-p}}{1+p} \right) \right) + \left(\frac{1}{q} \left(\frac{|\hbar'(\mho)|^q + \left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q}{s+1} \right) \right. \right. \\
&\left. \left. + \left(\frac{\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^q + |\hbar'(\eth)|^q}{s+1} \right) \right) \right].
\end{aligned}$$

Required result is obtained. □

COROLLARY 4. *If we choose $\alpha = 1$ and $B(0) = B(1) = 1$, in Theorem 7, we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\hbar(\mho) - \hbar\left(\frac{\mho+\eth}{2}\right) + 2\hbar(\eth)\right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth}\hbar(u)\,du\right| \\
\leq\;& \frac{\eth-\mho}{4}\left[\left(\frac{1}{p}\left(\frac{\left(4^{p+1}-1\right)3^{-1-p}}{1+p}\right)\right) + \left(\frac{1}{q}\left(\frac{\left|\hbar'(\mho)\right|^{q} + \left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^{q}}{s+1}\right.\right.\right. \\
&+\left.\left.\left.\left(\frac{\left|\hbar'\left(\frac{\mho+\eth}{2}\right)\right|^{q} + \left|\hbar'(\eth)\right|^{q}}{s+1}\right)\right)\right)\right].
\end{aligned}$$

THEOREM 8. *Let $\hbar : I \to \mathbb{R}$ be a differentiable function on I^{o}, $\mho, \eth \in I^{o}$ with $\mho < \eth$, where $\hbar' \in L[\mho, \eth]$. If there exits constants $-\infty < m < M < +\infty$ such that $m \leq \hbar'(\omega) \leq M$, for all $x \in [\mho, \eth]$, then we have*

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth)\right]\right. \\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right| \\
\leq\;& \frac{5(\eth-\mho)}{24}(M-m).
\end{aligned}$$

Proof. Using the Lemma 1, we have

$$\begin{aligned}
&\left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth)\right] \\
&-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \\
=\;& \frac{(\eth-\mho)}{4}\left[\int_0^1\left(\omega-\frac{4}{3}\right)\left(\hbar'\left((1-\omega)\mho + \omega\left(\frac{\mho+\eth}{2}\right)\right)d\omega\right)\right. \\
&\left.+\int_0^1\left(\omega+\frac{1}{3}\right)\left(\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)d\omega\right)\right] \\
=\;& \frac{(\eth-\mho)}{4}\left[\left(\int_0^1\left(\omega-\frac{4}{3}\right)\left(\hbar'\left((1-\omega)\mho+\omega\frac{\mho+\eth}{2}\right)\right)-\frac{m+M}{2}\right)d\omega\right. \\
&\left.+\left(\int_0^1\left(\omega+\frac{1}{3}\right)\left(\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)\right)-\frac{m+M}{2}\right)d\omega\right].
\end{aligned} \tag{2.5}$$

Taking the absolute value of inequality (2.5), we have

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right) + \frac{2}{3}\hbar(\eth)\right]\right. \\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar\right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar\right)(k)\right)\right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right| \\
\leq\;& \frac{(\eth-\mho)}{4}\left[\int_0^1\left|\omega-\frac{4}{3}\right|\left|\left(\hbar'\left((1-\omega)\mho+\omega\frac{\mho+\eth}{2}\right)\right)-\frac{m+M}{2}\right|d\omega\right. \\
&\left.+\int_0^1\left|\omega+\frac{1}{3}\right|\left|\left(\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)\right)-\frac{m+M}{2}\right|d\omega\right].
\end{aligned}$$

From $m \leq \hbar(\omega) \leq M$ for $\omega \in [\mho, \eth]$, we have

$$\left| \left(\hbar' \left((1-\omega)\mho + \omega \left(\frac{\mho+\eth}{2} \right) \right) \right) - \frac{m+M}{2} \right| \leq \frac{M-m}{2}, \tag{2.6}$$

and

$$\left| \left(\hbar' \left((1-\omega) \left(\frac{\mho+\eth}{2} \right) + \omega\eth \right) \right) - \frac{m+M}{2} \right| \leq \frac{M-m}{2}. \tag{2.7}$$

Using the inequality (2.6) and (2.7), we have

$$\begin{aligned} & \left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2} \right) + \frac{2}{3}\hbar(\eth) \right] \right. \\ & \left. -\frac{B(\alpha)}{\alpha(\eth-\mho)} \left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar \right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar \right)(k) \right) \right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \right| \\ \leq & \frac{\eth-\mho}{8}(M-m) \left(\int_0^1 \left| \omega - \frac{4}{3} \right| d\omega + \int_0^1 \left| \omega + \frac{1}{3} \right| d\omega \right) \\ \leq & \frac{5(\eth-\mho)}{24}(M-m). \end{aligned}$$

Hence we get our required result. □

Corollary 5. *If we choose $\alpha = 1$ and $B(0) = B(1) = 1$, in Theorem 8, we get*

$$\begin{aligned} & \left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2} \right) + \frac{2}{3}\hbar(\eth) \right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth} \hbar(u)\,du \right| \\ \leq & \frac{5(\eth-\mho)}{24}(M-m), \end{aligned}$$

which is obtained by Budak et al. in [29].

