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Determination of Unknown Time-Dependent Heat Source in Inverse Problems under Nonlocal Boundary Conditions by Finite Integration Method

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Abstract. In this study, we investigate the unknown time-dependent heat source function in inverse problems. We consider three general nonlocal conditions; two classical boundary conditions and one nonlocal over-determination, condition, these genereate six different cases. The finite integration method (FIM), based on numerical integration, has been adapted to solve PDEs, and we use it to discretize the spatial domain; we use backward differences for the time variable. Since the inverse problem is ill-posed with instability, we apply regularization to reduce the instability. We use the first-order Tikhonov's regularization together with the minimization process to solve the inverse source problem. Test examples in all six cases are presented in order to illustrate the accuracy and stability of the numerical solutions.

1. Introduction

Inverse problems are becoming crucial in several applications of physical and earth sciences, engineering and medicine. In particular, they are used in medical diagnosis and therapy/thermal equipment. In an inverse problem the goal is to identify inputs from the outputs of a forward process. The inverse source problem is a type of inverse problem that focuses on finding the unknown source function in the system; the source term normally depends on both time and spatial location. See [2, 7, 9, 14, 15, 17, 20, 22] for recent studies on the inverse source problem.

There are several methods that one uses for inverse problems. Among these

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are the finite difference method (FDM), finite element method (FEM), boundary element method (BEM), method of fundamental solution (MFS), see [1, 6, 10, 24]. Recently, in [23], Wen *et al* introduced the finite integration method (FIM) as a method potentially applicable to PDEs, and it seems this was first used in solving inverse problesm in [11] and [4]. Lesmana *et al.* in [11], applied the FIM to solve the inverse heat source problem under a Neumann boundary conditions together with given fixed space temperature, and used a direct numerical method to convert the inverse problem to a forward problem. Whereas Hazanee, in [4], used the FIM, together with Tikhonov regularization and the minimization technique, for an inverse heat source problem with a Neumann boundary condition and a nonlocal over-determination condition.

It is well-known that inverse problems tend to be ill-posed, stable solutions are not guaranteed. Many methods and techniques for stabilizing the solution have been tested, such as the Tikhonov's regularization, truncated singular decomposition (TSVD), iterative algorithms, variational methods, mollification methods, and smoothing spline approximation; see [3, 14, 16, 21]. Historically for inverse problems, the Tikhonov's regularization is the most popular and successful in stabilizing the solutions in inverse problems.

In this study, we consider the inverse problem of finding the time-dependent source function of the heat equation under two classical boundary conditions of types I, II and III, and one nonlocal over-determination condition. The additional over-determination condition is needed for the inverse problem, as there are two unknowns in the system, in order to guarantee existence and uniqueness of the solution. The FIM is utilized to discretize the space domain, whereas backward differences are considered for the time domain. The first-order Tikhonov's regularization based on the minimization of the linear least-squares is employed to stabilize the numerical solution. Furthermore, numerical examples are assessed to explore the efficiency of this procedure.

This paper is organized as follows. The statement of six cases of inverse heat source problems is introduced in the next section. In Section 3, FIM is introduced and the use of the FIM, with backward differences, to solve the inverse problem (2.1)-(2.3) is detailed. Section 4 describe how we use Tikhonov regularization to reduce instability of the solution. In Section 5, numerical examples of the six cases are simulated and discussed. Finally the study is summarized in Section 6.

2. Problem Statement

An interesting inverse source problem with two general boundary and over-determination conditions was presented by Ivanchov in [8]. The two general boundary conditions and one over-determination condition which combine to yield six distinct cases with first-, second- and third-type boundary conditions, are considered. This inverse problem deals with the identification of both the heat source $F(x,t) \in C(D_T)$ and the temperature $u(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ for $L > 0$, $T > 0$

and $D_T = (0, L) \times (0, T)$. The inverse problem statement is as follows,

$$u_t(x,t) = u_{xx}(x,t) + F(x,t), \ (x,t) \in D_T, \tag{2.1}$$

subject to the initial condition,

$$u(x,0) = \varphi(x), \ x \in [0, L], \tag{2.2}$$

and the following boundary and over-determination conditions,

$$\begin{cases} \gamma_{11}(t)\, u(0,t) + \gamma_{12}(t)\, u(L,t) + \gamma_{13}(t)\, u_x(0,t) + \gamma_{14}(t)\, u_x(L,t) = k_1(t), \\ \gamma_{21}(t)\, u(0,t) + \gamma_{22}(t)\, u(L,t) + \gamma_{23}(t)\, u_x(0,t) + \gamma_{24}(t)\, u_x(L,t) = k_2(t), \\ \gamma_{31}(t)\, u(0,t) + \gamma_{32}(t)\, u(L,t) + \gamma_{33}(t)\, u_x(0,t) + \gamma_{34}(t)\, u_x(L,t) = k_3(t), \end{cases} \tag{2.3}$$

where $t \in [0,T]$, $\varphi(x) \in C^2[0,L]$ and $k_i(t) \in C^1[0,T]$ are given functions, and the matrix $\gamma = (\gamma_{ij}) \in C^1[0,T]$ has rank 3 for $t \in [0,T]$, $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$. By assuming, without loss of generality, the same third-order minor of the matrix γ is non-zero and can state three of the four boundary data $u(0,t)$, $u(L,t)$, $u_x(0,t)$ and $u_x(L,t)$ in terms of the fourth one, the following six cases arise.

