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Anisotropic Variable Herz Spaces and Applications

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Abstract. In this study, we establish some new characterizations for a class of anisotropic Herz spaces in which all exponents are considered as variables. We also provide a description of these spaces based on bloc decomposition. As an application, we investigate the boundedness of certain sublinear operators within these function spaces.

1. Introduction and Preliminaries

The aim of this paper is to establish a characterization of the anisotropic variable Herz spaces $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(A;\mathbb{R}^n)$ associated with non-isotropic dilations A on $\mathbb{R}^n$ in terms of block decompositions. All exponents in the considered spaces are variable. First, we define the set of variable exponents as follows:

$$\mathcal{P}_0(\mathbb{R}^n) := \{p \text{ measurable: } p(\cdot) : \mathbb{R}^n \to [c,\infty) \text{ for some } c > 0\}.$$

The subset of variable exponents with a range of $[1,\infty)$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, we introduce the notation

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Now we give the definition of variable Lebesgue spaces.

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Definition 1.1. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, with a variable exponent is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This space is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \mu > 0 : \varrho_{p(\cdot)}(\frac{1}{\mu} f) \leqslant 1 \right\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space. We refer to the monographs [3] and [4] for further details and references on recent developments on variable Lebesgue spaces.

We present the most important condition on the exponent in the study of variable exponent spaces.

Definition 1.2. We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. In particular, if

$$|g(x) - g(0)| \leqslant \frac{c_{\log}}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at the origin (or has a log decay at the origin). Additionally, if there exist $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at infinity (or has a log decay at infinity).

For some examples of a function locally log-Hölder continuous, see E. Nakai and Y. Sawano [7, Example 1.3].

The sets $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ consist of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ that have a log decay at the origin and at infinity, respectively. The set $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ that are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$.

It is well known that if $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $p' \in \mathcal{P}^{\log}(\mathbb{R}^n)$, where p' denotes the conjugate exponent of p given by $1/p(\cdot) + 1/p'(\cdot) = 1$.

Definition 1.3. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\left\{\lambda_v > 0 : \varrho_{p(\cdot)}(\frac{f_v}{\lambda_v^{1/q(\cdot)}}) \leqslant 1\right\}.$$

A (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\gamma > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\frac{1}{\gamma}(f_v)_v) \leqslant 1\right\}. \tag{1.4}$$

If $q(\cdot)$ satisfies $q^+ < \infty$, then we can replace (1.4) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

In the following, we introduce some basic notation and definitions of non-isotropic spaces associated with general expansive dilations.

Definition 1.5. A dilation is $n \times n$ real matrix A, such that all eigenvalues λ of A satisfy $|\lambda| > 1$. We suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A so that $1 < |\lambda_1| \leqslant ... \leqslant |\lambda_n|$. Let λ_-, λ_+ be any numbers so that

$$1 < \lambda_- < |\lambda_1| \leqslant ... \leqslant |\lambda_n| < \lambda_+.$$

A set $\triangle \subset \mathbb{R}^n$ is said to be an ellipsoid if

$$\triangle = \{x \in \mathbb{R}^n : |Px| < 1\}$$

for some nondegenerate $n \times n$ matrix P, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

In [2, Lemma 2.2], it is demonstrated that for a dilation A, there exists an ellipsoid $\triangle$ and $r > 1$ satisfying

$$\triangle \subset r\triangle \subset A\triangle, \qquad \text{where} \quad |\triangle| = 1. \tag{1.6}$$

For convenience, we set

$$B_k = A^k\triangle \text{ for } k \in \mathbb{Z},$$

then, by (1.6) we obtain

$$B_k \subset rB_k \subset B_{k+1}, \quad |B_k| = b^k,$$

where $b = |\det A| > 1$.

Definition 1.7. A homogeneous quasi-norm associated with a dilation A is a measurable mapping $\sigma_A : \mathbb{R}^n \to [0, \infty)$, so that
- $\sigma_A(x) > 0$ for $x \neq 0$,
- $\sigma_A(Ax) = b\sigma_A(x)$ for all $x \in \mathbb{R}^n$,
- there is $c > 0$ so that $\sigma_A(x+y) \leqslant c(\sigma_A(x) + \sigma_A(y))$ for all $x, y \in \mathbb{R}^n$.

