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Existence Results for an Nonlinear Variable Exponents Anisotropic Elliptic Problems

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Abstract. In this paper, we prove the existence of distributional solutions in the anisotropic Sobolev space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with variable exponents and zero boundary, for a class of variable exponents anisotropic nonlinear elliptic equations having a compound nonlinearity $G(x,u)=\sum_{i=1}^{N}(\mid f\mid+\mid u\mid)^{p_i(x)-1}$ on the right-hand side, such that f is in the variable exponents anisotropic Lebesgue space $L^{\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p}(\cdot)=(p_1(\cdot),\dots,p_N(\cdot))\in(C(\overline{\Omega},]1,+\infty[))^N$.

1. Introduction

In this work, we demonstrate the existence of distributional solutions for a specific class of anisotropic nonlinear elliptic partial differential equations with variable exponents, of the type :

$$(1.1)\qquad -\sum_{i=1}^{N}\partial_i\big(\mid\partial_i u\mid^{p_i(x)-2}\partial_i u\big)=\sum_{i=1}^{N}(\mid f\mid+\mid u\mid)^{p_i(x)-1},\quad \text{in }\Omega,$$
$$u=0,\quad \text{on }\partial\Omega,$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is an open bounded Lipschitz domain (having a Lipschitz boundary $\partial\Omega$), $\partial_i u=\frac{\partial u}{\partial x_i}$, $i=1,\dots,N$, and the datum f belongs to the variable exponents anisotropic Lebesgue space $L^{\overrightarrow{p}(\cdot)}(\Omega)$, which is defined as follows:

$$L^{\overrightarrow{p}(\cdot)}(\Omega)=\bigcap_{i=1}^{N}L^{p_i(\cdot)}(\Omega).$$

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Here, problem (1.1) is $\overrightarrow{p}(x)$-Laplacian operator equations, which involve the anisotropic operator with variable exponents defined between the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(which will be further discussed in Section 2) and its dual, as follows:

$$u \mapsto -\sum_{i=1}^{N} \partial_i\big(\mid \partial_i u \mid^{p_i(x)-2} \partial_i u\big).$$

It is important to note that operators of this type have numerous applications in applied sciences. For instance, they are commonly used in electro-rheological fluids and image processing, as seen in references [6, 7, 2]. The noveltly of our work is that we take the right-hand side as a compound nonlinearity $G(x,u) = \sum_{i=1}^{N} (\mid f \mid + \mid u \mid)^{p_i(x)-1}$ that links the unknown u and the datum $f \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. One cannot seperate these to reduce the problem to the classical case where the right-hand side is in certain Sobolev spaces.

The proof is based on the usual method, which requires proving the existence of a sequence of suitable approximate solutions (u_n) using the Leray-Schauder's fixed point Theorem. Prior estimates are then used to show the boundedness of the solutions u_n and the almost everywhere convergence of their partial derivatives $\partial_i u_n$, $i = 1, \ldots, N$, which can be converted into strong L^1-convergence. With this convergence, we can pass to the limit by L^1-strongly sense in $\mid \partial_i u_n \mid^{p_i(x)-2} \partial_i u_n$, and in $(\mid f_n \mid + \mid u_n \mid)^{p_i(x)-1}$, and finally we conclude the convergence of u_n to the solution of (1.1).

The paper is divided into several sections, with Section 2 covering mathematical preliminaries. In this section, we discuss variable exponents anisotropic Lebesgue-Sobolev spaces and their key characteristics, as well as mentioning some embedding theorems. The main theorem and its proof can be found in Section 3.

2. Preliminaries

In this section, we will provide a brief reminder about variable exponent anisotropic Lebesgue and Sobolev spaces. We will mention their most important properties and facts that are relevant to this paper. For further information, please refer to sources [5, 1, 3]).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset, we define the set

$$\mathcal{C}_+(\overline{\Omega}) = \{p(\cdot) \in C\big(\overline{\Omega}, \mathbb{R}\big), \quad 1 < p^- \leq p^+ < \infty\},$$

where, $p^+ = \max_{x\in\overline{\Omega}} p(x)$, and $p^- = \min_{x\in\overline{\Omega}} p(x)$.

Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. Then the following Young's inequality holds true for all $a, b \in \mathbb{R}$ and all $\varepsilon > 0$,

$$|ab| \leq \varepsilon |a|^{p(x)} + c(\varepsilon)|b|^{p'(x)}, \tag{2.1}$$

where, $p'(\cdot)$ denotes the Sobolev conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$).