Case 1:

$$u_x(0,t) = \mu_1(t), u_x(L,t) = \mu_2(t), \tag{2.4}$$

$$v_1(t)\, u(0,t) + v_2(t)\, u(L,t) = k(t), \tag{2.5}$$

Case 2:

$$u(0,t) = \mu_1(t), u_x(L,t) = \mu_2(t), \tag{2.6}$$

$$v_1(t)\, u_x(0,t) + v_2(t)\, u(L,t) = k(t), \tag{2.7}$$

Case 3:

$$u(0,t) = \mu_1(t), u(L,t) = \mu_2(t), \tag{2.8}$$

$$v_1(t)\, u_x(0,t) + v_2(t)\, u_x(L,t) = k(t), \tag{2.9}$$

Case 4:

$$u(0,t) = \mu_1(t), u_x(L,t) + v_1(t)u(L,t) = \mu_2(t), \tag{2.10}$$

$$u_x(0,t) + v_2(t)\, u_x(L,t) = k(t), \tag{2.11}$$

Case 5:

(2.12) $$u_x(0,t) = \mu_1(t), u_x(L,t) + \upsilon_1(t)u(L,t) = \mu_2(t),$$

(2.13) $$u(0,t) + \upsilon_2(t)u(L,t) = k(t),$$

Case 6:

(2.14) $$u_x(0,t) - \upsilon_1(t)u(0,t) = \mu_1(t), u_x(L,t) + \upsilon_2(t)u(L,t) = \mu_2(t),$$

(2.15) $$\upsilon_3(t)u(0,t) + \upsilon_4(t)u(L,t) = k(t), t \in [0,T],$$

where $\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t), \mu_1(t), \mu_2(t), k(t) \in C^1[0,T]$ are given functions. The other cases can be reduced to these ones by the change $y = h - x$. In these six cases, it can be seen that the first two conditions of each case are general type boundary conditions: Case 1 is the Neumann conditions, Case 3 is the Dirichlet conditions, and the other cases are Robin conditions. In Cases 1- 6, we assume that $\upsilon_1^2(t) + \upsilon_2^2(t) > 0, t \in [0,T]$.

When the source function $F(x,t)$ of Eq. (2.1) is given, the problem of finding only $u(x,t)$ from the initial boundary value problem is called *the direct problem* or *forward problem*, whereas in the case of an unknown source function $F(x,t)$ it is called *the indirect problem* or *the inverse problem*. In this study, we represent the unknown function as $F(x,t) := r(t)f(x,t) + h(x,t)$ where the functions $f(x,t), h(x,t) \in C^{1,0}(\overline{D}_T)$ are given and $r(t) \in C[0,T]$ is unknown. Actually, this inverse problem (2.1)-(2.3) was assessed for existence and uniqueness of the solution (without numerical methods) by Ivanchov in [8].

Therefore in our study, we propose to identify the pair solution $(r(t), u(x,t))$ by using the FIM to discretize the space domain while using backward differences in the time domain. Actually, Hazanee and Lesnic in [5], studied these six cases of the inverse problem (2.1)-(2.3) numerically using BEM as the discretization tool. They also employed Tikhonov's regularization of orders zero, one and two, and found that the BEM together with Tikhonov's regularization was capable of providing stable and accurate results. However the BEM formulations give rise to a nonsymmetric fully populated matrix, and the coding for BEM is complicated. Hence, the aim of this study is to use the FIM for finding the time-dependent of heat source by solving the inverse problem (2.1)-(2.3), since the matrix in FIM is comparatively simple to generate and is triangular.

3. Finite Integration Method

Recently, in [23], Wen *et al.* revised a numerical method with potential applicability to solve PDEs, namely the finite integration method (FIM). FIM has been applied to one- and multi-dimensional partial differential equations, see [12, 13, 23, 25].

In this method, first take n-layer integrals over the n^{th} order PDE in order to obtain the solution by numerical integration. Many numerical integration methods have been studied, such as trapezoid rule, Simpson's rule, Newton-Cotes formulas, and Lagrange formulas. Among various numerical integration techniques adapted to FIM, it seems that the trapezoid rule is the simplest alternative to choose. Looking more closely at the use of trapezoid rule in FIM, it performs ordinary linear approximation (OLA) and makes the integration matrix obtained after taking integration over the system to be a lower triangular matrix. This is an advantage of using FIM based on OLA, denoted as FIM(OLA), since systems of linear equations with triangular matrix can be solved more quickly and with less memory, than general systems.