For a fixed dilation A, we define the "canonical" quasi-norm σ.

Definition 1.8. Define the step homogeneous quasi-norm σ on $\mathbb{R}^n$ induced by the dilation A as

$$\sigma(x) = \begin{cases} b^j & \text{if} \quad x \in B_{j+1} \backslash B_j, \quad j \in \mathbb{Z} \\ 0 & \text{if} \quad x = 0. \end{cases}$$

For any $x, y \in \mathbb{R}^n$, we have

$$\sigma(x+y) \leqslant b^\theta (\sigma(x) + \sigma(y)),$$

where θ is the smallest integer so that

$$2B_0 \subset A^\theta B_0 = B_\theta.$$

Also, we use the following notation

$$R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Now, we define the anisotropic Herz spaces with variable exponent.

Definition 1.9. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous anisotropic Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A; \mathbb{R}^n)$ associated with the dilation A is defined as the set of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)} := \left\| \left(b^{k\alpha(\cdot)} f \chi_k \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The non-homogeneous anisotropic Herz space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A; \mathbb{R}^n)$ associated with the dilation A consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)} := \|f \chi_{B_0}\|_{p(\cdot)} + \left\| \left(b^{k\alpha(\cdot)} f \chi_k \right)_{k \geqslant 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Clearly, $\dot{K}_{p(\cdot)}^{0,p(\cdot)}(A; \mathbb{R}^n) = K_{p(\cdot)}^{0,p(\cdot)}(A; \mathbb{R}^n) = L^{p(\cdot)}(A; \mathbb{R}^n)$. Recall that the anisotropic Herz spaces $K_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$, where q is constant, are introduced by H. Wang in [8]. A detailed discussion of the properties of these spaces may be found in [9] and [10].

By the same argument used in [6], we can establish the next result, which will be useful in the sequel.

Proposition 1.10. *Let* $\alpha \in L^{\infty}(\mathbb{R}^n)$ *and* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. *If* α *and* q *are* log-*Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(A;\mathbb{R}^n).$$

Additionally, if α *and* q *have a log decay at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)} \approx \Big(\sum_{k=-\infty}^{-1} \|b^{k\alpha(0)} f\,\chi_k\|_{p(\cdot)}^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} \|b^{k\alpha_\infty} f\,\chi_k\|_{p(\cdot)}^{q_\infty}\Big)^{1/q_\infty}.$$

Recall that the expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

2. Some Technical Lemmas

In this section, we introduce several lemmas used to prove the main theorems in sections 3 and 4. In the following, we denote by c as a generic positive constant, i.e. a constant whose value may change from line to line.

The following lemma plays an important role in the proof of the main results.

Lemma 2.1. *Let* $p \in \mathcal{P}(\mathbb{R}^n)$ *and* $R_k := B_k \setminus B_{k-1}, k \in \mathbb{Z}$. *If* $b^k \geqslant 2^{-n}$ *and* p *is* log-*Hölder continuous at infinity, then we have*

$$\|\chi_k\|_{p(\cdot)} \approx b^{\frac{k}{p_\infty}},$$

with the implicit constants independent of k.

Proof. Our proof based on an idea from [1, Lemma 2.2] where the case of the Euclidean ball was studied. First, we have

$$\|\chi_k\|_{p(\cdot)} \approx b^{\frac{k}{p_\infty}},$$

which is equivalent to

$$\|b^{-\frac{k}{p_\infty}}\chi_k\|_{p(\cdot)} \approx 1.$$

In particular, we will show that

$$\begin{aligned} \varrho_{p(\cdot)}(b^{-\frac{k}{p_\infty}}\chi_k) &:= \int_{\mathbb{R}^n} |b^{-\frac{k}{p_\infty}}\chi_k(x)|^{p(y)}\,dy \\ &= b^{-k}\int_{R_k} b^{k\left(\frac{p_\infty - p(y)}{p_\infty}\right)}\,dy \lesssim c, \end{aligned} \tag{2.2}$$

for some constant $c > 0$. For that, it is sufficient to prove that $b^{k\left(\frac{p_\infty - p(y)}{p_\infty}\right)}$ is bounded, i.e. $k\left(\frac{p_\infty - p(y)}{p_\infty}\right)\log b \leqslant c$ for all $y \in R_k$. Since p is log-Hölder continuous at infinity, then (2.2) is bounded by

$$b^{-k\left|\frac{p(y)-p_\infty}{p_\infty}\right|} \lesssim b^{-k\frac{1}{\log(e+|y|)}} \lesssim b^{-k/\log|y|}, \qquad y \in R_k. \tag{2.3}$$

