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Imaginary Bicyclic Biquadratic Number Fields with Class Number 5

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Abstract. An imaginary bicyclic biquadratic number field K is a field of the form $\mathbb{Q}\left(\sqrt{-m}, \sqrt{-n}\right)$ where m and n are squarefree positive integers. The ideal class number h_K of K is the order of the abelian group I_K/P_K, where I_K and P_K are the groups of fractional and principal fractional ideals in the ring of integers $\mathcal{O}_K$ of K, respectively. This provides a measure on how far is $\mathcal{O}_K$ from being a PID. We determine all imaginary bicyclic biquadratic number fields with class number 5. We show there are exactly 243 such fields.

1. Introduction

Given an algebraic number field K with ring of integers $\mathcal{O}_K$, we let I_K be the group of nonzero fractional and integral ideals of $\mathcal{O}_K$, and P_K be the subgroup of principal fractional ideals of I_K. Then, the factor group I_K/P_K is what we refer to as the *ideal class group* of K which we will denote by $H(K)$. The order of this group is called the *class number* of K which we will write as h_K.

$$1 \longrightarrow P_K \xrightarrow{f} I_K \xrightarrow{g} I_K/P_K \longrightarrow 1.$$

We have here a short exact sequence where $f\colon P_K \to I_K$ is the canonical injection, and $g\colon I_K \to I_K/P_K$ is the canonical projection. Moreover, I_K/P_K forms an abelian group under Minkowski product which is defined as

$$\mathfrak{a}\mathfrak{b} := \left\{ \sum_{i=1}^{r} a_i b_i \colon r \geq 1, a_i \in \mathfrak{a}, b_i \in \mathfrak{b} \right\}$$

where $\mathfrak{a}, \mathfrak{b}$ are ideals of $\mathcal{O}_K$.

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Observe that $H(K)$ is trivial when all ideals in $\mathcal{O}_K$ are principal. Thus, the class number of a field can be viewed as a measure of how "far" $\mathcal{O}_K$ is from being a principal ideal domain, and consequently, from being a unique factorization domain.

In this study, we focus on *imaginary bicyclic biquadratic fields* which are algebraic number fields of the form $\mathbb{Q}\left(\sqrt{-m}, \sqrt{-n}\right)$, where m and n are distinct positive squarefree integers.

For the case of imaginary biquadratic fields, Brown and Parry [1] determined all 47 such fields with class number 1. Buell, H. Williams, and K. Williams [2] identified all 160 such fields with class number 2, and Jung and Kwon [4] identified all 163 such fields with class number 3.

We follow the approach of Jung and Kwon by using Kuroda's class number formula which relates the class number of a multiquadratic number field to that of its subfields. We use the lists of imaginary quadratic fields with class numbers $1, 2, 5$, and 10 as we determine all imaginary bicyclic biquadratic number fields with class number 5.

2. Kuroda's Class Number Formula

Let $K = \mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ be an imaginary bicyclic biquadratic field with class number h_K. We denote its real quadratic subfield $\mathbb{Q}(\sqrt{n_1 n_2})$ by k_+ with class number h_+. The two imaginary quadratic subfields $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ are to be written as k_1 and k_2 with class numbers h_1 and h_2, respectively. From [5], we have Kuroda's class number formula:

$$h_K = \frac{Q}{2} h_+ h_1 h_2,$$

where the unit index $Q = [\mathcal{O}_K^\times : \mathcal{O}_{k_+}^\times \mathcal{O}_{k_1}^\times \mathcal{O}_{k_2}^\times]$ is 1 or 2.

To identify the value of Q, we use a theorem by Conner and Hurrelbrink [3].

Theorem 2.1. *The imaginary bicyclic biquadratic fields with odd class numbers are:*

1. $\mathbb{Q}(\sqrt{-p}, \sqrt{-pq})$ *with* $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = -1$,
2. $\mathbb{Q}(\sqrt{-1}, \sqrt{-q}), \mathbb{Q}(\sqrt{-2}, \sqrt{-2q})$ *with* $q \equiv 5 \pmod 8$
3. $\mathbb{Q}(\sqrt{-p}, \sqrt{-2p})$ *with* $p \equiv 3 \pmod 8$,
4. $\mathbb{Q}(\sqrt{-1}, \sqrt{-2})$,
5. $\mathbb{Q}(\sqrt{-1}, \sqrt{-p})$,
6. $\mathbb{Q}(\sqrt{-2}, \sqrt{-p})$,
7. $\mathbb{Q}(\sqrt{-p_1}, \sqrt{-p_2})$,

where p, p_1, p_2, q *are primes such that* $p, p_1, p_2 \equiv 3 \pmod 4$, *and* $q \equiv 1 \pmod 4$.

