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Generalized η-Ricci Solitons on Kenmotsu Manifolds associated to the General Connection

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Abstract. In this paper, we consider generalized η-Ricci solitons associated to the general connection on Kenmotsu manifolds. We confirm the existence of such solitons by constructing a non-trivial example, and we obtain some properties of Kenmotsu manifolds that admit the generalized η-Ricci solitons associated with the general connection.

1. Introduction

Kenmotsu manifolds arise naturally in the study of Sasakian manifolds, which are a special class of contact metric manifolds with additional geometric properties. They have been studied extensively in differential geometry and have important applications in physics, particularly in the study of supersymmetric field theory. The Kenmotsu manifold was introduced by Kenmotsu [10] in 1972 as a new class of almost contact metric manifolds which are very closely related to the warped product manifolds.

A Riemannian metric g on a manifold M is called Ricci soliton [9] if there exists a real constant λ and a vector field V satisfying

$$\mathcal{L}_V g + 2S + 2\lambda g = 0,$$

where $\mathcal{L}_V$ is the Lie derivative along V and S is the Ricci tensor. The notion of generalized Ricci soliton was introduced by Nurowski and Randall [14] as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0,$$

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where $V^\flat$ is defined by $V^\flat(X) = g(V, X)$ for any vector field X. Cho and Kimura [5] introduced the notion of η-Ricci soliton as follows

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0.$$

In particular, if $\rho = 0$, then the η-Ricci soliton equation becomes the Ricci soliton equation. Many authors studied generalizations of the Ricci solitons [1, 4, 6, 11, 12, 13, 15, 18]. Siddiqi [17] introduced the notion of generalized η-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0.$$

The main goal of this paper is to study generalized η-Ricci solitons on Kenmotsu manifolds assoicated to the general connection. We give an example of a generalized η-Ricci soliton on a Kenmotsu manifold associated to the general connection.

The organization of this paper is as follows. After some preliminaries on Kenmotsu manifolds with some necessary and fundamental concepts and formulas in Section 2, we give the main results and their proofs in Section 3. In Section 4, we give an example of Kenmotsu manifolds admitting generalized η-Ricci solitons with respect to the general connection.

2. Preliminaries

In this section, we recall some basic definitions and formulas of contact geometry which will be used in the sequel. Throughout the section, the vector fields X, Y and Z on manifold M are arbitrary unless otherwise stated.
A n-dimensional smooth pseudo-Riemannian manifold (M, g) is an almost contact manifold if there exist a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

In this case, say that (ϕ, ξ, η, g) is an almost contact metric structure and we have $\phi\xi = 0$, $\eta\circ\phi = 0$, and $\eta(X) = g(X, \xi)$. Suppose that ∇ is the Levi-Civita connection with respect to the metric g, R is the Riemannian curvature tensor, and S is the Ricci tensor of a Kenmotsu manifold M. In a Kenmotsu manifold, we have (see [7])

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{2.4}$$

Using (2.3) and (2.4), we find

$$\begin{aligned} R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\ R(\xi, X)Y &= -g(X, Y)\xi + \eta(Y)X, \\ S(X, \xi) &= -(n-1)\eta(X). \end{aligned} \tag{2.5}$$

Biswas and Baishya [2, 3] introduced the general connection $\bar{\nabla}$ on M with respect to Levi-Civita connection ∇ as follows

$$\bar{\nabla}_X Y = \nabla_X Y + a[((\nabla_X \eta)(Y))\xi - \eta(Y)\nabla_X \xi] + b\eta(X)\phi(Y), \tag{2.6}$$

for all vector fields X, Y on M where a, b are real constants. The general connection becomes Schouten-Van Kampen connection [16], quarter symmetric metric connection [8], and Zamkovoy connection [19] when $(a, b) = (1, 0)$, $(a, b) = (0, -i)$, and $(a, b) = (1, 1)$, respectively. On Kenmotsu manifolds, using (2.3), (2.4), and (2.6) we get

$$\bar{\nabla}_X Y = \nabla_X Y + a[g(X, Y)\xi - \eta(Y)X] + b\eta(X)\phi(Y). \tag{2.7}$$