To introduce FIM(OLA), we first consider the approximate integration of a one-dimensional function by FIM(OLA), based on the trapezoid rule, as

$$\int_0^L u(x)dx = \Delta x\left(\frac{u_0}{2} + u_1 + \ldots + u_{N-1} + \frac{u_N}{2}\right), \tag{3.1}$$

where $u_i = u(x_i)$, $x \in \{0 = x_0, x_1, x_2, \ldots, x_N = L\}$ and $\Delta x = \frac{L}{N}$. Then we approximate the discretised temperature as $\underline{U}^{(1)} = A\underline{u}$ where $\underline{u} = \left[u_0, u_1, \ldots, u_N\right]^T$, $\underline{U}^{(1)} = \left[\int_{x_0}^{x_0} u(x)dx, \int_{x_0}^{x_1} u(x)dx, \ldots, \int_{x_0}^{x_N} u(x)dx\right]^T$ and

$$A = \frac{L}{N}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & \frac{1}{2} \end{pmatrix}_{(N+1)\times(N+1)}$$

Whereas for n iterated integrals, the approximation $\underline{U}^{(n)}$ can be formed as $\underline{U}^{(n)} = A^n\underline{u}$, see Wen *et al.* in [23].

Another advantage of using FIM(OLA) is to computing the n^{th} power of A; A^n, which approximate n-fold iterated integration, $A^{(n)}$. This means that solving an n^{th} order differential equation utilises only one lower triangular integral matrix. Furthermore, the FIM can be applied to solve ordinary differential equations (ODEs) and PDEs whether they are time-dependent or not.

3.1. FIM for solving inverse problem

This subsection describes the use of FIM together with backward differences in order to solve the inverse problem (2.1)-(2.3). We first discretize time and space domains to M and N subintervals, respectively, as $t \in \{0 = t_0, t_1, \ldots, t_M = T\}$ and $x \in \{0 = x_0, x_1, \ldots, x_N = L\}$. The backward differences approximate the first order derivative as

$$u_t\left(x, t_j\right) = \frac{u\left(x, t_j\right) - u(x, t_{j-1})}{T/M}, j \in 1, 2, \ldots, M. \tag{3.2}$$

Here we denote the discrete temperature and source functions as $u_j(x) := u(x, t_j)$, $f_j(x) := f(x, t_j)$, $h_j(x) := h\left(x, t_j\right)$, $r_j := r(t_j)$, for $i \in \{0, 1, 2, \ldots, M\}$. Therefore the time-discretized form of the heat equation in Eq. (2.1) becomes

$$\frac{M}{T}\left(u_j(x) - u_{j-1}(x)\right) = \frac{\partial^2 u_j}{\partial x^2}(x) + r_j f_j(x) + h_j(x). \tag{3.3}$$

On using FIM, we integrate with respect to x on both sides and approximate the integrals by the matrix approximation introduced earlier. We obtain

$$\frac{M}{T}\left(A\underline{u}_j - A\underline{u}_{j-1}\right) = \underline{u}_{x,j} + r_j A\underline{f}_j + A\underline{h}_j + c_0\underline{i}, \tag{3.4}$$

where c_0 is an integration constant corresponding to the Eq. (3.4) and $\underline{i} = [1, 1, 1, \ldots, 1]^T$ is $N + 1$ column vector. Consider now integrating twice with respect to x on both sides, and Eq. (3.4) yields

$$\frac{M}{T}\left(A^2\underline{u}_j - A^2\underline{u}_{j-1}\right) = \underline{u}_j + r_j A^2\underline{f}_j + A^2 h_j + c_0\underline{x} + c_1\underline{i}, \tag{3.5}$$

where $\underline{x} = [x_0, x_1, x_2, \ldots, x_N]^T$ is $N + 1$ column vector of space nodes and c_0, c_1 are integration constants corresponding to the Eq. (3.5). Rearranging this gives

$$\left(\frac{M}{T}A^2 - I\right)\underline{u}_j - c_0\underline{x} - c_1\underline{i} = \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2\underline{f}_j + A^2\underline{h}_j. \tag{3.6}$$

The solutions of the inverse problem (2.1)-(2.3) with the distinct boundary and over-determination conditions from (2.3) separated to the six cases shown in (2.4)-(2.15), can be discretized as follows.