Theorem 9. *Let $\hbar : I \to \mathbb{R}$ be a differentiable function on I^o, $\mho, \eth \in I^o$ with $\mho < \eth$. If $\hbar'$ is an L-Lipschitz function on $[\mho, \eth]$, then*

$$\begin{aligned} & \left| \left[\frac{2}{3}\hbar(\mho) - \frac{1}{3}\hbar\left(\frac{\mho+\eth}{2} \right) + \frac{2}{3}\hbar(\eth) \right] \right. \\ & \left. -\frac{B(\alpha)}{\alpha(\eth-\mho)} \left[\left(\left({}^{CF}_{\mho}I^{\alpha}\hbar \right)(k) + \left({}^{CF}I^{\alpha}_{\eth}\hbar \right)(k) \right) \right] + \frac{2(1-\alpha)}{B(\alpha)}\hbar(k) \right| \\ \leq & \frac{7(\eth-\mho)^2}{24}L. \end{aligned}$$

Proof. By using the Lemma 1, and since $\hbar'$ is an L-Lipschitz function, we have

$$\begin{aligned}
&\left|\left[\frac{2}{3}\hbar(\mho)-\frac{1}{3}\hbar\left(\frac{\mho+\eth}{2}\right)+\frac{2}{3}\hbar(\eth)\right]\right.\\
&\left.-\frac{B(\alpha)}{\alpha(\eth-\mho)}\left[\left(\left({}_{\mho}^{CF}I^{\alpha}\hbar\right)(k)+\left({}^{CF}I_{\eth}^{\alpha}\hbar\right)(k)\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}\hbar(k)\right|\\
\leq\ & \frac{(\eth-\mho)}{4}\left[\int_0^1\left|\omega-\frac{4}{3}\right|\left|\hbar'\left((1-\omega)\mho+\omega\left(\frac{\mho+\eth}{2}\right)\right)-|\hbar'(\mho)|\right|d\omega\right.\\
&\left.+\int_0^1\left|\omega+\frac{1}{3}\right|\left|\hbar'\left((1-\omega)\left(\frac{\mho+\eth}{2}\right)+\omega\eth\right)-|\hbar'(\eth)|\right|d\omega+\frac{5}{6}\left(|\hbar'(\eth)|-|\hbar'(\mho)|\right)\right]\\
\leq\ & \frac{(\eth-\mho)}{8}\left(\int_0^1\left|\omega-\frac{4}{3}\right|\omega d\omega+\int_0^1\left|\omega+\frac{1}{3}\right|(1-\omega)\,d\omega+\frac{5}{3}\right)L(\eth-\mho)\\
\leq\ & \frac{7(\eth-\mho)^2}{24}L.
\end{aligned}$$

This completes the proof. □

3. Application to special means

We shall use the following special means.
(a) The Arithmetic Mean

$$A=A(\mho,\eth):=\frac{\mho+\eth}{2},\ \ \mho,\eth\geq 0.$$

(b)The Logarithmic Mean

$$L(\mho,\eth):=\frac{\eth-\mho}{\ln\eth-\ln\mho}\quad \mho,\eth>0,\mho\neq\eth.$$

(c) The Generalized logarithmic Mean

$$L_r^r=\ L_r^r(\mho,\eth):=\left[\frac{\eth^{r+1}-\mho^{r+1}}{(r+1)(\eth\ -\ \mho)}\right]^{1/r}\ r\in\mathbb{R}\backslash\{-1,0\},\ \mho,\eth>0.$$

Proposition 3. *Let* $\mho,\eth\in\mathbb{R}$ *with* $0<\mho<\eth$, *then we have*

$$\left|4A\left(\mho^2,\eth^2\right)-A^2(\mho,\eth)-3L_2^2(\mho,\eth)\right|\leq\frac{5(\eth-\mho)}{24}\left[\left(\frac{4\mho^q+\eth^q}{5}\right)^{\frac{1}{q}}+\left(\frac{\mho^q+4\eth^q}{5}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 3, with $q\geq 2$ applying the $\hbar(x)=\frac{1}{2}x^2$ and $\alpha=1$, $B(0)=B(1)=1$. □