We can distinguish two cases as follows:
Case 1: For every integer $k \geqslant 0$, due to [2, Lemma 3.2] and Definition 1.8, we deduce that

$$b^{k\log\lambda_-/\log b} \lesssim |y| \lesssim \varrho(x)^{\log\lambda_+/\log b} = b^{k\log\lambda_+/\log b} \text{ for } k \geqslant 0,$$

for all $y \in R_k$. This implies that (2.3) is bounded by

$$b^{\frac{-k}{k\log(b)(\log\lambda_+/\log b)}} = e^{-\frac{\log\lambda_+}{\log b}} \lesssim c.$$

Case 2: Considering the case $\left[-n\frac{\log 2}{\log b}\right] \leqslant k \leqslant -1$, i.e. $2^{-n} \leqslant b^k < 1$, we have (2.3) bounded by

$$\begin{aligned} -k\left|\frac{p(y)-p_\infty}{p_\infty}\right|\log b &\leqslant |p(y)-p_\infty|\log\frac{1}{b^k} \\ &\leqslant 2np^+\log 2 \\ &\lesssim c. \end{aligned}$$

In either case, we obtain that (2.2) is bounded by

$$b^{-k}\int_{R_k} dy \lesssim c.$$

Now, we show that $b^{\frac{k}{p_\infty}} \lesssim \|\chi_k\|_{p(\cdot)}$. This is a consequence of Hölder's inequality and the estimate $\|\chi_k\|_{p'(\cdot)} \lesssim b^{\frac{k}{p'_\infty}}$ which was already proved. In fact, we have

$$b^{\frac{k}{p_\infty}} = b^{\frac{k}{p_\infty}-k}\int_{\mathbb{R}^n}\chi_k(y)dy \leqslant 2b^{-\frac{k}{p'_\infty}}\|\chi_k\|_{p(\cdot)}\|\chi_k\|_{p'(\cdot)} \lesssim \|\chi_k\|_{p(\cdot)}.$$

This finishes the proof. □

Remark 2.4. It is known that for $p \in \mathcal{P}^{\log}$, we have

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \approx |B|.$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B,$$

for small balls $B \subset \mathbb{R}^n$ and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}}$$

for large balls ($|B| \geqslant 1$), with constants only depending on the log-Hölder constant of p. See, for example, [3, Corollary 4.5.9].

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 2.5 ([5]). *Let $\gamma > 1$, $\kappa > 0$ and $0 < q \leqslant \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then, the sequences

$$\left\{\delta_k : \delta_k = \sum_{j\leqslant k} \gamma^{-(k-j)\kappa}\varepsilon_j\right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \left\{\eta_k : \eta_k = \sum_{j\geqslant k} \gamma^{-(j-k)\kappa}\varepsilon_j\right\}_{k\in\mathbb{Z}}$$

belong to ℓ^q, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leqslant c\, I,$$

with $c > 0$ only depending on γ and q.

The following lemma presents the Hölder inequality in $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.6 ([3]). *Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then, there exists a constant c such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, $f \cdot g \in L^1(\mathbb{R}^n)$, and*

$$\|f \cdot g\|_1 \leqslant c\, \|f\|_{p(\cdot)}\, \|g\|_{p'(\cdot)}\,.$$

3. Bloc Decomposition of Anisotropic Variable Herz Spaces

Now, we establish characterizations of the spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$ in terms of central bloc decompositions, which will be convenient for the study of the boundedness of operators on these spaces.

Let us first recall the definition of bloc decomposition.