Case 1:

$$u_{x,j}(0) = \mu_{1j},\ u_{x,j}(L) = \mu_{2j}, \tag{3.7}$$

$$v_{1j}u_j(0) + v_{2j}u_j(L) = k_j, \tag{3.8}$$

Case 2:

$$u_j(0) = \mu_{1j},\ u_{x,j}(L) = \mu_{2j}, \tag{3.9}$$

$$v_{1j}u_{x,j}(0) + v_{2j}u_j(L) = k_j, \tag{3.10}$$

Case 3:

$$u_j\,(0) = \mu_{1j},\ u_j\,(L) = \mu_{2j}, \tag{3.11}$$

$$\upsilon_{1j}u_{x,j}\,(0) + \upsilon_{2j}u_{x,j}\,(L) = k_j, \tag{3.12}$$

Case 4:

$$u_j\,(0) = \mu_{1j},\ u_{x,j}\,(L) + \upsilon_{1j}u_j(L) = \mu_{2j}, \tag{3.13}$$

$$u_{x,j}\,(0) + \upsilon_{2j}u_{x,j}\,(L) = k_j, \tag{3.14}$$

Case 5:

$$u_{x,j}\,(0) = \mu_{1j},\ u_{x,j}\,(L) + \upsilon_{1,j}u_j(L) = \mu_{2j}, \tag{3.15}$$

$$u_j\,(0) + \upsilon_{2j}u_j\,(L) = k_j, \tag{3.16}$$

Case 6:

$$u_{x,j}\,(0) - \upsilon_{1,j}u_j\,(0) = \mu_{1j},\ u_{x,j}\,(L) + \upsilon_{2,j}u_j(L) = \mu_{2j}, \tag{3.17}$$

$$\upsilon_{3j}u_j\,(0) + \upsilon_{4j}u_j\,(L) = k_j, \tag{3.18}$$

by denoting $\mu_{1j} := \mu_1\left(t_j\right)$, $\mu_{2j} := \mu_2\left(t_j\right)$, $\upsilon_{1j} := \upsilon_1(t_j)$, $\upsilon_{2j} := \upsilon_2(t_j)$ and $k_j := k(t_j)$. Considering the approximate integral Eq. (3.4) with discretized boundary conditions (3.7), (3.9), (3.11), (3.13), (3.15), (3.17) for cases 1-6, respectively, each case gives the matrix equation written in general form as

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -\underline{i} \\ \hline [***] & * & * \\ {[***]} & * & * \end{array}\right]\left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_jA^2\underline{f}_j + A^2\underline{h}_j \\ \hline -\mu_{1j} \\ * \end{array}\right] \tag{3.19}$$

where $*$ depends on boundary conditions for each cases.

At this point, if we were considering the direct problem, the temperature $u(x,t)$ can be approximated by solving the system of linear equations (3.19) using the initial condition (2.2). When $j = 1$, the temperature $\underline{u}_1$ is calculated using known $\underline{u}_0$ given by the initial condition. Then at the time step j, for $j \in \{1, 2, \ldots, M\}$, the temperature $\underline{u}_j$ can be calculated by using the known temperature $\underline{u}_{j-1}$ given from the previous time step.

For the inverse problem the heat source function $r(t)$ is unknown. The over-determination conditions (3.8), (3.10), (3.12), (3.14), (3.16), (3.18) for cases 1-6, respectively, need to be used together with minimizing the linear least-squares as the objective function,

$$\Gamma(r) := \sum_{j=0}^{M} \left[\left(\gamma_j u_j(*) + \gamma_j u_{x,j}(*)\right) - k_j\right]^2, \tag{3.20}$$

where $\gamma_j u_j(*) + \gamma_j u_{x,j}(*)$ is the time-discretized over-determination condition corresponding to the objective functions for each cases. Here, the boundary temperatures $u_j(0)$ and $u_j(L)$ can be approximated by solving the system of linear equations (3.19) with an initial guess of the heat source function $r_j^{(0)} := r^{(0)}\left(t_j\right)$ to be prescribed, whereas the flux $u_{x,j}(0)$ and $u_{x,j}(L)$ can be obtained by considering (3.4) at $x = 0$ and $x = L$.

As the above overall consideration, the solution $r(t_j)$ of Cases 1-6 can be approximated by solving the matrix equation in Eq. (3.19) and minimizing the objective function (3.20) with following matrix equations and objective functions of each cases, $\Gamma_1 - \Gamma_6$, as

Case 1:

$$\Gamma_1(r) := \sum_{j=0}^{M} \left[v_{1j} u_j(0) + v_{2j} u_j(L) - k_j\right]^2, \tag{3.21}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -\underline{i} \\ \hline \underline{0} & 1 & 0 \\ \frac{M}{T}\underline{A}_N & -1 & 0 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N \underline{u}_{j-1} + r_j \underline{A}_N \underline{f}_j + \underline{A}_N \underline{h}_j \end{array}\right],$$