4. Application to quadrature formula

Considering Z is the partition of the points $\mho=\ell_0<\ell_1<\ldots\ldots\ldots.<\ell_n=\eth$ of the interval $[\mho,\eth]$ and let

$$\int_{\mho}^{\eth}\hbar(\ell)\,d\ell=\mu(\hbar,Z)+R(\hbar,Z),$$

where

$$\mu(\hbar, Z) = \sum_{i=0}^{n-1} \left(\frac{\ell_{i+1} - \ell_i}{3}\right) \left(2\hbar(\ell_i) - \hbar\left(\frac{\ell_i + \ell_{i+1}}{2}\right) + 2\hbar(\ell_{i+1})\right)$$

and $R(\hbar, Z)$ constitute the considering approximation error.

Proposition 4. *Let* $\hbar : [\mho, \eth] \to \mathbb{R}$ *is a differentiable function on* $(\mho, \eth)$ *with* $0 \le \mho < \eth$ *and* $\hbar' \in L[\mho, \eth]$. *If* $|\hbar'|$ *is s-convex function in the second sense for some fixed* $s \in (0, 1]$, *we have*

$$\begin{aligned} |R(\hbar, Z)| &\le \sum_{i=0}^{n-1} \frac{(\ell_{i+1} - \ell_i)^2}{4} \left(\frac{1}{3(2 + 3s + s^2)}\right) [(5 + 4s)|\hbar'(\ell_i)| \\ &\quad + (5 + 4s)|\hbar'(\ell_{i+1})| + 2(5 + s)\left|\hbar'\left(\frac{\ell_i + \ell_{i+1}}{2}\right)\right|\right]. \end{aligned}$$

Proof. The assertion follows from Theorem 4 on the subintervals $[\ell_i, \ell_{i+1}](i = 0, 1, ..., n-1)$ of the partition of Z and $\alpha = 1$, $B(0) = B(1) = 1$, we have

$$\begin{aligned} &\left|\frac{1}{3}\left(2\hbar(\ell_i) - \hbar'\left(\frac{\ell_i + \ell_{i+1}}{2}\right) + 2\hbar(\ell_{i+1})\right) - \frac{1}{\ell_{i+1} - \ell_i}\int_{\ell_i}^{\ell_{i+1}} \hbar(u)\,du\right| \\ &\le \frac{(\ell_{i+1} - \ell_i)}{4}\left(\frac{1}{3(2 + 3s + s^2)}\right)[(5 + 4s)|\hbar'(\ell_i)| + (5 + 4s)|\hbar'(\ell_{i+1})| \\ &\quad + 2(5 + s)\left|\hbar'\left(\frac{\ell_i + \ell_{i+1}}{2}\right)\right|\right]. \end{aligned} \tag{4.1}$$

By multiplying both sides of (4.1) by $(\ell_{i+1-}\ell_i)$, summing the resulting inequalities for $i = 0, 1, ..., n-1$, and applying the triangular inequality, the desired result is obtained. □

5. q-digamma Functions

Let $0 < \psi < 1$, the $q-$digamma (psi) functions $\varphi_{\psi,}$ is the ψ- analogue of the digamma function ψ defined as [31].

$$\begin{aligned} \varphi_\psi &= -\ln(1 - \psi) + \ln\psi \sum_{k=0}^{\infty} \frac{\psi^{k+\ell}}{1 - \psi^{k+\ell}} \\ &= -\ln(1 - \psi) + \ln\psi \sum_{k=0}^{\infty} \frac{\psi^{k\ell}}{1 - \psi^{k\ell}}. \end{aligned}$$

For $\psi > 1$ and $\ell > 0$, the q-digamma functions φ_ψ defined as

$$\begin{aligned} \varphi_\psi &= -\ln(\psi - 1) + \ln\psi\left[\ell - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\psi^{-(k+\ell)}}{1 - \psi^{-(k+\ell)}}\right] \\ &= -\ln(\psi - 1) + \ln\psi\left[\ell - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\psi^{-k\ell}}{1 - \psi^{-k\ell}}\right]. \end{aligned}$$