In addition, for any two real a, b $((a,b) \neq (0,0))$:

$$(|a|^{p(x)-2}a - |b|^{p(x)-2}b)(a-b) \geq \begin{cases} 2^{2-p^+}|a-b|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|a-b|^2}{(|a|+|b|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{2.2}$$

Also, we will recall this elementary inequality:

$$(a_1 + \ldots + a_m)^r \leq \max\{1, m^{r-1}\}(a_1^r + \ldots + a_m^r), \tag{2.3}$$

which is valid for $a_i \geq 0,\ i = 1, \ldots, m$ and $r \geq 0$.
The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R};\ \rho_{p(\cdot)}(u) < \infty\},$$

where,

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx, \quad \text{the convex modular of } u.$$

It is a Banach space, and reflexive if $p^- > 1$, under the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\gamma > 0 \mid \rho_{p(\cdot)}(u/\gamma) \leq 1\right\}.$$

The Hölder type inequality:

$$|\int_\Omega uv\, dx| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

holds true.
The variable exponents Sobolev space $W^{1,p(\cdot)}(\Omega)$ defined as fellows

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |Du| \in L^{p(\cdot)}(\Omega)\right\},$$

it becomes a Banach space when equipped with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|Du\|_{p(\cdot)}. \tag{2.4}$$

We define also the Banach space $W_0^{1,p(\cdot)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

endowed with the norm (2.4). Moreover, is reflexive and separable if $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.
The following results came in [1, 3].
If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then we have

$$\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right), \tag{2.5}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.6}$$

Now, we will introduce the concept of anisotropic Sobolev spaces with variable exponents $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, as we need them to solve our problem (1.1).

Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1,+\infty)\big),\ i = 1,\dots,N$, and we set for every x in $\overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x),\dots,p_N(x)),\quad p_+(x) = \max_{1\le i\le N} p_i(x),\, p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$\overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N}\frac{1}{p_i(x)}},\quad p_+(x) = \max_{1\le i\le N} p_i(x),$$

$$p_+^+ = \max_{x\in\overline{\Omega}} p_+(x),\quad p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$p_-^- = \min_{x\in\overline{\Omega}} p_-(x),\quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \ge N. \end{cases}$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \Big\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1,\dots,N\Big\},$$

under the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

The spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are defined as follow

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},\quad \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

The following embedding results given in [4, 5].
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

Lemma 2.1. *If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}, r(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact.} \tag{2.7}$$

Lemma 2.2. *If we have*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{2.8}$$

Then the following inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.9}$$

where $C > 0$ independent of u. Thus,

$$u \mapsto \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)} \text{ is an equivalent norm on } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{2.10}$$

3. Statement of Results and Proof

Definition 3.1. The function u is a solution of the problem (1.1) in the sense of distributions if and only if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^{\infty}(\Omega)$,

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx = \sum_{i=1}^{N} \int_{\Omega} (| \, f \, | + | \, u \, |)^{p_i(x)-1} \varphi \, dx.$$

Our main result is the following.

Theorem 3.1. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ such that $\overline{p} < N$ and (2.8) holds, and assume that $f \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then the problem (1.1) has at least one distributional solution $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

3.1. Existence of approximate solutions

Let (f_n) be a sequence of bounded functions defined in Ω which converges to f in $L^{\overrightarrow{p}(\cdot)}(\Omega)$. Since $f_n \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, from (2.5) we obtain

$$\|f_n\|_{p_i(\cdot)} \leq 1 + \rho_{p_i(\cdot)}^{\frac{1}{p_i^-(\cdot)}}(f_n) \leq 2 + \rho_{p_i}^{\frac{1}{p_-}}(f_n) < \infty.$$

Through this, we conclude that

$$f_n \text{ is bounded in } L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N. \tag{3.1}$$

Lemma 3.1. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ such that $\overline{p} < N$ and (2.8) holds, and assume that $f \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$-\sum_{i=1}^{N} \partial_i \big(| \, \partial_i u_n \, |^{p_i(x)-2} \, \partial_i u_n \big) = \sum_{i=1}^{N} (| \, f_n \, | + | \, u_n \, |)^{p_i(x)-1}, \quad \text{in } \Omega, \qquad u_n = 0, \quad \text{on } \partial\Omega, \tag{3.2}$$

in the following sense

$$\sum_{i=1}^{N} \int_{\Omega} | \, \partial_i u_n \, |^{p_i(x)-2} \, \partial_i u_n \partial_i \varphi \, dx = \sum_{i=1}^{N} \int_{\Omega} (| \, f_n \, | + | \, u_n \, |)^{p_i(x)-1} \varphi \, dx, \tag{3.3}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$.