Definition 3.1. Let $\alpha \in L^\infty(\mathbb{R}^n)$, be log-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}(\mathbb{R}^n)$. A function a_k is said to be a central $(\alpha(\cdot), p(\cdot))$-bloc, if
(i) $\operatorname{supp} a_k = \overline{\{x \in \mathbb{R}^n : a_k(x) \neq 0\}} \subset B_k$.
(ii) $\|a_k\|_{p(\cdot)} \leqslant b^{-k\alpha(0)}, \qquad k < 0.$
(iii) $\|a_k\|_{p(\cdot)} \leqslant b^{-k\alpha_\infty}, \qquad k \geqslant 0.$

A function a_k on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-bloc of restricted type, if it satisfies the condition (iii) and $\operatorname{supp} a_k \subset B_k, k \geqslant 0$.

Remark 3.2. If α and p are constants, then we recover the classical case.

One of the main results of this paper will be the following theorem. It generalizes Theorem 2.3 of H. Wang [8] by taking q as a constant.

Theorem 3.3. *Let $\alpha \in L^\infty(\mathbb{R}^n), p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log-*Hölder continuous, both at the origin and at infinity with $\alpha(0), \alpha_\infty > 0$, then the following two statements are equivalentes*

(i) $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$
(ii) f *can be represented by*

$$f(x) = \sum_{k=-\infty}^{\infty} \beta_k a_k(x), \tag{3.4}$$

where $\beta_k \geqslant 0$, *each* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-block with support contained in* B_k *and*

$$\left(\sum_{k=-\infty}^{-1} |\beta_k|^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{\infty} |\beta_k|^{q_\infty}\right)^{\frac{1}{q_\infty}} \leqslant c\,\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}$ *and*

$$\inf\left(\left(\sum_{k=-\infty}^{-1} |\beta_k|^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{\infty} |\beta_k|^{q_\infty}\right)^{\frac{1}{q_\infty}}\right)$$

are equivalent, where the infimum is taken over all decompositions of f *as in (3.4).*

Proof. The idea of the proof is borrowed from [5], where the variable Herz-type Hardy spaces case is studied.

First, we show that (i) implies (ii). For every $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$, we have

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \\ &= \sum_{k=-\infty}^{\infty} \left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)} \frac{f(x)\chi_k(x)}{\left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}} = \sum_{k=-\infty}^{\infty} \beta_k a_k(x), \end{aligned}$$

where

$$\beta_k = \left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)} \quad \text{and} \quad a_k(x) = \frac{f(x)\chi_k(x)}{\left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}}.$$

It is obvious that $\operatorname{supp} a_k \subset B_k$ and

$$\|a_k\|_{p(\cdot)} \approx \begin{cases} b^{-k\alpha(0)}, & \text{if } k \leqslant -1, \\ b^{-k\alpha_\infty}, & \text{if } k \geqslant 0. \end{cases}$$

Thus, each a_k is a central $(\alpha(\cdot), p(\cdot))$- bloc with the support B_k and

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{-1} |\beta_k|^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{\infty} |\beta_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
= &\Big(\sum_{k=-\infty}^{-1} \left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{\infty} \left\|b^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
\approx &\Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \left\|f\chi_k\right\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty} \left\|f\chi_k\right\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
\approx &\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}.
\end{aligned}$$

It remains to prove that (ii) implies (i). For this purpose, let $f(x) = \sum_{k=-\infty}^{\infty} \beta_k a_k(x)$ be a decomposition of f that satisfies the hypothesis (ii) of Theorem 3.3, by the Minkowski inequality, we obtain

$$\|f\chi_j\|_{p(\cdot)} \leqslant \sum_{k=j}^{\infty} |\beta_k| \, \|a_k\|_{p(\cdot)} \quad \text{for each } j \in \mathbb{Z}. \tag{3.5}$$

From this, (3.5), and Proposition 1.10, it follows that $\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}$ is bounded by

$$\begin{aligned}
&c\Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big(\sum_{j=k}^{\infty} |\beta_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&+c\Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{j=k}^{\infty} |\beta_j| \, \|a_j\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} = I_1 + I_2.
\end{aligned}$$

