

FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR ANALYTIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH GREGORY COEFFICIENTS

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ABSTRACT. In this work, we consider the function

$$\Psi(z) = \frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n$$

whose coefficients G_n are the Gregory coefficients related to Stirling numbers of the first kind and introduce a new subclass $\mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ of analytic bi-univalent functions subordinate to the function Ψ .

For functions belong to this class, we investigate the estimates for the general Taylor-Maclaurin coefficients by using the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coefficient bounds.

1. Introduction

The generating equation of the Gregory coefficients G_n (see [5, 24]) is given by

$$(1) \quad \frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n$$

$$(z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}, \quad G_0 = 1, \quad \ln 1 = 0),$$

where

$$G_n = \frac{1}{n!} \sum_{l=1}^n \frac{S_1(n, l)}{l+1}$$

and $S_1(n, l)$ is Stirling numbers of the first kind given by

$$S_1(n, l) = \begin{cases} \frac{(2n-l)!}{(l-1)!} \sum_{k=0}^{n-l} \frac{1}{(n+k)(n-l-k)(n-l+k)!} \sum_{r=0}^k \frac{(-1)^r r^{n-l+k}}{r!(k-r)!} & , \quad l \in [1, n] \\ 1 & , \quad n = 0, l = 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(see also [6]). In particular, we have:

$$\begin{aligned} S_1(1, 1) &= 1, & S_1(2, 1) &= -1, & S_1(2, 2) &= 1, \\ S_1(3, 1) &= 2, & S_1(3, 2) &= -3, & S_1(3, 3) &= 1. \end{aligned}$$

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Thus, initial values of G_n for $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ are

$$(2) \quad G_1 = \frac{1}{2}, \quad G_2 = -\frac{1}{12}, \quad G_3 = \frac{1}{24}, \quad G_4 = -\frac{19}{720}, \quad G_5 = \frac{3}{160}.$$

Let \mathcal{H} be the class of analytic functions in the open unit disc \mathbb{U} , and consider the classes \mathcal{P} , \mathcal{A} and \mathcal{S} defined by

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{H} : p(0) = 1 \text{ and } \Re(p(z)) > 0 \ (z \in \mathbb{U})\}, \\ \mathcal{A} &= \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}, \\ \mathcal{S} &= \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}, \end{aligned}$$

respectively.

For two functions $\mathbf{f}, \mathbf{g} \in \mathcal{H}$, we say that the function \mathbf{f} is subordinate to \mathbf{g} in \mathbb{U} , and write

$$\mathbf{f}(z) \prec \mathbf{g}(z), \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \text{ and } |\omega(z)| < 1, \ (z \in \mathbb{U})\},$$

such that

$$\mathbf{f}(z) = \mathbf{g}(\omega(z)), \quad (z \in \mathbb{U}).$$

It is clear that the function $f \in \mathcal{A}$ can be expressed as

$$(3) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}).$$

By Koebe One-Quarter Theorem [12], every function $f \in \mathcal{S}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r^*; \quad r^* \geq \frac{1}{4} \right).$$

For the inverse function $g := f^{-1}$, we obtain

$$\begin{aligned} g(w) &:= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \\ (4) \quad &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned}$$

If the functions f and f^{-1} are univalent in \mathbb{U} , then f is called bi-univalent function. Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (3).

The class Σ of analytic bi-univalent functions was first introduced by Lewin [20], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [7] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [22] proved that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. For more details and examples of functions belong to the class Σ , see [28] (see also [11, 18, 27, 28, 31, 32]).

Brannan and Taha [8] and Taha [29] investigated certain subclasses of Σ and found non-sharp estimates on $|a_2|$ and $|a_3|$. Furthermore, Sivasubramanian et al. [26] verified Brannan and Clunie's conjecture $|a_2| \leq \sqrt{2}$ for some subclasses of Σ . But the bounds on the general coefficient $|a_n|$ for $n > 3$ are not much known. In this study, we use the Faber polynomial expansions introduced by Faber [13] to determine general coefficient

bound $|a_n|$ of analytic bi-univalent functions whose coefficients are related to Gregory coefficients. Some works investigated the general coefficient bounds $|a_n|$ using Faber polynomial expansions can be found in [3, 4, 9, 15, 16, 19, 25, 33], (see also [10, 14, 17]).

2. The Class $\mathcal{G}_\Sigma^{\lambda, \mu}(\Psi)$

Throughout this paper, we assume that $\lambda \geq 1$ and $\mu \geq 0$. By considering the function Ψ ,

$$\Psi(z) = \frac{z}{\ln(1+z)}$$

given by (1), we introduce a new subclass of analytic bi-univalent functions as follows.