Case 2:

$$\Gamma_2(r) := \sum_{j=0}^{M} \left[v_{1j} u_{x,j}(0) + v_{2j} u_j(L) - k_j\right]^2, \tag{3.22}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -\underline{i} \\ \hline \underline{0} & 0 & 1 \\ \frac{M}{T}\underline{A}_N & -1 & 0 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N \underline{u}_{j-1} + r_j \underline{A}_N \underline{f}_j + \underline{A}_N \underline{h}_j \end{array}\right],$$

Case 3:

$$\Gamma_3(r) := \sum_{j=0}^{M} \left[v_{1j} u_{x,j}(0) + v_{2j} u_{x,j}(L) - k_j\right]^2, \tag{3.23}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -\underline{i} \\ \hline \underline{0} & 0 & 1 \\ \frac{M}{T}\underline{A}_N^{(2)} & -L & -1 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N^{(2)} \underline{u}_{j-1} + r_j \underline{A}_N^{(2)} \underline{f}_j + \underline{A}_N^{(2)} \underline{h}_j \end{array}\right],$$

Case 4:

$$\Gamma_4(r) := \sum_{j=0}^{M} \left[u_{x,j}(0) + v_{2j} u_{x,j}(L) - k_j\right]^2, \tag{3.24}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -i \\ \hline \underline{0} & 0 & 1 \\ \frac{M}{T}\underline{A}_N + \underline{V}_{1j}^{(L)} & -1 & 0 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N \underline{u}_{j-1} + r_j \underline{A}_N \underline{f}_j + \underline{A}_N \underline{h}_j \end{array}\right],$$

Case 5:

$$\Gamma_5(r) := \sum_{j=0}^{M} \left[u_j(0) + v_{2j} u_j(L) - k_j\right]^2, \tag{3.25}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -\underline{i} \\ \hline \underline{0} & 1 & 0 \\ \frac{M}{T}\underline{A}_N + \underline{V}_{1j}^{(L)} & -1 & 0 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N \underline{u}_{j-1} + r_j \underline{A}_N \underline{f}_j + \underline{A}_N \underline{h}_j \end{array}\right],$$

Case 6:

$$\Gamma_6(r) := \sum_{j=0}^{M} \left[v_{3j} u_j(0) + v_{4j} u_j(L) - k_j\right]^2, \tag{3.26}$$

$$\left[\begin{array}{c|cc} \frac{M}{T}A^2 - I & -\underline{x} & -i \\ \hline \underline{V}_{1j}^{(0)} & 1 & 0 \\ \frac{M}{T}\underline{A}_N + \underline{V}_{2j}^{(L)} & -1 & 0 \end{array}\right] \left[\begin{array}{c} \underline{u}_j \\ \hline c_0 \\ c_1 \end{array}\right] = \left[\begin{array}{c} \frac{M}{T}A^2\underline{u}_{j-1} + r_j A^2 \underline{f}_j + A^2 \underline{h}_j \\ \hline -\mu_{1j} \\ \mu_{2j} + \frac{M}{T}\underline{A}_N \underline{u}_{j-1} + r_j \underline{A}_N \underline{f}_j + \underline{A}_N \underline{h}_j \end{array}\right],$$

where $\underline{A}_N = \left[a_{N0}, a_{N1}, a_{N2}, \ldots, a_{NN}\right]$ is an $N+1$ row vector and a_{Ni} is an element of the N^{th}-row of the matrix A, $\underline{A}_N^{(2)} = \left[a_{N0}^{(2)}, a_{N1}^{(2)}, a_{N2}^{(2)}, \ldots, a_{NN}^{(2)}\right]$ is an $N+1$ row vector and $a_{Ni}^{(2)}$ is an element of the N^{th}-row of integration matrix $A^2 = A \times A$, and $\underline{V}_{1j}^{(L)} = \left[0, 0, 0, \ldots, 0, v_{1j}\right]$, $\underline{V}_{2j}^{(0)} = \left[v_{2j}, 0, 0, \ldots, 0\right]$ are $N+1$ row vectors.

4. Regularization

The inverse heat source problem (2.1)-(2.3) is ill-posed, so that small errors in input data result in large errors in the numerical solution. In this study, noises/contaminations of the input data are added randomly to $k(t)$ in the over-determination condition,

$$\underline{k}^{\varepsilon} = \underline{k} + \underline{\varepsilon}, \tag{4.1}$$

where $\underline{k} = \left[k_0, k_1, k_2, \ldots, k_N\right]^T$, $\underline{k}^\epsilon$ is a vector of contaminated input data, and $\underline{\epsilon}$ is a vector of noisy input randomized by Gaussian normal distribution with mean 0 and standard deviation σ and defined by $\sigma = p \times max_j \left|k(t_j)\right|$ where p is the percentage of noise to be added. Therefore the objective function is perturbed as

$$\Gamma^\epsilon(r) = \sum_{j=0}^{M} \left[\left(\gamma_j u_j(*) + \gamma_j u_{x,j}(*)\right) - k_j^\epsilon\right]^2. \tag{4.2}$$