Then we deal with I_1 and I_2, separatly. For I_1, we divide the sum $\sum_{j=k}^{\infty} \cdots$ into two parts,

$$\sum_{j=k}^{-1} \cdots + \sum_{j=0}^{\infty} \cdots.$$

I_1 is bounded by $I_1^a + I_1^b$, where

$$I_1^a := c\Big(\sum_{k=-\infty}^{-1} \Big(b^{k\alpha(0)} \sum_{j=k}^{-1} |\beta_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}$$

and

$$I_1^b := c\Big(\sum_{k=-\infty}^{-1} \Big(b^{k\alpha(0)} \sum_{j=0}^{\infty} |\beta_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}.$$

Since $0 < \alpha(0) < \infty$, then by Lemma 2.5 (with $\gamma = b^{\alpha(0)} > 1$), we get

$$I_1^a \leqslant c\Big(\sum_{k=-\infty}^{-1}\Big(\sum_{j=k}^{-1}|\beta_j|b^{-(j-k)\alpha(0)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \leqslant c\Big(\sum_{k=-\infty}^{-1}|\beta_k|^{q(0)}\Big)^{\frac{1}{q(0)}}.$$

By Hölder's inequality in ℓ^1 with $\frac{1}{q_\infty} + \frac{1}{q'_\infty} = 1$ and since $\alpha(0), \alpha_\infty > 0$, we obtain

$$\begin{aligned} I_1^b &\leqslant c\Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big(\sum_{j=0}^{\infty}|\beta_j|b^{-j\alpha_\infty}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\leqslant c\Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big)^{\frac{1}{q(0)}}\Big(\sum_{j=0}^{\infty}|\beta_j|^{q_\infty}\Big)^{\frac{1}{q_\infty}}\Big(\sum_{j=0}^{\infty}b^{-j\alpha_\infty q'_\infty}\Big)^{\frac{1}{q'_\infty}} \\ &\leqslant c\Big(\sum_{j=0}^{\infty}|\beta_j|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leqslant c\|f\|_{\dot K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

Thus, we have the desired estimate for I_1.

Next, we deal with I_2. We have

$$\begin{aligned} I_2 &= \Big(\sum_{k=0}^{\infty}\Big(b^{k\alpha_\infty}\sum_{j=k}^{\infty}|\beta_j|\,\|a_j\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leqslant \Big(\sum_{k=0}^{\infty}\Big(\sum_{j=k}^{\infty}|\beta_j|b^{-(j-k)\alpha_\infty}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}. \end{aligned}$$

Since $0 < \alpha_\infty < \infty$, then by Lemma 2.5 (with $\gamma = b^{\alpha_\infty} > 1$), we deduce that

$$\begin{aligned} I_2 &\leqslant \Big(\sum_{k=0}^{\infty}|\beta_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\leqslant \Big(\sum_{k=-\infty}^{-1}|\beta_k|^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{\infty}|\beta_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}}. \end{aligned}$$

This finishes the estimation of I_2 and the proof of Theorem 3.3. □

Remark 3.6. A non-homogeneous counterpart of Theorem 3.3 is available. Since $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(A;\mathbb{R}^n)$, its proof is an immediate consequence of [8, Theorem 2.5].

4. Some Applications

The next result concerns the boundedness, on anisotropic variable Herz spaces, of some sublinear operators T satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\varrho(x-y)}\mathrm{d}y, \quad x \notin \operatorname{supp} f \tag{4.1}$$

for integrable and compactly supported functions f.

Theorem 4.2. *Let $\alpha \in L^{\infty}(\mathbb{R}^n), p \in \mathcal{P}(\mathbb{R}^n), q \in \mathcal{P}_0(\mathbb{R}^n)$, and if α, p and q are* log-*Hölder continuous, both at the origin and at infinity such that*

$$0 < \alpha(0) < 1 - 1/p(0) \quad \textit{and} \quad 0 < \alpha_{\infty} < 1 - 1/p_{\infty}.$$

Then every sublinear operator T satisfying (4.1) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$, respectively.