Before proving Lemma 3.1 we must prove the following lemma:

Lemma 3.2. *Let $n \in \mathbb{N}^*$ fixed, and for all $(v, \theta) \in X \times [0,1]$ where $X = L^{p_+(\cdot)}(\Omega)$ we consider the problem*

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(| \partial_i u |^{p_i(x)-2} \partial_i u \right) = \theta \sum_{i=1}^{N} (| f_n | + | v |)^{p_i(x)-1}, \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega. \end{cases} \tag{3.4}$$

For all $(v, \theta) \in X \times [0,1]$ the problem (3.4) has only the weak solution u satisfying for all $\varphi \in \overset{\circ}{W}{}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the weak formulation

$$\sum_{i=1}^{N} \int_\Omega | \partial_i u |^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx = \theta \sum_{i=1}^{N} \int_\Omega (| f_n | + | v |)^{p_i(x)-1} \varphi \, dx. \tag{3.5}$$

Moreover, the operator $\Psi : X \times [0,1] \longrightarrow X$ defined by :

$$\Psi(v, \theta) = u \Leftrightarrow (u \text{ is the only weak solution of the problem (3.4)}),$$

is continuous and compact.

Proof Using (2.3) and the fact that $f_n, v \in L^{p_i(\cdot)}(\Omega)$ we get for all $(v, \theta) \in X \times [0,1]$ that

$$\int_\Omega \left((| f_n | + | v |)^{p_i(x)-1} \right)^{p_i'(x)} dx \le c \int_\Omega (| f_n |^{p_i(x)} + | v |^{p_i(x)}) \, dx \le C. \tag{3.6}$$

Therefore, from (3.6) we obtain for all $i = 1, \dots, N$,

$$(| f_n | + | v |)^{p_i(x)-1} \in L^{p_i'(\cdot)}(\Omega),$$

and this implies that

$$\theta \sum_{i=1}^{N} (| f_n | + | v |)^{p_i(x)-1} \in L^{\overrightarrow{p}'(\cdot)}(\Omega) \left(= \bigcap_{i=1}^{N} L^{p_i'(\cdot)}(\Omega) \right).$$

Here, $\overrightarrow{p}'(\cdot) = (p_1'(\cdot), \dots, p_N'(\cdot))$ where $p_i'(\cdot)$ denotes the Sobolev congugate of $p_i(\cdot)$. The existence of the weak solution u of the problem (3.4) in $L^{p_+(\cdot)}(\Omega)$ is directly produced by the main Theorem on monotone operators, and the uniqueness of this solution is a direct result of the homogeneous problem and this by assuming the existence of two weak solutions.

Now we give an estimate of the solution u of the problem (3.4). Taking $\varphi = u$ as test function, and using (2.8), (2.3), (2.5), (2.6), the fact that $p_i(\cdot) \le \overline{p}^*(\cdot)$ (from

(2.8)), Lemma 2.1, boundedness of $f_n \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, and Hölder inequality, we have

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u \mid^{p_i(x)} \ dx &\leq \sum_{i=1}^{N}\int_{\Omega} (\mid f_n \mid + \mid v \mid)^{p_i(x)-1} \mid u_n \mid \ dx \\
&\leq 2\Big\| \sum_{i=1}^{N} (\mid f_n \mid + \mid v \mid)^{p_i(x)-1} \Big\|_{p_i'(x)} \|u\|_{p_i(\cdot)} \\
&\leq c\left(\sum_{i=1}^{N} \| \mid f \mid^{p_i(x)-1} \|_{p_i'(x)} + \sum_{i=1}^{N} \| \mid v \mid^{p_i(x)-1} \|_{p_i'(x)}\right)\|u\|_{\overrightarrow{p}(x)} \\
&\leq c\left(2N + \sum_{i=1}^{N}\left(\int_{\Omega} \mid f_n \mid^{p_i(x)} \ dx\right)^{\frac{1}{p_i^-}} + \sum_{i=1}^{N}\left(\int_{\Omega} \mid v \mid^{p_i(x)} \ dx\right)^{\frac{1}{p_i^-}}\right)\|u\|_{\overrightarrow{p}(x)} \\
&\leq c\left(c' + \sum_{i=1}^{N} \|v\|_{p_i(\cdot)}^{\frac{p_i^+}{p_i^-}}\right)\|u\|_{\overrightarrow{p}(x)}
\end{aligned}
$$

(3.7)

$$
\leq C\left(1 + \|v\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}\right)\|u\|_{\overrightarrow{p}(x)},
$$

where for the last equality we used that

$$
c' + \sum_{i=1}^{N} \|v\|_{p_i(\cdot)}^{\frac{p_i^+}{p_i^-}} \leq c'' + \sum_{i=1}^{N} \|v\|_{p_i(\cdot)}^{\frac{p_+^+}{p_-^-}} \leq c'' + c'''\|v\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}
$$

for appropriate constants c'' and c'''.