DEFINITION 2.1. A function $f \in \Sigma$ given by (3) is said to be in the class $\mathcal{G}_\Sigma^{\lambda, \mu}(\Psi)$ if the following conditions are satisfied:

$$(5) \quad (1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \prec \Psi(z)$$

and

$$(6) \quad (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \prec \Psi(w)$$

where $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (4).

REMARK 2.2. We obtain the following bi-univalent function classes which consists of functions $f \in \Sigma$ for the special values of λ and μ .

(i) For $\mu = 1$:

$$\mathcal{H}_\Sigma^\lambda(\Psi) = \left\{ f \in \Sigma : (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \prec \Psi(z) \text{ and } (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \prec \Psi(z) \right\}.$$

(ii) For $\mu = 1$ and $\lambda = 1$:

$$\mathcal{N}_\Sigma(\Psi) = \{ f \in \Sigma : f'(z) \prec \Psi(z) \quad \text{and} \quad g'(w) \prec \Psi(w) \}.$$

introduced by Murugusundaramoorthy et al. [21].

(iii) For $\mu = 0$ and $\lambda = 1$:

$$\mathcal{S}_\Sigma(\Psi) = \left\{ f \in \Sigma : \frac{zf'(z)}{f(z)} \prec \Psi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \prec \Psi(w) \right\}.$$

(iv) For $\lambda = 1$:

$$\mathcal{B}_\Sigma^\mu(\Psi) = \left\{ f \in \Sigma : \frac{z^{1-\mu} f'(z)}{(f(z))^{1-\mu}} \prec \Psi(z) \quad \text{and} \quad \frac{w^{1-\mu} g'(w)}{(g(w))^{1-\mu}} \prec \Psi(w) \right\}.$$

REMARK 2.3. Since the function Ψ is univalent (see [21]), the class $\mathcal{G}_\Sigma^{\lambda, \mu}(\Psi)$ is related to the class $\mathcal{N}(\mu)$ of non-Bazilevič functions defined by Obradović [23] as follows:

$$\Re \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right) > 0 \quad (0 < \mu < 1, z \in \mathbb{U}).$$

3. Coefficient estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (3), the coefficients of its inverse function $g = f^{-1}$ can be expressed as, [1]:

$$(7) \quad g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n,$$

where

$$(8) \quad \begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables a_2, a_3, \dots, a_n , [2]. For $n = 2, 3, 4$, we obtain

$$(9) \quad K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3), \quad K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4),$$

respectively. In general case, for any $p \in \mathbb{N}$, K_n^p is, [1],

$$(10) \quad K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \dots + \frac{p!}{(p-n)!n!} D_n^n,$$

where

$$D_n^p = D_n^p(a_2, a_3, \dots),$$

and by [30],

$$D_n^m(a_1, a_2, \dots, a_n) = \sum \frac{m!}{i_1! \dots i_n!} a_1^{i_1} \dots a_n^{i_n}$$

while $a_1 = 1$, and the sum is taken over all non-negative integers i_1, \dots, i_n satisfying

$$\begin{cases} i_1 + i_2 + \dots + i_n = m \\ i_1 + 2i_2 + \dots + ni_n = n \end{cases}.$$

It is clear that

$$D_n^n(a_1, a_2, \dots, a_n) = a_1^n.$$

Consequently, for functions $f \in \mathcal{A}$ of the form (3), we can write

$$(11) \quad (1-\lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \dots, a_n) z^{n-1},$$

where

$$\begin{aligned} F_1 &= (\mu + \lambda) a_2, \\ F_2 &= (\mu + 2\lambda) \left[\frac{\mu-1}{2} a_2^2 + a_3 \right], \\ F_3 &= (\mu + 3\lambda) \left[\frac{(\mu-1)(\mu-2)}{3!} a_2^3 + (\mu-1) a_2 a_3 + a_4 \right]. \end{aligned}$$

In general,

$$(12) \quad F_{n-1}(a_2, a_3, \dots, a_n) = [\mu + (n - 1)\lambda] \times [(\mu - 1)!] \times \sum \frac{a_2^{i_1} a_3^{i_2} \dots a_n^{i_{n-1}}}{i_1! i_2! \dots i_n! [\mu - (i_1 + i_2 + \dots + i_{n-1})]!}$$

is a Faber polynomial of degree $(n - 1)$.

LEMMA 3.1. [12] Let $p(z) = 1 + c_1z + c_2z^2 + \dots \in \mathcal{P}$, then $|c_k| \leq 2$ for $k \in \mathbb{N}$.