Proof. It suffices to prove that T is bounded on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$. The non-homogeneous case can be proved similarly. We must show that

$$\|Tf\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)} \leqslant c\,\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}$$

for all $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)$. Thanks to Theorem 3.3, it holds that

$$f = \sum_{i=-\infty}^{\infty} \beta_i a_i$$

where $\beta_i \geqslant 0$ and a_i's are $(\alpha(\cdot), p(\cdot))$- bloc with $\operatorname{supp} a_i \subseteq B_i$. Hence, we obtain

$$\begin{aligned}
\|Tf\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)} &\approx \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big(\sum_{i=-\infty}^{\infty}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q(0)}\Big)^{1/q(0)} \\
&\quad + \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{i=-\infty}^{\infty}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty} \\
&\lesssim \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big(\sum_{i=-\infty}^{k-\theta-1}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q(0)}\Big)^{1/q(0)} \\
&\quad + \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)}\Big(\sum_{i=k-\theta}^{\infty}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q(0)}\Big)^{1/q(0)} \\
&\quad + \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{i=-\infty}^{k-\theta-1}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty} \\
&\quad + \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{i=k-\theta}^{\infty}\beta_i\,\|Ta_i\cdot\chi_k\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty} \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

First, we estimate J_1. Since

$$\sigma(x) \leqslant b^{\theta}\left(\sigma(x-y)+\sigma(y)\right),$$

and taking $x \in R_k, y \in B_i$ with $k \geqslant i+\theta+1$, then $x \in B_{i+\theta+1} \backslash B_{i+\theta}$, and we get

$$\begin{aligned}\sigma(x-y) &\geqslant b^{-\theta}\sigma(x)-\sigma(y)=b^{-\theta}\sigma(x)-b^{i-1}\\ &= b^{-\theta}\sigma(x)-b^{-\theta-1}\sigma(x)=b^{-\theta}\Big(1-\frac{1}{b}\Big)\sigma(x).\end{aligned}$$

The condition (4.1) gives

$$\begin{aligned}|Ta_i(x)| &\lesssim \int_{B_i}\frac{|a_i(y)|}{\varrho(x)}\mathrm{d}y\\ &\leqslant cb^{-k}\int_{B_i}|a_i(y)|\,\mathrm{d}y.\end{aligned}$$

By Lemma 2.6 and the condition (ii) in Definition 3.1, we get

$$\begin{aligned}|Ta_i(x)| &\lesssim b^{-k}\left\|a_i\right\|_{p(\cdot)}\left\|\chi_{B_i}\right\|_{p'(\cdot)}\\ &\leqslant cb^{-k-i(\alpha(0)-1+1/p(0))},\end{aligned}$$

which implies that

$$\begin{aligned}\|Ta_i\cdot\chi_k\|_{p(\cdot)} &\leqslant cb^{-k-i(\alpha(0)-1+1/p(0))}\|\chi_k\|_{p(\cdot)}\\ &\leqslant cb^{k(-1+1/p(0))-i(\alpha(0)-1+1/p(0))}.\end{aligned}$$

By Lemma 2.5 (with $\gamma=b^{-\alpha(0)+1-1/p(0)}>1$), we have

$$\begin{aligned}J_1 &\lesssim \Big(\sum_{k=-\infty}^{-1}\Big(\sum_{i=-\infty}^{k-\theta-1}\beta_i b^{-(k-i)(-\alpha(0)+1-1/p(0))}\Big)^{q(0)}\Big)^{1/q(0)}\\ &\lesssim \Big(\sum_{k=-\infty}^{-1}\beta_k^{q(0)}\Big)^{1/q(0)}\\ &\leqslant c\|f\|_{\dot K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}.\end{aligned}$$

To estimate J_2, we distinguish two cases, $k-\theta<0$ and $k-\theta\geqslant 0$. Here we assume that $k-\theta<0$. The other case will follow in the same way.

We divide the sum $\sum_{i=k-\theta}^{\infty}\cdots$ into two parts

$$\sum_{i=k-\theta}^{-1}\cdots+\sum_{i=0}^{\infty}\cdots,$$

then J_2 is bounded by $J_2^a + J_2^b$, where

$$J_2^a := \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=k-\theta}^{-1} \beta_i \left\| Ta_i \cdot \chi_k \right\|_{p(\cdot)} \Big)^{q(0)} \Big)^{1/q(0)}$$

$$J_2^b := \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=0}^{\infty} \beta_i \left\| Ta_i \cdot \chi_k \right\|_{p(\cdot)} \Big)^{q(0)} \Big)^{1/q(0)}.$$