On the other hand, by (2.6), we get

$$
1 + \int_{\Omega} \mid \partial_i u \mid^{p_i(x)} \ dx \geq \big\|\partial_i u\big\|_{p_i(\cdot)}^{p_i^-}, \ i = 1, \ldots, N,
$$

and we have

$$
1 + \big\|\partial_i u\big\|_{p_i(\cdot)}^{p_i^-} \geq \big\|\partial_i u\big\|_{p_i(\cdot)}^{p_-^-}, \ i = 1, \ldots, N.
$$

Through this, we find that

$$
2 + \int_{\Omega} \mid \partial_i u \mid^{p_i(x)} \ dx \geq \big\|\partial_i u\big\|_{p_i(\cdot)}^{p_-^-}, \ i = 1, \ldots, N.
$$

Then, we conclude

$$
2N \mid \Omega \mid + \sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u \mid^{p_i(x)} \ dx \geq \sum_{i=1}^{N} \big\|\partial_i u\big\|_{p_i(\cdot)}^{p_-^-}.
$$

So, we get

$$\sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u \mid^{p_i(x)}\, dx \geq \left(\frac{1}{N}\sum_{i=1}^{N}\left\|\partial_i u\right\|_{p_i(\cdot)}\right)^{p_-^-} - 2N \mid \Omega \mid . \tag{3.8}$$

By combining (3.7) and (3.8), we obtain

$$\left\|u\right\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq c\left(1+\|v\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}\right)\|u\|_{\overrightarrow{p}(\cdot)} + c', \tag{3.9}$$

where, $c > 0$ and $c' > 0$. Since $\left\|u\right\|_{\overrightarrow{p}(\cdot)} > 1$, from (3.9) we have

$$\left\|u\right\|_{\overrightarrow{p}} \leq \left(c\left(1+\|v\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}\right)+c'\right)^{\frac{1}{p_-^- -1}}. \tag{3.10}$$

Since $\left\|u\right\|_{\overrightarrow{p}(\cdot)} \leq 1$, we find that (3.10) only holds in this case with under certain conditions, such as $c \geq 1$ or $c' \geq 1$. The goal is to combine the two cases $\left\|u\right\|_{\overrightarrow{p}(\cdot)} > 1$, and $\left\|u\right\|_{\overrightarrow{p}(\cdot)} \leq 1$ into same result (3.10).
We will now prove the continuity of Ψ. Let (v_m, θ_m) be a sequence of $L^{p_+(\cdot)}(\Omega) \times [0, 1]$ converging to (v, θ) in this space. Then,

$$v_m \longrightarrow v, \quad \text{Strongly in } L^{p_+(\cdot)}(\Omega), \tag{3.11}$$

$$\theta_m \longrightarrow \theta, \quad \text{in } \mathbb{R}. \tag{3.12}$$

After considering the sequence (u_m) defined by $u_m = \Psi(v_m, \theta_m)$, $m \in \mathbb{N}^*$, we obtain for n fixed in $\mathbb{N}^*$ and all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N}\int_{\Omega} |\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i \varphi\, dx = \theta_m \sum_{i=1}^{N}\int_{\Omega}(\mid f_n \mid + |v_m|)^{p_i(x)-1}\varphi\, dx. \tag{3.13}$$

For v, θ defined in (3.11), (3.12), we put $u = \Psi(v, \theta)$, then we have for n fixed in $\mathbb{N}^*$ and all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N}\int_{\Omega} \mid \partial_i u \mid^{p_i(x)-2} \partial_i u \partial_i \varphi\, dx = \theta \sum_{i=1}^{N}\int_{\Omega}(\mid f_n \mid + \mid v \mid)^{p_i(x)-1}\varphi\, dx. \tag{3.14}$$

By(3.10) and the boundedness of (v_m) in $L^{p_+(\cdot)}(\Omega)$ (from (3.11)):

$$\left\|u_m\right\|_{\overrightarrow{p}(\cdot)} = \left\|\Psi(v_m, \theta_m)\right\|_{\overrightarrow{p}(\cdot)} \leq \left(c\left(1+\|v_m\|_{p_+(\cdot)}^{\frac{p_+^+}{p_-^-}}\right)+c'\right)^{\frac{1}{p_-^- -1}} \leq \varrho, \tag{3.15}$$

with $\varrho > 0$ independent of m.
From (3.15) we conclude that the sequence (u_m) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. So, there exists a function $w \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_m)) such that

$$u_m \rightharpoonup w \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{3.16}$$