LEMMA 3.2. [34] Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then

$$|(k + l)z_1 + (k - l)z_2| \leq \begin{cases} 2R|k| & , \quad |k| \geq |l| \\ 2R|l| & \quad |k| \leq |l| \end{cases} .$$

THEOREM 3.3. Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ be given by (3). If $a_k = 0$ ($2 \leq k \leq n - 1$), then

$$|a_n| \leq \frac{1}{2[\mu + (n - 1)\lambda]} \quad (n \geq 3).$$

Proof. Let $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ be of the form (3). Then we have the expansion (11). For the inverse map $g = f^{-1}$ given by (4), we get

$$(13) \quad (1 - \lambda) \left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \dots, A_n) w^{n-1},$$

with

$$(14) \quad A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n).$$

On the other hand, since $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$, by the subordination principle, there exist the Schwarz's function $\varkappa(z)$,

$$\varkappa \in \mathcal{H} : \varkappa(0) = 0 \quad \text{and} \quad |\varkappa(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$(15) \quad (1 - \lambda) \left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1} = \Psi(\varkappa(z)) \quad (z \in \mathbb{U}).$$

Since Ψ is univalent in the open unit disk \mathbb{U} , by (15), the function

$$(16) \quad p(z) := \frac{1 + \varkappa(z)}{1 - \varkappa(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$$

belongs to the class \mathcal{P} . Solving $\varkappa(z)$ in terms of $p(z)$ in (16), we obtain

$$(17) \quad \begin{aligned} \varkappa(z) = & \frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 \\ & + \frac{1}{2}\left(c_4 - c_1c_3 + \frac{3}{4}c_1^2c_2 - \frac{1}{2}c_2^2 - \frac{1}{8}c_1^4\right)z^4 + \dots . \end{aligned}$$

Using (17) in (1) and considering (2), we find

$$\begin{aligned}
 \Psi(\varkappa(z)) &= 1 + \frac{1}{4}c_1z + \frac{1}{4}\left(c_2 - \frac{7}{12}c_1^2\right)z^2 + \frac{1}{4}\left(c_3 - \frac{7}{6}c_1c_2 + \frac{17}{48}c_1^3\right)z^3 \\
 (18) \quad &+ \frac{1}{4}\left(c_4 - \frac{7}{6}c_1c_3 + \frac{17}{16}c_1^2c_2 - \frac{7}{12}c_2^2 - \frac{649}{2880}c_1^4\right)z^4 + \dots
 \end{aligned}$$

Define

$$\varkappa(z) := \sum_{n=1}^{\infty} \varphi_n z^n.$$

Thus, we can write

$$(19) \quad \Psi(\varkappa(z)) = 1 + \sum_{n=1}^{\infty} \sum_{j=1}^n G_j D_n^j(\varphi_1, \varphi_2, \dots, \varphi_n) z^n.$$

Similarly, since $g \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$, there exist the Schwarz's function $\tau(w)$,

$$\tau \in \mathcal{H} : \tau(0) = 0 \quad \text{and} \quad |\tau(w)| < 1 \quad (w \in \mathbb{U})$$

such that

$$(20) \quad (1 - \lambda) \left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1} = \Psi(\tau(w)) \quad (w \in \mathbb{U}).$$

Since Ψ is univalent in the open unit disk \mathbb{U} , by (20), the function

$$(21) \quad q(w) := \frac{1 + \tau(w)}{1 - \tau(w)} = 1 + d_1w + d_2w^2 + d_3w^3 + \dots$$

belongs to the class \mathcal{P} . Solving $\tau(w)$ in terms of $q(w)$ in (21), we obtain

$$\begin{aligned}
 \tau(w) &= \frac{1}{2}d_1w + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)w^2 + \frac{1}{2}\left(d_3 - d_1d_2 + \frac{d_1^3}{4}\right)w^3 \\
 (22) \quad &+ \frac{1}{2}\left(d_4 - d_1d_3 + \frac{3}{4}d_1^2d_2 - \frac{1}{2}d_2^2 - \frac{1}{8}d_1^4\right)w^4 + \dots
 \end{aligned}$$

Using (22) in (1) and considering (2), we find

$$\begin{aligned}
 \Psi(\tau(w)) &= 1 + \frac{1}{4}d_1w + \frac{1}{4}\left(d_2 - \frac{7}{12}d_1^2\right)w^2 + \frac{1}{4}\left(d_3 - \frac{7}{6}d_1d_2 + \frac{17}{48}d_1^3\right)w^3 \\
 (23) \quad &+ \frac{1}{4}\left(d_4 - \frac{7}{6}d_1d_3 + \frac{17}{16}d_1^2d_2 - \frac{7}{12}d_2^2 - \frac{649}{2880}d_1^4\right)w^4 + \dots
 \end{aligned}$$