For J_2^a, the numbers k and i are negatives numbers. Then, by the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of T, Definition 3.1 and Lemma 2.5 (with $\gamma = b^{\alpha(0)} > 1$), we deduce that

$$\begin{aligned} J_2^a &\lesssim \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=k-\theta}^{-1} \beta_i \left\| a_i \right\|_{p(\cdot)} \Big)^{q(0)} \Big)^{1/q(0)} \\ &\lesssim \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{i=k-\theta}^{-1} \beta_i b^{-(i-k)\alpha(0)} \Big)^{q(0)} \Big)^{1/q(0)} \\ &\lesssim \Big(\sum_{k=-\infty}^{-1} \beta_k^{q(0)} \Big)^{1/q(0)} \\ &\leqslant c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

For J_2^b, we have $k \leqslant -1$ and $i \geqslant 0$. By the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of T and Definition 3.1, we have

$$\begin{aligned} J_2^b &:= \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=0}^{\infty} \beta_i \left\| Ta_i \cdot \chi_k \right\|_{p(\cdot)} \Big)^{q(0)} \Big)^{1/q(0)} \\ &\lesssim \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=0}^{\infty} \beta_i \left\| a_i \right\|_{p(\cdot)} \Big)^{q(0)} \Big)^{1/q(0)} \\ &\lesssim \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big(\sum_{i=0}^{\infty} \beta_i b^{-i\alpha_\infty} \Big)^{q(0)} \Big)^{1/q(0)}. \end{aligned}$$

By Hölder's inequality in ℓ^1 with $\frac{1}{q_\infty} + \frac{1}{q'_\infty} = 1$ and since $\alpha(0), \alpha_\infty > 0$, we obtain

$$\begin{aligned} J_2^b &\lesssim \Big(\sum_{k=-\infty}^{-1} b^{k\alpha(0)q(0)} \Big)^{1/q(0)} \Big(\sum_{i=0}^{\infty} \beta_i^{q_\infty} \Big)^{1/q_\infty} \Big(\sum_{i=0}^{\infty} b^{-i\alpha_\infty q'_\infty} \Big)^{1/q'_\infty} \\ &\lesssim \Big(\sum_{i=0}^{\infty} \beta_i^{q_\infty} \Big)^{1/q_\infty} \\ &\leqslant c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

Next, we estimate J_3. We distinguish two cases, $k-\theta-1 \geqslant 0$ and $k-\theta-1<0$. Here we assume that $k-\theta-1 \geqslant 0$. The other case follows similarly.
Let us decompose the sum $\sum_{i=-\infty}^{k-\theta-1} \cdots$ into two parts

$$\sum_{i=-\infty}^{-1} \cdots + \sum_{i=0}^{k-\theta-1} \cdots.$$

Then J_3 is bounded by $J_3^a + J_3^b$, where

$$\begin{aligned} J_3^a &:= \left(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty} \left(\sum_{i=-\infty}^{-1} \beta_i \left\|Ta_i \cdot \chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{1/q_\infty}, \\ J_3^b &:= \left(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty} \left(\sum_{i=0}^{k-\theta-1} \beta_i \left\|Ta_i \cdot \chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{1/q_\infty}. \end{aligned}$$

For J_3^a, we have $k \geqslant 0$ and $i \leqslant -1$. By the condition (ii) in Definition 3.1 and Lemma 2.1, we obtain

$$\begin{aligned} \left\|Ta_i \cdot \chi_k\right\|_{p(\cdot)} &\lesssim b^{-k-i(\alpha(0)-1+1/p(0))} \|\chi_{\mathfrak{B}_k}\|_{p(\cdot)} \\ &\lesssim b^{k(-1+1/p_\infty)-i(\alpha(0)-1+1/p(0))}, \end{aligned}$$

which gives

$$J_3^a \lesssim \Big(\sum_{k=0}^{\infty} b^{k(\alpha_\infty-1+1/p_\infty)q_\infty} \Big(\sum_{i=-\infty}^{-1} \beta_i b^{-i(\alpha(0)-1+1/p(0))}\Big)^{q_\infty}\Big)^{1/q_\infty}.$$