By (3.16), (2.8), and Lemma 2.1, we obtain that

$$u_m \longrightarrow w \quad \text{Strongly in } L^{p_+(\cdot)}(\Omega). \tag{3.17}$$

Since the function $s \mapsto (| f_n | + |s|)^{p_i(x)-1}$ is continuous on $L^{p_+(\cdot)}(\Omega)$, we can pass to the limit in (3.13) as $m \longrightarrow +\infty$, then we get for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i w|^{p_i(x)-2} \partial_i w \partial_i \varphi \, dx = \theta \sum_{i=1}^{N} \int_\Omega (| f_n | + | v |)^{p_i(x)-1} \varphi \, dx, \tag{3.18}$$

and this implies that $w = \Psi(v, \theta)$.
The uniqueness of the weak solution of problem (3.4) then shows that $w = u = \Psi(v, \theta)$. So,

$$\Psi(v_m, \theta_m) = u_m \longrightarrow u = \Psi(v, \theta) \quad \text{Strongly in } L^{p_+(\cdot)}(\Omega).$$

Which shows the continuity of Ψ.
We now move on to prove the compactness of Ψ. Let $\tilde{B}$ be a bounded of $L^{p_+(\cdot)}(\Omega) \times [0, 1]$. Thus $\tilde{B}$ is contained in a product of the type $B \times [0, 1]$ with B a bounded set of $L^{p_+(\cdot)}(\Omega)$, which can be assumed to be a ball of center O and of radius $r > 0$.
For $u \in \Psi(\tilde{B})$, thanks to (3.10), we get

$$\left\|u\right\|_{\overrightarrow{p}(\cdot)} \leq \left(c \left(1 + r^{\frac{p_+^+}{p_-^-}} \right) + c' \right)^{\frac{1}{p_-^- - 1}} = \rho.$$

For $u = \Psi(v, \theta)$ with $(v, \theta) \in B \times [0, 1]$ ($\left\|v\right\|_{p_+(\cdot)} \leq r$). This proves that Ψ applies $\tilde{B}$ in the closed ball in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset L^{p_+(\cdot)}(\Omega)$ of center O and radius ρ. Let u_n be a sequence of elements of $\Psi(\tilde{B})$, therefore $u_n = \Psi(v_n, \theta_n)$ with $(v_n, \theta_n) \in \tilde{B}$. Since u_n remains in a bounded of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, it is possible to extract a subsequence which converges strongly to an element u of $L^{p_+(\cdot)}(\Omega)$. This proves that $\overline{\Psi(\tilde{B})}^{L^{p_+(\cdot)}(\Omega)}$ is compact. So Ψ is compact. □

Proof (of the Lemma 3.1):
It is clear that

$$\Psi(v, 0) = 0, \quad \forall v \in X, \tag{3.19}$$

because $u = 0 \in L^{p_+(\cdot)}(\Omega)$ the only weak solution of the problem (3.4) in the case $\theta = 0$.
Now we show that there is an $M > 0$ such that

$$\forall (v,\theta) \in X \times [0,1] : v = \Psi(v,\theta) \Rightarrow \|v\|_X \leq M. \tag{3.20}$$

For that, we give the estimate for $v \in L^{p_+(\cdot)}(\Omega)$ such that $v = \Psi(v,\theta)$, then we have for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)-2} \partial_i v \partial_i \varphi \, dx = \theta \sum_{i=1}^{N} \int_\Omega (|\, f_n \,| + |\, v \,|)^{p_i(x)-1} \varphi \, dx. \tag{3.21}$$

After choosing $\varphi = v$ in (3.21), and using (2.3), the fact that $f_n \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, $v \in L^{p_+(\cdot)}(\Omega)$, and Young's inequality, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\, \partial_i v \,|^{p_i(x)} \, dx &\leq c \sum_{i=1}^{N} \int_\Omega (|\, f_n \,|^{p_i(x)-1} |\, v \,| + |\, v \,|^{p_i(x)}) \, dx \\
&\leq c \sum_{i=1}^{N} \int_\Omega |\, v \,|^{p_i(x)} \, dx + c \left(C(\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, f_n \,|^{p_i(x)} \, dx + \varepsilon \sum_{i=1}^{N} \int_\Omega |\, v \,|^{p_i(x)} \, dx \right) \\
&= c(1+\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, v \,|^{p_i(x)} \, dx + cC(\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, f_n \,|^{p_i(x)} \, dx \\
&\leq c(1+\varepsilon) \left(N \,|\, \Omega \,| + \sum_{i=1}^{N} \int_\Omega |\, v \,|^{p_+(x)} \, dx \right) + C'(\varepsilon) \leq C''(\varepsilon).
\end{aligned} \tag{3.22}$$