Let $\bar{R}$ and $\tilde{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively, that is,

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \; \tilde{S}(X, Y) = \sum_{i=1}^{n} g(\bar{R}(e_i, X)Y, e_i).$$

On Kenmotsu manifolds, using (2.7) and the above relation, we have

$$\begin{aligned} \bar{R}(X, Y)Z \;=\;& R(X, Y)Z + (ab - b)[\eta(Y)g(X, \phi Z)\xi - \eta(X)g(Y, \phi Z)\xi] \\ & + (ab - b)[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y] \\ & + a(1 - a)[g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi] \\ & + a(2 - a)[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \tilde{S}(X, Y) \;=\;& S(X, Y) + b(1 - a)g(X, \phi Y) + a(1 - a)\eta(X)\eta(Y) \\ & + (2na - na^2 - 3a + 2a^2)g(X, Y), \end{aligned} \tag{2.9}$$

where S denotes the Ricci tensor of the connection ∇. Using (2.9), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX - b(1 - a)\phi X + (2na - na^2 - 3a + 2a^2)X + a(1 - a)\eta(X)\xi.$$

If r and $\bar{r}$ are the scalar curvature of the Levi-Civita connection ∇ and the general connection $\bar{\nabla}$, respectively, then the equation (2.9) yields

$$\bar{r} = r + a(1 - a) + n(2na - na^2 - 3a + 2a^2).$$

Let $\bar{S}(X, Y) = \frac{\tilde{S}(X,Y) + \tilde{S}(Y,X)}{2}$. We define the generalized η-Ricci soliton associated to the general connection by

$$\alpha\bar{S} + \frac{\beta}{2}\bar{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta \otimes \eta + \lambda g = 0, \tag{2.10}$$

where λ is a smooth function on M, α, β, μ, ρ are real constants such that $(\alpha, \beta, \mu) \neq (0, 0, 0)$, and

$$(\overline{\mathcal{L}}_V g)(Y, Z) := g(\bar{\nabla}_Y V, Z) + g(Y, \bar{\nabla}_Z V).$$

The generalized η-Ricci soliton equation becomes

(1) the almost η-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,

(2) the almost Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,

(3) the generalized Ricci soliton equation when $\rho = 0$ and λ is real constant.

By the virtue of the definition of $\overline{\mathcal{L}}_V g$, we have

$$\begin{aligned}(\overline{\mathcal{L}}_V g)(X, Y) &= g(\bar{\nabla}_X V, Y) + g(X, \bar{\nabla}_Y V)\\ &= g(\nabla_X V, Y) + a[g(X, V)\eta(Y) - \eta(V)g(X, Y)] + b\eta(X)g(\phi(V), Y)\\ &\quad +g(X, \nabla_Y V) + a[g(V, Y)\eta(X) - \eta(V)g(X, Y)] + b\eta(Y)g(X, \phi(V))\\ &= \mathcal{L}_V g(X, Y) - 2a\eta(V)g(X, Y) + ag(X, V)\eta(Y) + ag(Y, V)\eta(X)\\ &\quad +b\eta(X)g(\phi V, Y) + b\eta(Y)g(X, \phi V).\end{aligned} \tag{2.11}$$

3. Main Results and Their Proofs

In this section, we give the our results and their proofs. Also, all vector fields X, Y, Z, and W on manifold M will be used in this section are arbitrary unless otherwise stated.

A Kenmotsu manifold is said to be η-Einstein with respect to the general connection if

$$\bar{S} = kg + l\eta \otimes \eta,$$

for some smooth functions k and l on manifold. Suppose that Kenmotsu manifold M satisfies the generalized η-Ricci soliton (2.10) associated to the general connection and $V = f\xi$ for some function f on M. We have

$$\begin{aligned}\mathcal{L}_{f\xi} g(X, Y) &= g(\nabla_X f\xi, Y) + g(X, \nabla_Y f\xi)\\ &= (Xf)\eta(Y) + fg(X - \eta(X)\xi, Y) + (Yf)\eta(X) + fg(X, Y - \eta(Y)\xi)\\ &= (Xf)\eta(Y) + (Yf)\eta(X) + 2f(g(X, Y) - \eta(X)\eta(Y)).\end{aligned}$$