Define

$$\tau(w) := \sum_{n=1}^{\infty} \psi_n w^n.$$

Thus, we can write

$$(24) \quad \Psi(\tau(w)) = 1 + \sum_{n=1}^{\infty} \sum_{j=1}^n G_j D_n^j(\psi_1, \psi_2, \dots, \psi_n) w^n.$$

Considering (11) and (19) in equality (15), for any $n \geq 2$, we get

$$(25) \quad F_{n-1}(a_2, a_3, \dots, a_n) = \sum_{j=1}^{n-1} G_j D_{n-1}^j(\varphi_1, \varphi_2, \dots, \varphi_{n-1}),$$

and similarly, (13) and (24) in equality (20), we obtain

$$(26) \quad F_{n-1}(A_2, A_3, \dots, A_n) = \sum_{j=1}^{n-1} G_j D_{n-1}^j(\psi_1, \psi_2, \dots, \psi_{n-1}).$$

By hypothesis, since $a_k = 0$ ($2 \leq k \leq n - 1$), we have

$$A_n = -a_n,$$

$$\varphi_1 = \dots = \varphi_{n-2} = 0, \quad \varphi_{n-1} = \frac{1}{2}c_{n-1}$$

and

$$\psi_1 = \dots = \psi_{n-2} = 0, \quad \psi_{n-1} = \frac{1}{2}d_{n-1}.$$

So (25) and (26) imply that

$$[\mu + (n - 1)\lambda] a_n = G_1 \varphi_{n-1} \quad \text{and} \quad -[\mu + (n - 1)\lambda] a_n = G_1 \psi_{n-1},$$

respectively. Using the fact that $G_1 = \frac{1}{2}$ and Lemma 3.1, we obtain

$$|a_n| = \frac{G_1 |\varphi_{n-1}|}{\mu + (n - 1)\lambda} = \frac{G_1 |\psi_{n-1}|}{\mu + (n - 1)\lambda} \leq \frac{1}{2[\mu + (n - 1)\lambda]},$$

which completes the proof of the Theorem 3.3. □

THEOREM 3.4. *Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ be given by (3). Then*

$$(27) \quad |a_2| \leq \sqrt{\frac{3}{3(\mu + 2\lambda)(\mu + 1) + 14(\mu + \lambda)^2}}$$

and

$$(28) \quad |a_3| \leq \begin{cases} \frac{1}{2(\mu + 2\lambda)} & , \quad 3(\mu + 2\lambda)(1 - \mu) \leq 14(\mu + \lambda)^2 \\ \frac{3}{3(\mu + 2\lambda)(\mu + 1) + 14(\mu + \lambda)^2} & , \quad 3(\mu + 2\lambda)(1 - \mu) \geq 14(\mu + \lambda)^2 \end{cases}.$$

Proof. If we set $n = 2$ and $n = 3$ in (25) and (26), respectively, we get

$$(29) \quad (\mu + \lambda) a_2 = G_1 \varphi_1,$$

$$(30) \quad (\mu + 2\lambda) \left[\frac{\mu - 1}{2} a_2^2 + a_3 \right] = G_1 \varphi_2 + G_2 \varphi_1^2,$$

$$(31) \quad -(\mu + \lambda) a_2 = G_1 \psi_1,$$

$$(32) \quad (\mu + 2\lambda) \left[\frac{\mu + 3}{2} a_2^2 - a_3 \right] = G_1 \psi_2 + G_2 \psi_1^2,$$

where

$$\begin{aligned} \varphi_1 &= \frac{1}{2}c_1, & \varphi_2 &= \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right), \\ \psi_1 &= \frac{1}{2}d_1, & \psi_2 &= \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right). \end{aligned}$$

Note that

$$\varphi_1 = -\psi_1 \quad \text{and} \quad \varphi_1^2 + \psi_1^2 = 8(\mu + \lambda)^2 a_2^2$$

or equivalently

$$(33) \quad c_1 = -d_1 \quad \text{and} \quad c_1^2 + d_1^2 = 32(\mu + \lambda)^2 a_2^2.$$

From (29) and (31), we find

$$(34) \quad |a_2| = \frac{G_1 |\varphi_1|}{\mu + \lambda} = \frac{G_1 |\psi_1|}{\mu + \lambda} \leq \frac{1}{2(\mu + \lambda)}.$$