So, for any fixed choice of ε in (3.22), we obtain

$$\sum_{i=1}^{N} \int_\Omega |\, \partial_i v \,|^{p_i(x)} \, dx \leq C. \tag{3.23}$$

Using similar arguments to those used for (3.8), we get

$$\sum_{i=1}^{N} \int_\Omega |\, \partial_i v \,|^{p_i(x)} \, dx \geq \left(\frac{1}{N} \sum_{i=1}^{N} \|\partial_i v\|_{p_i(\cdot)} \right)^{p_-^-} - 2N \,|\, \Omega \,| \,. \tag{3.24}$$

By combining (3.23) and (3.24) with using (2.8), we obtain that

$$\frac{C'}{N^{p_-^-}} \|v\|_{p_+(\cdot)}^{p_-^-} \leq C''. \tag{3.25}$$

From this, we conclude that, there exist $c > 0$, such that

$$\|v\|_{p_+(\cdot)} \leq c. \tag{3.26}$$

This implies (3.20). Through, (3.19), (3.20), and Lemma 3.2, we can apply the Leray-Schauder Theorem. So, the operator $\Psi_1 : X \longrightarrow X$ defined by $\Psi_1(u) = \Psi(u, 1)$ has a fixed point, which shows the existence of a solution of the approximated problems (3.2) in the sense of (3.3). □

3.1.1. A Priori Estimates

Lemma 3.3. *Let f, a and p_i, $i = 1, \dots, N$ be restricted as in Theorem 3.1. Then there exist $C > 0$ independent of n, such that*

$$\|u_n\|_{\overrightarrow{p}(\cdot)} \leq C. \tag{3.27}$$

Proof After choosing $\varphi = u_n$ in (3.3), and using (2.3), the fact that $f_n \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and Young's inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |\, \partial_i u_n \,|^{p_i(x)} \, dx \leq c \sum_{i=1}^{N} \int_\Omega (|\, f_n \,|^{p_i(x)-1} |\, u_n \,| + |\, u_n \,|^{p_i(x)}) \, dx \\
&\leq c \sum_{i=1}^{N} \int_\Omega |\, u_n \,|^{p_i(x)} \, dx + c \left(C(\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, f_n \,|^{p_i(x)} \, dx + \varepsilon \sum_{i=1}^{N} \int_\Omega |\, u_n \,|^{p_i(x)} \, dx \right) \\
&= c(1+\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, u_n \,|^{p_i(x)} \, dx + cC(\varepsilon) \sum_{i=1}^{N} \int_\Omega |\, f_n \,|^{p_i(x)} \, dx \\
&\leq c(1+\varepsilon) \left(N \,|\, \Omega \,|\, + \sum_{i=1}^{N} \int_\Omega |\, u_n \,|^{p_+(x)} \, dx \right) + C'(\varepsilon) \leq C''(\varepsilon).
\end{aligned} \tag{3.28}$$

So, for any fixed choice of ε in (3.28), we obtain

$$\sum_{i=1}^{N} \int_\Omega |\, \partial_i u_n \,|^{p_i(x)} \, dx \leq C. \tag{3.29}$$

By following the same arguments like in (3.8), we get

$$\sum_{i=1}^{N} \int_\Omega |\, \partial_i u_n \,|^{p_i(x)} \, dx \geq \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \partial_i u_n \right\|_{p_i(\cdot)} \right)^{p_-^-} - 2N \,|\, \Omega \,|\, . \tag{3.30}$$

By combining (3.29) and (3.30) with using (2.8), we obtain that

$$\frac{C'}{N^{p_-^-}} \|u_n\|^{p_-^-}_{\overrightarrow{p}(\cdot)} \leq C''. \tag{3.31}$$

From this, we conclude that, there exist $c > 0$ independent of n, such that

$$\left\| u_n \right\|_{\overrightarrow{p}(\cdot)} \leq c. \tag{3.32}$$

Therefore, (3.27) has been proven. □

Lemma 3.4. *There exists a subsequence (still denoted (u_n)) such that, for all $i = 1, \dots, N$*

$$\partial_i u_n \longrightarrow \partial_i u \text{ a.e. in } \Omega. \tag{3.33}$$

Proof From (3.27) the sequence (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
So, there exists a function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega. \tag{3.34}$$