When there are errors in the input data, the results from inverse problem diverge. In order to stabilize the solution, the first-order Tikhonov regularization is added to the perturbed objective function (3.20) as

$$\Gamma^\epsilon\left(r_\lambda\right) = \sum_{j=0}^{M} \left[\left(\gamma_j u_j(*) + \gamma_j u_{x,j}(*)\right) - k_j^\epsilon\right]^2 + \lambda \sum_{j=1}^{M} \left[\frac{T}{M+1}\left(r_j - r_{j-1}\right)\right]^2, \tag{4.3}$$

where $\lambda > 0$ is a regularization parameter to be prescribed. Note that the term of $\lambda \sum_{j=1}^{M} \left[\frac{T}{M+1}\left(r_j - r_{j-1}\right)\right]^2$ in (4.3) is called the regularization term, and when no regularization is imposed, $\lambda = 0$, the objective function (4.3) becomes (3.20). The regularized objective function (4.3) has the first-order Tikhonov regularization because the first-order derivative is applied in the regularization term with function r, see [19]. We used the *fminunc* routine from MATLAB Optimization toolbox for calculating the objective function (4.3) with default tolerance setting as $TolFun = 10^{-6}$ and $MaxFunctionEvaluation = 1.01 \times 10^{-4}$.

Hence for all the six cases, a stable pair solution $\left(r(t_j), u(x_i, t_j)\right)$ of the inverse problem (2.1)-(2.3) can now be found by solving the matrix equation (3.19) and minimizing the regularized objective function (4.3).

5. Numerical Examples

In order to analyze the accuracy of the source function $r(t)$ of inverse problem (2.1)-(2.3), produced by applying FIM(OLA) together with backward difference and stabilizing the solution by using Tikhonov regularization as detailed earlier in Sections 3 and 4, let us introduce the root mean square error (RMSE) defined as

$$\text{RMSE}(u) = \sqrt{\frac{L}{N+1} \sum_i (Exact\left(x_i\right) - Approx(x_i))^2},$$

$$\text{RMSE}(r) = \sqrt{\frac{T}{M+1} \sum_j (Exact(t_j) - Approx(t_j))^2},$$

where $Exact\left(x_i\right)$ and $Approx\left(x_i\right)$ represent the analytical and numerical solutions of $u(x_i, 1)$, respectively, whereas $Exact\left(t_j\right)$ and $Approx\left(t_j\right)$ represent the analytical and numerical solutions of $r(t_j)$, respectively.

For numerical example, we consider a test example as appeared in [4] and [5] for the inverse source problem (2.1)-(2.3) with boundary and over-determination conditions for 6 cases as shown in (2.4)-(2.15) with $T = 1$, $L = 1$, $u(x, 0) = \varphi(x) = 1 + x - x^2$, $f(x, t) = \left(1-x^2\right) e^{-t}$, and $h(x, t) = (2+x)\, e^t$. The boundary and over-determination conditions given in each case are as follows,

Case 1:

$$u_x(0, t) = e^t,\ u_x(1, t) = -e^t,\ u(0, t) + u(1, t) = 2e^t, \tag{5.1}$$

Case 2:

$$u(0, t) = e^t,\ u_x(1, t) = -e^t,\ u_x(0, t) + u(1, t) = 2e^t, \tag{5.2}$$

Case 3:

$$u(0, t) = e^t,\ u(1, t) = e^t,\ e^t u_x(0, t) + t u_x(1, t) = (e^t - t)e^t, \tag{5.3}$$

Case 4:

$$u(0, t) = e^t, \quad u_x(1,t) + (1+t)\, u(1,t) = te^t,\ u_x(0, t) + e^{-t} u_x(1,t) = e^t - 1, \tag{5.4}$$

Case 5:

$$u_x(0, t) = e^t,\ u_x(1,t) + e^t u(1,t) = 1-e^t,\ u(0, t) + (1+t)\, u(1,t) = (2+t)\, e^t, \tag{5.5}$$

Case 6:

$$\begin{aligned} u_x(0, t) - e^t u(0, t) = e^t - e^{2t},\ u_x(1,t) + (1+t)\, u(1,t) = te^t, \\ tu(0, t) + (1-t)\, u(1,t) = e^t, \end{aligned} \tag{5.6}$$

for $x \in (0, 1)$ and $t \in (0, T)$. The RMSE for these examples uses the analytical solutions as $u(x, t) = \left(1+x - x^2\right) e^t$ and $r(t) = e^{2t}$. Note that these examples have been checked for existence and uniqueness of solution for the inverse problem (2.1)-(2.3) corresponding to theorems shown in [8].