PROPOSITION 5. *Let* $\mho, \eth \in \mathbb{R}$ *with* $0 < \mho < \eth$, *then we have*

$$\left|\frac{1}{3}\left[2\varphi'_{\psi}(\mho) - \varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right) + 2\varphi'_{\psi}(\eth)\right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth}\varphi_{\psi}(u)\,du\right|$$
$$\leq \frac{\eth-\mho}{4}\left[\frac{1}{2}\left(\left|\varphi'_{\psi}(\mho)\right| + \left|\varphi'_{\psi}(\eth)\right|\right) + \frac{2}{3}\left|\varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right)\right|\right].$$

Proof. The assertion follows from Theorem 4, $\hbar(\epsilon) = \varphi_{\psi}(\epsilon)$, $\epsilon > 0$, $\hbar'(\epsilon) = \varphi'_{\psi}(\epsilon)$ is convex $(0, \infty)$ and $s = \alpha = 1$ & $B(0) = B(1) = 1$. □

PROPOSITION 6. *Let* $\mho, \eth \in \mathbb{R}$ *with* $0 < \mho < \eth$, *and* $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ *then we have*

$$\left|\frac{1}{3}\left[2\varphi'_{\psi}(\mho) - \varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right) + 2\varphi'_{\psi}(\eth)\right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth}\varphi_{\psi}(u)\,du\right|$$
$$\leq \frac{(\eth-\mho)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{2}\left(\left|\varphi'_{\psi}(\mho)\right|^{q} + \left|\varphi'_{\psi}(\eth)\right|^{q}\right) + \left|\varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right)\right|^{q}\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5, $\hbar(\epsilon) = \varphi_{\psi}(\epsilon)$, $\epsilon > 0$, $\hbar'(\epsilon) = \varphi'_{\psi}(\epsilon)$ is convex $(0, \infty)$ and $s = \alpha = 1$ & $B(0) = B(1) = 1$. □

PROPOSITION 7. *Let* $\mho, \eth \in \mathbb{R}$ *with* $0 < \mho < \eth$, *and* $\frac{1}{p} + \frac{1}{q} = 1$, $q \geq 1$ *then we have*

$$\left|\frac{1}{3}\left[2\varphi'_{\psi}(\mho) - \varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right) + 2\varphi'_{\psi}(\eth)\right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth}\varphi_{\psi}(u)\,du\right|$$
$$\leq \frac{\eth-\mho}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\left|\varphi'_{\psi}(\mho)\right|^{q} + \left|\varphi'_{\psi}(\eth)\right|^{q}\right) + \frac{2}{3}\left|\varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right)\right|^{q}\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 6, $\hbar(\epsilon) = \varphi_{\psi}(\epsilon)$, $\epsilon > 0$, $\hbar'(\epsilon) = \varphi'_{\psi}(\epsilon)$ is convex $(0, \infty)$ and $s = \alpha = 1$ & $B(0) = B(1) = 1$. □

PROPOSITION 8. *Let* $\mho, \eth \in \mathbb{R}$ *with* $0 < \mho < \eth$, *and* $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ *then we have*

$$\left|\frac{1}{3}\left[2\varphi'_{\psi}(\mho) - \varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right) + 2\varphi'_{\psi}(\eth)\right] - \frac{1}{\eth-\mho}\int_{\mho}^{\eth}\varphi_{\psi}(u)\,du\right|$$
$$\leq \frac{\eth-\mho}{4}\left[\left(\frac{1}{p}\left(\frac{(4^{p+1}-1)\,3^{-1-p}}{1+p}\right)\right) + \left(\frac{1}{q}\left(\frac{\left|\varphi'_{\psi}(\mho)\right|^{q} + \left|\varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right)\right|^{q}}{2}\right)\right.\right.$$
$$\left.\left.\left(\frac{\left|\varphi'_{\psi}\left(\frac{\mho+\eth}{2}\right)\right|^{q} + \left|\varphi'_{\psi}(\eth)\right|^{q}}{2}\right)\right)\right].$$

Proof. The assertion follows from Theorem 7, $\hbar(\epsilon) = \varphi_{\psi}(\epsilon)$, $\epsilon > 0$, $\hbar'(\epsilon) = \varphi'_{\psi}(\epsilon)$ is convex $(0, \infty)$ and $s = \alpha = 1$ & $B(0) = B(1) = 1$. □

6. Conclusions

In this paper, we established an important identity for the Caputo-Fabrizio fractional integral operator. We also generalized the Milne-type inequality for s-convex function, whose derivatives in absolute value at particular powers are convex. By considering the identity as an auxiliary results, we obtained some new results by using well known inequalities such as Hölder, power-mean and Young. Additionally, we also derived some applications to quadrature formula, special means, and q-digamma functions. In the future, scholars may explore new results with modified k-Riemann-Liouville and modified A-B fractional operators shown in this article.

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