Inserting $V = f\xi$ in (2.11), we infer

$$\overline{\mathcal{L}}_{f\xi} g(X, Y) = (Xf)\eta(Y) + (Yf)\eta(X) + 2f(1 - a)(g(X, Y) - \eta(X)\eta(Y)).$$

Also, we obtain

$$\xi^\flat \otimes \xi^\flat(X, Y) = \eta(X)\eta(Y). \tag{2.1}$$

Applying $V = f\xi$, (2.9) and (2.1) in the equation (2.10), we arrive at

$$\alpha\bar{S}(X,Y) + \frac{\beta}{2}\left[(Xf)\eta(Y) + (Yf)\eta(X) + 2f(1-a)(g(X,Y) - \eta(X)\eta(Y))\right] + (\mu f^2 + \rho)\eta(X)\eta(Y) + \lambda g(X,Y) = 0. \tag{2.2}$$

If in the equation (2.2) we consider $Y = \xi$ and use (2.5) and (2.9) then we get

$$\alpha[1-n+a+a^2+2an-na^2-3a]\eta(X) + \frac{\beta}{2}Xf + \frac{\beta}{2}(\xi f)\eta(X) + (\mu f^2+\rho+\lambda)\eta(X) = 0. \tag{2.3}$$

Substituting $X = \xi$ in (2.3) gives

$$\beta\xi f = -(\mu f^2 + \rho + \lambda) - \alpha[1-n+a+a^2+2an-na^2-3a]. \tag{2.4}$$

Combining equations (2.4) and (2.3), we deduce

$$\beta X f = \left\{-(\mu f^2+\rho+\lambda) - \alpha[1-n+a+a^2+2an-na^2-3a]\right\}\eta(X),$$

which implies that

$$\beta df = \left\{-(\mu f^2+\rho+\lambda) - \alpha[1-n+a+a^2+2an-na^2-3a]\right\}\eta. \tag{2.5}$$

Applying (2.5) in (2.2), we infer

$$\alpha\bar{S} = [\lambda + \alpha(1-n+a+a^2+2an-na^2-3a) + 2\beta f(1-a)]\eta\otimes\eta - (\lambda + 2\beta f(1-a))g.$$

Therefore, we get the following theorem:

Theorem 3.1. *Suppose that Kenmotsu manifold (M, g, ϕ, ξ, η) admits a generalized η-Ricci soliton $(g, f\xi, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the general connection for some smooth function f on M. Then M is an η-Einstein manifold with respect to the general connection.*

Let $\bar{S} = kg + l\eta\otimes\eta$ for some constants k and l on M. From (2.11), we have

$$\overline{\mathcal{L}}_\xi g(X,Y) \quad = \quad 2(1-a)(g(X,Y) - \eta(X)\eta(Y)).$$

Therefore,

$$\begin{aligned}
&\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_\xi g + \mu\xi^\flat\otimes\xi^\flat + \rho\eta\otimes\eta + \lambda g \\
&= k\alpha g + l\alpha\eta\otimes\eta + \beta(1-a)(g - \eta\otimes\eta) + \mu\eta\otimes\eta + \rho\eta\otimes\eta + \lambda g \\
&= (k\alpha + \lambda + \beta(1-a))g + (l\alpha + \mu + \rho - \beta(1-a))\eta\otimes\eta.
\end{aligned}$$

Hence, this leads to the following theorem:

Theorem 3.2. *Let M be a Kenmotsu manifold such that $\bar{S} = kg + l\eta \otimes \eta$ for some constants k and l. Then manifold M admitts a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -l\alpha - \mu + \beta(1-a), -k\alpha - \beta(1-a))$ with respect to the general connection.*

Definition 3.3. *Let M be a Kenmotsu manifold with the general connection $\bar{\nabla}$. The quasi-conformal like curvature tensor $\bar{\omega}$ [3] with respect to the general connection on M is defined by*

$$\begin{aligned} \bar{\omega}(X,Y)Z &= \bar{R}(X,Y)Z + k_1\left(\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\right) \\ &\quad -\frac{k_2\bar{r}}{n}\left(\frac{1}{n-1} + k_1 + k_3\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ &\quad +k_3\left(g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y\right), \end{aligned}$$

where k_1, k_2 and k_3 are constants. A Kenmotsu manifold M is called quasi-conformal like flat with respect to the general connection if $\bar{\omega}(X,Y)Z = 0$.