We consider the function

$$\Theta_n = \sum_{i=1}^{N} \int_\Omega \Big(| \partial_i u_n |^{p_i(x)-2} \partial_i u_n - | \partial_i u |^{p_i(x)-2} \partial_i u \Big) (\partial_i u_n - \partial_i u)\, dx,$$

and let's prove that,

$$\lim_{n \to +\infty} \Theta_n = 0. \tag{3.35}$$

We can write Θ_n in the following form

$$\begin{aligned} \Theta_n &= \sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx \\ &- \sum_{i=1}^{N} \int_\Omega | \partial_i u |^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx = I_n - J_n, \end{aligned}$$

where,

$$\begin{aligned} I_n &= \sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx, \\ J_n &= \sum_{i=1}^{N} \int_\Omega | \partial_i u |^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx. \end{aligned}$$

After choosing $\varphi = u_n - u$ in (3.3), with the use of (3.34), and boundedness of $(| f_n | + | u_n |)^{p_i(x)-1}$ in $L^{p_i'(\cdot)}$ (($p_i'(\cdot)$ is the Sobolev conjugate of $p_i(\cdot)$)), we can obtain

$$\lim_{n \to +\infty} I_n = 0. \tag{3.36}$$

Since $(\partial_i u_n)$ is bounded in $L^{p_i(\cdot)}$ (due (3.27)), then there exists a function $w \in L^{p_i(\cdot)}$ and a subsequence (still denoted by $(\partial_i u_n)$) such that

$$\partial_i u_n \rightharpoonup w \quad \text{weakly in } L^{p_i(x)}. \tag{3.37}$$

Through (3.37) and the boundedness of $| \partial_i u_n |^{p_i(x)-2} \partial_i u_n$ in $L^{p_i'(x)}$ we conclude that

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-2} \partial_i u_n (\partial_i u_n - w)\, dx = 0. \tag{3.38}$$

Combining (3.36) and (3.38), we get

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-2} \partial_i u_n (\partial_i u - w)\, dx = 0. \tag{3.39}$$

Equation (3.39) implies that $w = \partial_i u$, and so

$$\partial_i u_n \rightharpoonup \partial_i u \quad \text{weakly in } L^{p_i(x)}. \tag{3.40}$$

From (3.40) and the boundedness of $| \partial_i u |^{p_i(x)-2} \partial_i u$ in $L^{p_i'(x)}$ we conclude that

$$\lim_{n\to+\infty} J_n = 0. \tag{3.41}$$

From (3.36) and (3.41) we get (3.35).
We put for all $i = 1, \ldots, N$

$$\lambda_{i,n}(x) = \Big(| \partial_i u_n |^{p_i(x)-2} \partial_i u_n - | \partial_i u |^{p_i(x)-2} \partial_i u\Big) (\partial_i u_n - \partial_i u).$$

Through (2.2) we conclude that, for all $i = 1, \ldots, N$

$$\lambda_{i,n}(x) > 0. \tag{3.42}$$

Then, (3.42) and (3.35) gives us, for all $i = 1, \ldots, N$

$$\lambda_{i,n}(x) \longrightarrow 0, \quad \text{strongly in } L^1(\Omega). \tag{3.43}$$

Therefore, for a subsequence (still denoted by (u_n)), we get for every $i = 1, \ldots, N$

$$\lambda_{i,n}(x) \longrightarrow 0 \quad \text{a.e. in } \Omega. \tag{3.44}$$

Then there exists a subset $\Omega_0 \subset \Omega$, such that, $| \Omega_0 |= 0$ and for all $x \in \Omega\backslash\Omega_0$

$$| \partial_i u(x) |< \infty, \text{ and } \lambda_{i,n}(x) \longrightarrow 0.$$

From (3.44), we have for some functions k

$$\lambda_{i,n}(x) \leq k(x).$$

Let us prove that, there exists a function g such that

$$| \partial_i u_n(x) |\leq g(x). \tag{3.45}$$

From (2.2), we obtain

$$k(x) \geq \begin{cases} c\Big((|\partial_i u_n| - |\partial_i u|)^{p_-^-} - 1\Big), & \text{if } p_i(x) \geq 2, \\ c'\left(\frac{|\partial_i u_n| - |\partial_i u|}{1+|\partial_i u_n| + |\partial_i u|}\right)^2, & \text{if } 1 < p_i(x) < 2. \end{cases} \tag{3.46}$$

Through (3.46), we obtain (3.45). Now, we proceed by contradiction to prove that

$$\partial_i u_n(x) \longrightarrow \partial_i u(x) \text{ in } \Omega \backslash \Omega_0. \tag{3.47}$$