The direct problem of Case 1 with above input data has been considered with $M, N \in \{20, 40, 60, 80, 100\}$ and it was found that $M = 100$ with various N gives sufficient accuracy. Here we decided to use $M = 100$, $N = 20$ and the RMSE(u) of six cases are shown in Table 1.

Case	1	2	3	4	5	6
RMSE(u)	1.004E-2	3.999E-3	1.318E-3	2.408E-3	8.423E-3	3.887E-3

Table 1: RMSEs of $u(x, 1)$ for the direct problem obtained by $M = 100$ and $N = 20$ of Cases 1-6.

In what follows, the node counts $M = 100$ and $N = 20$ are used and the initial guess of the source function is set to zero, i.e. $\underline{r}^{(0)} = \underline{0}$.

5.1. Case of exact input data

For the numerical results, we first consider the case of exact data, no noise inputs to the system, and found that the results can be analysed as follows.

Cases 1, 2 and 5

The approximate solution $r(t_j)$ in Case 1 with no contamination obtained by solving matrix equation (3.19) and minimizing the objective function (3.20), i.e. $\Gamma_1(r)$, is shown in Figure 1 it is obviously to see that the approximate solution showing in dash-cross line ($-\times$) converged to the analytical solution showing in solid line ($-$) but slightly deviated at $t = 0$ and $t \in (0.8, 1]$ with RMSE(r)=3.928E-1. One thing to remark that at the starting point $t = 0$ the result is imprecision because the initial guess is zero $\underline{r}^{(0)} = \underline{0}$. Note that the computation setting as default tolerance which let the solver stop various states depending on the reach of tolerance set. In order to reduce the accuracy happened, the regularized objective function $\Gamma^{\epsilon}\left(r_\lambda\right)$ in (4.3) needs to be employed. The trial and error technique was used to find a suitable regularization parameter from 10^{-9} to 10^{+2}, i.e. testing $\lambda \in \{10^{-9}, 10^{-8}, \dots, 10^{+2}\}$, and $\lambda = 10^{-5}$ has been found to be the most appropriate regularization parameter. This gives the more stable result as shown in Figure 1 with circles line ($\circ\circ\circ$) and RMSE(r)=2.985E-1.

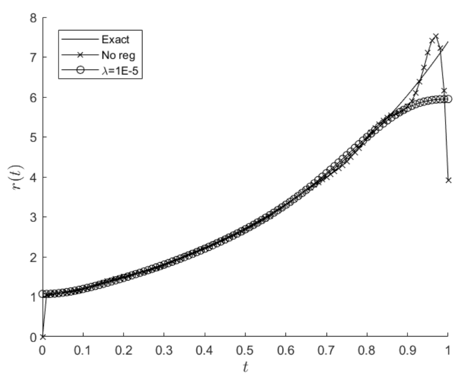

Figure 1: The solution of $r(t)$ obtained by minimizing the objective function $\Gamma_1\left(r_\lambda\right)$ without noisy input and $\lambda \in \{0, 10^{-5}\}$ obtained by $M = 100$ and $N = 20$, for Case 1.

Note that the slight inaccuracy displayed in circle line around the end point is commonly obtained with the use of regularization or stabilizing techniques such as the mollification method, or Tikhonov regularization of a Fredholm integral equation as presented in detail in [18].

As we have tested, without illustration shown, we found that the approximate solution in Cases 2 and 5 without regularization displayed similar to Case 1, and the use of regularization can be reduced the inaccuracy occurred as can be seen by RMSE(r) detailed in Table 2.

Case	1	2	3	4	5	6
RMSE(r); $\lambda = 0$	3.928E-1	1.050E-1	1.027E-1	1.039E-1	3.379E-1	1.231E-1
RMSE(r); $\lambda = 10^{-5}$	2.985E-1	1.023E-1	6.350E-2	1.527E-1	2.644E-1	4.861E-1

Table 2: RMSEs of the source function $r(t)$ for the inverse problem with no noisy input and $\lambda \in \{0, 10^{-5}\}$, for Cases 1-6.

Cases 3-4 and 6

In Case 3, the result $r(t)$ obtained by minimizing the objective function $\Gamma_3(r)$ for the case without noisy input data and no regularization, is illustrated in Figure 2 showing good agreement of exact ($-$) and numerical ($-\times$) solutions, except at time $t = 0$, with RMSE(r)=1.027E-1. We further consider the solution of this case using regularization with $\lambda = 10^{-5}$ as it was sought to be the most suitable regularization parameter among $\lambda \in \{10^{-9}, 10^{-8}, \ldots, 10^{+2}\}$, and found that the regularized numerical solution with RMSE(r)=6.350E-2 behaves well at $t = 0$ but slight inaccuracies near $t = 1$ as displayed in Figure 2 with circle line ($\circ\circ\circ$).