The class numbers of $\mathbb{Q}(\sqrt{-pq}), \mathbb{Q}(\sqrt{-q}), \mathbb{Q}(\sqrt{-2q})$, and $\mathbb{Q}(\sqrt{-2p})$ are known to be congruent to $2 \pmod 4$. Meanwhile, all other proper quadratic subfields from the listed cases have odd class numbers [3]. By comparison of parities through Kuroda's class number formula, we obtain

$$\text{odd} = \frac{Q}{2} \times \text{odd} \times \text{odd} \times (\text{odd or even}).$$

Thus, cases $1, 2$, and 3 must have $Q = 1$ while cases $4, 5, 6$, and 7 have $Q = 2$.

3. Determination of Fields with Class Number 5

We now determine all imaginary bicyclic biquadratic fields with class number 5. Using Kuroda's class number formula, we set $h_K = 5$. Furthermore, as imaginary biquadratic fields are CM-fields, h_+ should divide h_K [4]. We are then left with the following cases, up to permutation of indices for the imaginary quadratic fields.

1. $h_+ = h_1 = 1, h_2 = 10$ and $Q = 1$,
2. $h_+ = 1, h_1 = 2, h_2 = 5$ and $Q = 1$,
3. $h_+ = 5, h_1 = 1, h_2 = 2$ and $Q = 1$,
4. $h_+ = h_1 = 1, h_2 = 5$ and $Q = 2$,
5. $h_+ = 5, h_1 = h_2 = 1$ and $Q = 2$.

The table below contains a complete list of imaginary quadratic fields with class numbers $1, 2, 5$, and 10 ([6],[7],[9],[10]). Meanwhile, the L-functions and modular forms database (LMFDB) [8] was used to identify the class numbers of real quadratic fields.

$h_{\mathbb{Q}(\sqrt{-n})}$	n
1	1, 2, 3, 7, 11, 19, 43, 67, 163
2	5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427
5	47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683
10	74, 86, 119, 122, 143, 159, 166, 181, 197, 218, 229, 303, 317, 319, 346, 373, 394, 415, 421, 422, 538, 541, 611, 613, 635, 694, 699, 709, 757, 779, 803, 851, 853, 877, 923, 982, 1093, 1115, 1213, 1318, 1643, 1707, 1779, 1819, 1835, 1891, 1923, 2363, 2643, 2899, 3091, 3139, 3147, 3291, 3635, 3667, 3683, 3811, 3859, 4083, 4227, 4435, 4579, 4627, 4915, 5131, 5163, 5515, 5611, 5667, 5803, 6115, 6259, 6403, 6667, 7123, 7363, 7387, 7435, 7483, 7627, 8227, 8947, 9307, 10147, 10483, 13843

Table 1: Class number of quadratic field $\mathbb{Q}(\sqrt{-n})$

The determination of all imaginary bicyclic biquadratic fields with class number 5 is as follows:

Case 1: $h_+ = h_1 = 1, h_2 = 10$ and $Q = 1$.

From our list of imaginary quadratic fields $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ with class numbers 1 and 10, respectively, we determine the real quadratic fields $\mathbb{Q}(\sqrt{n_1 n_2})$ with class number 1. There are 58 such fields. From these, we determine the biquadratic fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ that satisfy any of conditions 1, 2, and 3 in Theorem 2.1. There are exactly 48 such fields. We verify that the ten other biquadratic fields have class number 10 through a theorem of Buell, H. Williams, and K. Williams [2]:

Lemma 3.1. *Let $N(\epsilon)$ be the norm of the fundamental unit ϵ of $\mathbb{Q}(\sqrt{n_1 n_2})$. Suppose $N(\epsilon) = 1$. If $\sqrt{-1}$ or $\sqrt{-2} \in K = \mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$, then $Q = 1$ if and only if $\sqrt{2\epsilon} \notin \mathcal{O}_K$.*

$\mathbb{Q}(\sqrt{-1}, \sqrt{-86})$

$$\epsilon = 10405 + 1122\sqrt{86}$$
$$2\epsilon = 20810 + 2244\sqrt{86} = (102 + 11\sqrt{86})^2$$

$\mathbb{Q}(\sqrt{-1}, \sqrt{-166})$

$$\epsilon = 1700902565 + 132015642\sqrt{166}$$
$$2\epsilon = 3401805130 + 264031284\sqrt{166} = (41242 + 3201\sqrt{166})^2$$

$\mathbb{Q}(\sqrt{-1}, \sqrt{-422})$

$$\epsilon = 7022501 + 341850\sqrt{422}$$
$$2\epsilon = 14045002 + 683700\sqrt{86} = (2650 + 129\sqrt{422})^2$$

$\mathbb{Q}(\sqrt{-1}, \sqrt{-694})$

$$\epsilon = 38782105445014642382885 + 1472148590903997672114\sqrt{694}$$
$$2\epsilon = 77564210890029284765770 + 2944297181807995344228\sqrt{694}$$
$$= (196931727878 + 7475426163\sqrt{694})^2$$