Now consider a Kenmotsu manifold M is quasi-conformal like flat with respect to the general connection. From [3], we have $\bar{S} = Ag$

$$A = \left(-k_1 + \frac{(n-1)k_2}{n}\left(\frac{1}{n-1} + k_1 + k_3\right)\right)\frac{\bar{r}}{1 - k_1 + k_3 n - k_3}.$$

Therefore, we can state the following corollary:

Corollary 3.4. *Let M be a quasi-M-projectively like flat Kenmotsu manifold with respect to the general connection. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu + \beta(1-a), -A\alpha - \beta(1-a))$ with respect to the general connection.*

Let M satisfies $\bar{R}(X,Y)\bar{S} = 0$ be on M. Then, we get

$$\bar{S}(\bar{R}(X,Y)Z, W) + \bar{S}(Z, \bar{R}(X,Y)W) = 0.$$

Plugging $X = Z = \xi$ in the last equation and using (2.8), we deduce

$$\bar{S}(Y,W) + b\bar{S}(\phi Y, W) = (1-n)\left(g(Y,W) - bg(Y,\phi W)\right). \tag{2.6}$$

Substituting $Y = \phi Y$ in (2.6), we arrive at

$$\bar{S}(\phi Y, W) - b\bar{S}(Y,W) = (1-n)\left(g(\phi Y, W) - bg(Y,W) + (-a+b+1)\eta(Y)\eta(W)\right). \tag{2.7}$$

Combining equations (2.6) and (2.7), we infer

$$\bar{S}(Y,W) = \frac{1-n}{1+b^2}\left((1+b^2)g(Y,W) - b(-a+b+1)\eta(Y)\eta(W)\right).$$

Therefore, we have the following corollary.

Corollary 3.5. *Let M be a Kenmotsu manifold such that $\bar{R}(X,Y)\bar{S}=0$. Then manifold M satisfies a generalized η-Ricci soliton*

$$\left(g,\xi,\alpha,\beta,\mu,\frac{1-n}{1+b^2}b(-a+b+1)\alpha-\mu+\beta(1-a),(n-1)(1+b^2)\alpha-\beta(1-a)\right)$$

with respect to the general connection.

Suppose that V is a conformal Killing vector field with respect to the general connection, that is, satisfies in $(\overline{\mathcal{L}}_V g)(X,Y)=2hg(X,Y)$ for some smooth function h on M. Using (2.9) and (2.10), we have

$$\alpha\bar{S}(X,Y)+\beta hg(X,Y)+\mu V^\flat(X)V^\flat(Y)+\rho\eta(X)\eta(Y)+\lambda g(X,Y)=0. \tag{2.8}$$

By considering $Y=\xi$ in the equation (2.8), we obtain

$$g(-(n-1)\alpha\xi+\beta h\xi+\mu\eta(V)V+\rho\xi+\lambda\xi,X)=0.$$

Since X is arbitrary vector field we get the following theorem:

Theorem 3.6. *If the metric g of a Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g,V,\alpha,\beta,\mu,\rho,\lambda)$ such that $\overline{\mathcal{L}}_V g=2hg$ then*

$$(-(n-1)\alpha+\beta h+\rho+\lambda)\xi+\mu\eta(V)V=0.$$

Now assume that $(g,V,\alpha,\beta,\mu,\rho,\lambda)$ is a generalized η-Ricci soliton with respect to the general connection on a Kenmotsu manifold where V is a torse-forming vector filed with respect to the general connection, that is,

$$\bar{\nabla}_X V=fX+\omega(X)V, \tag{2.9}$$

where f is a smooth function and ω is a 1-form. Then

$$(\overline{\mathcal{L}}_V g)(X,Y)=2fg(X,Y)+\omega(X)g(V,Y)+\omega(Y)g(Y,X). \tag{2.10}$$