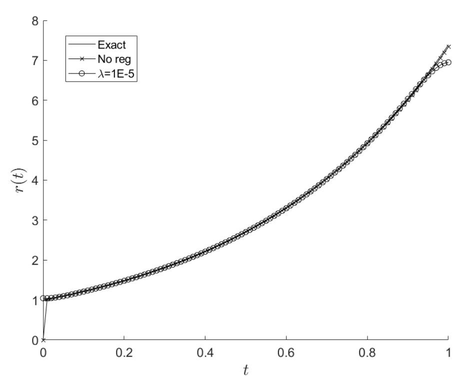

Figure 2: The solution of $r(t)$ obtained by minimizing the objective function $\Gamma_3\left(r_\lambda\right)$ with no noisy input obtained by $M = 100$ and $N = 20$, for Case 3.

Looking more closely at the non-regularized result in Figure 2 with dash-cross line ($-\times$), we found that on ignoring $t = 0$ this numerical solution is very accurate elsewhere, with RMSE(r)=2.548E-2. This starting point is inaccurate because the initial guess is zero $\underline{r}^{(0)} = \underline{0}$ whilst without regularization it stays as $r(0) = 0$. Then we concluded that the regularization is no need for this case as the use of regularization does not reduce $\mathrm{RMSE}(r)$ properly, moreover the inaccuracy occurred only at $t = 0$ when no regularization, whilst it happened around $t \in (0.8, 1]$ when the regularization employed.

The study of Cases 4 and 6 found that the solutions with no noisy input were similar to Case 3 that the numerical solutions obtained by solving the matrix equation (3.19) together with minimizing the objective function (3.20) without regularization, $\lambda = 0$, had very good accuracy except at time $t = 0$. Whereas when the regularization was applied the $\mathrm{RMSE}(r)$ could not reduce, and the error around $t \in (0.8, 1]$ still come up. Like in Case 3, we concluded that the use of regularization is not necessary as we found that the result with no regularization provided sufficiently accurate results, see Table 2 for $\mathrm{RMSE}(r)$.

5.2. Case of noisy input $p = 1\%$

We next consider the case of a noisy system. The contamination was added to the system as $\underline{k}^{\varepsilon}$ in (4.1) with $p = 1\%$, and the solution obtained by minimizing the perturbed objective function $\Gamma^{\varepsilon}(r_{\lambda})$ had tested. In Figure 3 the dash-dot line ($-\bullet$) shows the result in Case 1 without regularization, i.e. $\lambda = 0$, whereas the result with $\lambda = 10^{-5}$ is displayed with circles ($\circ\circ\circ$). It is obvious that the Tikhonov regularization reduced inaccuracy and stabilized the solution, and the regularized result converge to the exact solution with $\mathrm{RMSE}(r)$ reduced from 6.175 to 2.748E-1.

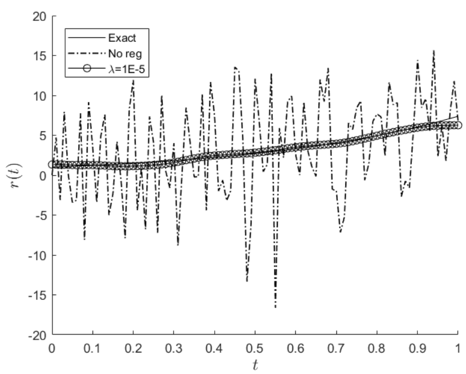

Figure 3: The solution of $r(t)$ obtained by minimizing the objective function $\Gamma_1^{\varepsilon}(r_{\lambda})$ with $p = 1\%$ noisy input and $\lambda \in \{0, 10^{-5}\}$, for Case 1.

Considering the other cases found that when noise is added to the input data $\underline{k}^{\varepsilon}$ with $p = 1\%$, the use of the first–order Tikhonov regularization could deal with the instability of the system and the approximate result with regularization much improved accuracies by using $\lambda = 10^{-5}$ as can be seen from the RMSE(r) shown in Table 3.

Case	1	2	3	4	5	6
RMSE(u); $\lambda = 0$	6.175	9.617E-1	7.740E-1	3.461	6.400	2.19E+1
RMSE(u); $\lambda = 10^{-5}$	2.748E-1	1.530E-1	1.436E-1	2.588E-1	2.519E-1	4.734E-1

Table 3: RMSEs of the source function $r(t)$ for the inverse problem with $p = 1\%$ noisy input and $\lambda \in \{0, 10^{-5}\}$, for Cases 1-6.

6. Conclusion

In this study, the inverse problem of finding a time-dependent source function for the heat equation with nonlocal boundary conditions and over-determination conditions were investigated. The finite integration method based on linear ordinary approximation together with finite differences were employed to discretize the system and to find approximate numerical solutions. In the examples, the numerical solutions obtained by minimizing linear least-squares were inaccurate and unstable, but with Tikhonov's regularization minimization the results converged to the exact solution in all cases and gave stable solutions with and without noise corrupting the input data.

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