Also from (30) and (32), we obtain

$$(\mu + 2\lambda)(\mu + 1)a_2^2 = G_1(\varphi_2 + \psi_2) + G_2(\varphi_1^2 + \psi_1^2)$$

or equivalently

$$(35) \quad (\mu + 2\lambda)(\mu + 1)a_2^2 = \frac{1}{4}(c_2 + d_2) - \frac{7}{48}(c_1^2 + d_1^2).$$

Substituting the value of $c_1^2 + d_1^2$ from (33) in the right hand side of the above equality, we deduce that

$$(36) \quad a_2^2 = \frac{3(c_2 + d_2)}{4[3(\mu + 2\lambda)(\mu + 1) + 14(\mu + \lambda)^2]}.$$

Using the Lemma 3.1, we get

$$(37) \quad |a_2| \leq \sqrt{\frac{3}{3(\mu + 2\lambda)(\mu + 1) + 14(\mu + \lambda)^2}},$$

and combining this with the inequality (34), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (27).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (32) from (30). We thus get

$$(\mu + 2\lambda)(-2a_2^2 + 2a_3) = \frac{G_1}{2}(c_2 - d_2)$$

or

$$(38) \quad a_3 = a_2^2 + \frac{c_2 - d_2}{8(\mu + 2\lambda)}.$$

Upon substituting the value of a_2^2 from (36) into (38), it follows that

$$a_3 = \left(\Lambda(\lambda, \mu) + \frac{1}{8(\mu + 2\lambda)} \right) c_2 + \left(\Lambda(\lambda, \mu) - \frac{1}{8(\mu + 2\lambda)} \right) d_2,$$

where

$$\Lambda(\lambda, \mu) = \frac{3}{4[3(\mu + 2\lambda)(\mu + 1) + 14(\mu + \lambda)^2]}.$$

By Lemma 3.2, we get the desired estimate on the coefficient $|a_3|$ as asserted in (28). \square

By setting $\lambda = 1$ and $\mu = 1$ in Theorem 3.4, we obtain the following consequence.

COROLLARY 3.5. *Let the function $f \in \mathcal{N}_\Sigma(\Psi)$ be given by (3). Then*

$$|a_2| \leq \sqrt{\frac{3}{74}} \quad \text{and} \quad |a_3| \leq \frac{1}{6}.$$

REMARK 3.6. Note that the above corollary gives an improvement the result on $|a_3|$ given in [21, Theorem 1].

THEOREM 3.7. Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ be given by (3). Then for any $\delta \in \mathbb{R}$, we have

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{1}{2(\mu+2\lambda)} & , \delta \in [\rho, \eta] \\ \frac{3|1-\delta|}{3(\mu+2\lambda)(\mu+1)+14(\mu+\lambda)^2} & , \delta \in (-\infty, \rho] \cup [\eta, \infty) \end{cases} ,$$

where

$$\rho = \frac{3(\mu+2\lambda)(1-\mu) - 14(\mu+\lambda)^2}{6(\mu+2\lambda)}$$

and

$$\eta = \frac{3(\mu+2\lambda)(\mu+3) + 14(\mu+\lambda)^2}{6(\mu+2\lambda)} .$$

Proof. For the function $f \in \mathcal{G}_{\Sigma}^{\lambda, \mu}(\Psi)$ of the form (3), from (36) and (38) we have

$$a_3 - \delta a_2^2 = \left(\mathfrak{h}(\delta) + \frac{1}{8(\mu+2\lambda)} \right) c_2 + \left(\mathfrak{h}(\delta) - \frac{1}{8(\mu+2\lambda)} \right) d_2 ,$$

where

$$\mathfrak{h}(\delta) = \frac{3(1-\delta)}{4[3(\mu+2\lambda)(\mu+1) + 14(\mu+\lambda)^2]} .$$

Then by Lemma 3.1 and Lemma 3.2, we conclude that

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{1}{2(\mu+2\lambda)} & , |\mathfrak{h}(\delta)| \leq \frac{1}{8(\mu+2\lambda)} \\ 4|\mathfrak{h}(\delta)| & , |\mathfrak{h}(\delta)| \geq \frac{1}{8(\mu+2\lambda)} \end{cases} .$$

□

By setting $\lambda = 1$ and $\mu = 1$ in Theorem 3.7, we obtain the following consequence.

COROLLARY 3.8. [21] Let the function $f \in \mathcal{N}_{\Sigma}(\Psi)$ be given by (3). Then for any $\delta \in \mathbb{R}$, we have

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{1}{6} & , \delta \in \left[-\frac{28}{9}, \frac{46}{9}\right] \\ \frac{3|1-\delta|}{74} & , \delta \in (-\infty, -4\frac{28}{9}] \cup \left[\frac{46}{9}, \infty\right) \end{cases} .$$

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