For this reason, we assume that there exists $x_0 \in \Omega\backslash\Omega_0$ such that $\partial_i u_n(x_0)$ does not converge to $\partial_i u(x_0)$. The Bolzano Weierstrass theorem implies that

$$\partial_i u_n(x_0) \longrightarrow b \in \mathbb{R}.$$

By the passage to the limit in $\lambda_{i,n}(x_0)$ when $n \longrightarrow +\infty$, we obtain

$$\Big(| b |^{p_i(x_0)-2} b - | \partial_i u(x_0) |^{p_i(x_0)-2} \partial_i u(x_0)\Big)(b - \partial_i u(x_0)) = 0.$$

From (2.2), we get that $b = \partial_i u(x_0)$. Therefore, we find that (3.33) has been proven. □

3.2. Proof of the Theorem 3.1

From (3.33) and (3.27), Vitali's theorem gives , for all $i = 1, \ldots, N$

$$\partial_i u_n \longrightarrow \partial_i u \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega. \tag{3.48}$$

So, we have

$$| \partial_i u_n |^{p_i(x)-2} \partial_i u_n \longrightarrow | \partial_i u |^{p_i(x)-2} \partial_i u \;\text{ a.e. in } \Omega. \tag{3.49}$$

By (3.27) we can get, for all $i = 1, \ldots, N$

$$\int_\Omega || \partial_i u_n |^{p_i(x)-2} \partial_i u_n |^{p_i'(x)} \, dx = \int_\Omega | \partial_i u_n |^{p_i(x)} \, dx \leq c, \quad p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot)-1}. \tag{3.50}$$

Equation (3.50) implies that for all $i = 1, \ldots, N$

$$\Big(| \partial_i u_n |^{p_i(x)-2} \partial_i u_n\Big) \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.51}$$

By Young's inequality and since $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we get for all $\varepsilon > 0$

$$\begin{aligned} \int_\Omega || \partial_i u_n |^{p_i(x)-2} \partial_i u_n | \, dx &= \int_\Omega | \partial_i u_n |^{p_i(x)-1} \, dx \\ &\leq C(\varepsilon) + \varepsilon \int_\Omega | \partial_i u_n |^{p_i(x)} \, dx \\ &\leq C(\varepsilon) + \varepsilon c = C'(\varepsilon). \end{aligned} \tag{3.52}$$

For any fixed choice for ε, we conclude that, for all $i = 1, \ldots, N$

$$\Big(| \partial_i u_n |^{p_i(x)-2} \partial_i u_n\Big) \in L^1(\Omega). \tag{3.53}$$

So by (3.53), (3.49), (3.51), and Vitali's theorem, we derive, for all $i = 1, \ldots, N$

$$| \partial_i u_n |^{p_i(x)-2} \partial_i u_n \longrightarrow | \partial_i u |^{p_i(x)-2} \partial_i u \text{ strongly in } L^1(\Omega). \tag{3.54}$$

Now from (3.34) we conclude that

$$(| f_n | + | u_n |)^{p_i(x)-1} \longrightarrow (| f | + | u |)^{p_i(x)-1} \quad \text{a.e. in } \Omega. \tag{3.55}$$

On the other hand, by (2.3) and since f_n, $u_n \in L^{p_i(\cdot)}(\Omega)$, we obtain for all $i = 1, \ldots, N$

$$\begin{aligned} \int_\Omega | (| f_n | + | u_n |)^{p_i(x)-1} |^{p_i'(x)} \, dx &= \int_\Omega (| f_n | + | u_n |)^{p_i(x)} \, dx \\ &\leq c \int_\Omega \Big(| f_n |^{p_i(x)} + | u_n |^{p_i(x)}\Big) \, dx \leq C. \end{aligned} \tag{3.56}$$

Equation (3.56) implies that, for all $i = 1, \ldots, N$

$$(| f_n | + | u_n |)^{p_i(x)-1} \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.57}$$

Like in the proof of (3.53), using the inequality (2.3) and that f_n, $u_n \in L^{p_i(\cdot)}(\Omega)$, we can obtain for all $i = 1, \ldots, N$

$$\Big((| f_n | + | u_n |)^{p_i(x)-1}\Big) \in L^1(\Omega). \tag{3.58}$$

So by (3.58), (3.55), (3.57), and Vitali's theorem, we derive, for all $i = 1, \ldots, N$

$$(| f_n | + | u_n |)^{p_i(x)-1} \longrightarrow (| f | + | u |)^{p_i(x)-1} \text{ strongly in } L^1(\Omega). \tag{3.59}$$

So we can pass to the limit in (3.3). Thus, we have proven the theorem 3.1.

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