$\mathbb{Q}(\sqrt{-1}, \sqrt{-1318})$

$$\epsilon = 11393611468262768176276517 + 313836679699540895794554\sqrt{1318}$$
$$2\epsilon = 22787222936525536352553034 + 627673359399081791589108\sqrt{1318}$$
$$= (3375442410746 + 92976458049\sqrt{1318})^2$$

$\mathbb{Q}(\sqrt{-2}, \sqrt{-86})$

$$\epsilon = 3482 + 531\sqrt{43}$$
$$2\epsilon = 6964 + 1062\sqrt{43} = (59 + 9\sqrt{43})^2$$

$\mathbb{Q}(\sqrt{-2}, \sqrt{-166})$

$$\epsilon = 82 + 9\sqrt{83}$$
$$2\epsilon = 164 + 18\sqrt{83} = (9 + \sqrt{83})^2$$

$\mathbb{Q}(\sqrt{-2}, \sqrt{-422})$

$\epsilon = 278354373650 + 19162705353\sqrt{211}$

$2\epsilon = 556708747300 + 38325410706\sqrt{211} = (527593 + 36321\sqrt{211})^2$

$\mathbb{Q}(\sqrt{-2}, \sqrt{-694})$

$\epsilon = 641602 + 34443\sqrt{347}$

$2\epsilon = 1283204 + 68886\sqrt{347} = (801 + 43\sqrt{347})^2$

$\mathbb{Q}(\sqrt{-2}, \sqrt{-982})$

$\epsilon = 93628044170 + 4225374483\sqrt{491}$

$2\epsilon = 187256088340 + 8450748966\sqrt{491} = (305987 + 13809\sqrt{491})^2$

Thus, these ten imaginary biquadratic fields have class number 10.

Case 2: $h_+ = 1, h_1 = 2, h_2 = 5$ and $Q = 1$.

There is only one such field with $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ are of class numbers 2 and 5, respectively, and their corresponding real quadratic field $\mathbb{Q}(\sqrt{n_1 n_2})$ has class number 1. This field is $\mathbb{Q}(\sqrt{-235}, \sqrt{-47})$. This satisfies condition 1 of Theorem 2.1. Thus, this field has class number 5.

Case 3: $h_+ = 5, h_1 = 1, h_2 = 2$ and $Q = 1$.

There is no such field satisfying the given conditions.

Case 4: $h_+ = h_1 = 1, h_2 = 5$ and $Q = 2$.

There are 193 fields such that $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ have class numbers 1 and 5, respectively, and their corresponding real quadratic field $\mathbb{Q}(\sqrt{n_1 n_2})$ has class number 1. All of these fields satisfy one of conditions 5, 6, and 7 of Theorem 2.1. Thus, these fields have class number 5.

Case 5: $h_+ = 5, h_1 = h_2 = 1$ and $Q = 2$.

There is only one such field with $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ both have class number 1, and their corresponding real quadratic field $\mathbb{Q}(\sqrt{n_1 n_2})$ has class number 5. This field is $\mathbb{Q}(\sqrt{-19}, \sqrt{-43})$. This satisfies condition 7 of Theorem 2.1. Thus, this field has class number 5.

With these, we have proven the following proposition.

Proposition 3.2. *There are exactly 243 imaginary bicyclic biquadratic fields with class number 5.*

The imaginary bicyclic biquadratic fields with class number 5 are listed as follows for each case:

n_1	n_2
1	181, 197, 317, 373, 421, 541, 613, 709, 757, 853, 877, 1213
2	74, 122, 218, 346, 394, 538
3	159, 303, 699, 1707, 1779, 1923, 2643, 3147, 3291, 4083, 4227, 5163, 5667
7	119, 4627, 5803, 7483
11	143, 319, 803, 3091, 6259, 10483
19	779, 3667, 4579, 6403, 8227
43	86, 3139

Table 2: Fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = h_1 = 1, h_2 = 10$

n_1	n_2
235	47

Table 3: Fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = 1, h_1 = 2, h_2 = 5$

n_1	n_2
1	47, 103, 127, 131, 179, 227, 347, 523, 571, 619, 683, 691, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683
2	47, 79, 103, 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683
3	47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 739, 787, 947, 1051, 1123, 1723, 1867, 2203, 2347
7	47, 79, 103, 127, 131, 179, 227, 443, 523, 619, 683, 691, 739, 787, 947, 1051, 1723, 1747, 2203, 2347, 2683
11	47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683
19	47, 79, 127, 131, 179, 227, 347, 443, 571, 619, 691, 739, 787, 947, 1051, 1747, 2203, 2347, 2683
43	79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 691, 739, 787, 947, 1123, 1747, 1867, 2203, 2347, 2683
67	47, 79, 127, 131, 179, 227, 571, 619, 683, 691, 739, 787, 947, 1123, 1723, 1747, 1867, 2347
163	47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 619, 691, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683

Table 4: Fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = h_1 = 1, h_2 = 5$

n_1	n_2
19	43

Table 5: Fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = 5, h_1 = h_2 = 1$

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