Applying (2.10) into (2.10), we arrive at

$$\begin{aligned}&\alpha\bar{S}(X,Y)+\frac{\beta}{2}\left[2fg(X,Y)+\omega(X)g(V,Y)+\omega(Y)g(Y,X)\right]\\&+\mu V^\flat(X)V^\flat(Y)+\rho\eta(X)\eta(Y)+\lambda g(X,Y)=0.\end{aligned}$$

We take contraction of the above equation over X and Y to obtain

$$\alpha\bar{r}+n\left[\beta f+\lambda\right]+\rho+\beta\omega(V)+\mu|V|^2=0.$$

Therefore we have the following theorem.

Theorem 3.7. *If the metric g of a Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ where V is the torse-forming vector filed with respect to the general connection and satisfied in (2.9), then*

$$\lambda = -\frac{1}{n}\left[\alpha\bar{r} + \rho + \beta\omega + \mu|V|^2\right] - \beta f.$$

4. Example

In this section, we give an example of Kenmotsu manifolds with respect to the general connection to prove the existence of generalized η-Ricci soliton on Kenmotsu manifolds with respect to the general connection.

Example 4.1. Let $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ where (x, y, z) is the standard coordinates in $\mathbb{R}^3$. We consider the vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{-z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

and the metric g as follows

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

We define an almost contact structure (ϕ, ξ, η) on M by

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all vector field X. Note the relations (2.1) and (2.2) hold. Thus (M, ϕ, ξ, η, g) defines an almost contact structure on M. By direct computation we get

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The Levi-Civita connection ∇ of M is determined by

$$\nabla_{e_i} e_j = \begin{pmatrix} -e_3 & 0 & e_1 \\ 0 & -e_3 & e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the formula (2.3) and (2.4) are true and (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold. Now, using (2.8) ,we get the general connection on M as follows

$$\bar{\nabla}_{e_i} e_j = \begin{pmatrix} (a-1)e_3 & 0 & (1-a)e_1 \\ 0 & (a-1)e_3 & (1-a)e_2 \\ -be_2 & be_1 & 0 \end{pmatrix}.$$

The all components of curvature tensor with respect to the general connection can be caculated as in the following:

$$\bar{R}(e_1,e_2)e_1=(1-a)^2e_2, \qquad \bar{R}(e_1,e_2)e_2=-(1-a)^2e_1,$$
$$\bar{R}(e_1,e_2)e_3=0, \qquad \bar{R}(e_1,e_3)e_1=(1-a)e_3,$$
$$\bar{R}(e_1,e_3)e_2=(-b+ab)e_3, \qquad \bar{R}(e_1,e_3)e_3=b(1-a)e_2-(1-a)e_1,$$
$$\bar{R}(e_2,e_3)e_1=-b(1-a)e_3, \qquad \bar{R}(e_2,e_3)e_2=(1-a)e_3,$$
$$\bar{R}(e_2,e_3)e_3=b(a-1)e_1+(a-1)e_2.$$

Thus, we get

$$\bar{S}(e_i,e_j)=\begin{cases} a^2+3a-2, & \text{if } i=j \text{ and } i,j\in\{1,2\} \\ 2a-2, & \text{if } i=j=3 \\ 0, & \text{otherwise.}\end{cases}$$

Then $\bar{S}=(a^2+3a-2)g-(a^2+a)\eta\otimes\eta$. Now if we consider $V=\xi$ then $\overline{\mathcal{L}}_V g=2(1-a)(g-\eta\otimes\eta)$. Then $(g,\xi,\alpha,\beta,\mu,\rho=-\mu+\beta(1-a)+\alpha(a^2+a),\lambda=-\beta(1-a)-\alpha(a^2+3a-2))$ is a generalized η-Ricci soliton on manifold M with respect to the general connection.

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