Thanks to Hölder's inequality in ℓ^1, with $\frac{1}{q(0)} + \frac{1}{q'(0)} = 1$, we easily obtain

$$\begin{aligned} J_3^a &\lesssim \Big(\sum_{k=0}^{\infty} b^{k(\alpha_\infty-1+\frac{1}{p_\infty})q_\infty}\Big)^{\frac{1}{q_\infty}} \Big(\sum_{i=-\infty}^{-1} \beta_i^{q(0)}\Big)^{\frac{1}{q(0)}} \Big(\sum_{i=-\infty}^{-1} b^{-i(\alpha(0)-1+\frac{1}{p(0)})q'(0)}\Big)^{\frac{1}{q'(0)}} \\ &\lesssim \Big(\sum_{i=-\infty}^{-1} \beta_i^{q(0)}\Big)^{1/q(0)} \\ &\leqslant c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

Concerning J_3^b, where k and i are non-negatives numbers, we have by the condition (iii) in Definition 3.1 and Lemma 2.1

$$\begin{aligned} \left\|Ta_i \cdot \chi_k\right\|_{p(\cdot)} &\lesssim b^{-k-i(\alpha_\infty-1+1/p_\infty)} \|\chi_{\mathfrak{B}_k}\|_{p(\cdot)} \\ &\lesssim b^{k(-1+1/p_\infty)-i(\alpha_\infty-1+1/p_\infty)}, \end{aligned}$$

which gives

$$\begin{aligned} J_3^b &:= \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{i=0}^{k-\theta-1} \beta_i \left\|Ta_i \cdot \chi_k\right\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty} \\ &\lesssim \Big(\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k-\theta-1} \beta_i b^{-(k-i)(-\alpha_\infty+1-1/p_\infty)}\Big)^{q_\infty}\Big)^{1/q_\infty}, \end{aligned}$$

by Lemma 2.5 (with $\gamma = b^{-\alpha_\infty+1-1/p_\infty} > 1$), we obtain

$$J_3^b \lesssim \Big(\sum_{k=0}^{\infty} \beta_k^{q_\infty}\Big)^{1/q_\infty} \leqslant c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}.$$

Finally, we estimate J_4. In this case k and i are non-negatives numbers, then by the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of T, the condition (iii) in Definition 3.1 and Lemma 2.5 (with $\gamma = b^{\alpha_\infty} > 1$), we easily obtain that

$$\begin{aligned} J_4 &\lesssim \Big(\sum_{k=0}^{\infty} b^{k\alpha_\infty q_\infty}\Big(\sum_{i=k-\theta}^{\infty} \beta_i \left\|a_i\right\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty} \\ &\lesssim \Big(\sum_{k=0}^{\infty}\Big(\sum_{i=k-\theta}^{\infty} \beta_i b^{-(i-k)\alpha_\infty}\Big)^{q_\infty}\Big)^{1/q_\infty} \\ &\lesssim \Big(\sum_{k=0}^{\infty} \beta_k^{q_\infty}\Big)^{1/q_\infty} \\ &\leqslant c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

Combing the estimations of J_1, J_2, J_3 and J_4, we finish the proof of Theorem 4.2. □

Remark 4.3. We would like to mention that if $q(\cdot)$ is constant, then the statements corresponding to Theorem 4.2 can be found in Theorem 3.1 of [8].

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References

[1] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl., **394(2)**(2012), 781–795.

[2] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc., **164**(2003), 1–122.

[3] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, 2011.

[4] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta and T. Shimomura, *Maximal functions in variable exponent spaces: limiting cases of the exponent*, Ann. Acad. Sci. Fenn. Math., **34**(2009), 503–522.

[5] D. Drihem and R. Heraiz, *Boundedness of some bilinear operators on variable Herz-type Hardy spaces*, J. Pseudo-Differ. Oper. Appl., **10**(2019), 601–648.

[6] D. Drihem and F. Seghiri, *Notes on the Herz-type Hardy spaces of variable smoothness and integrability*, Math. Inequal. Appl., **19(1)**(2016), 145–165.

[7] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal., **262(9)**(2012), 3665–3748.

[8] H. Wang, *Anisotropic Herz spaces with variable exponents*, Commun. Math. Anal., **18(2)**(2015), 1–14.

[9] H. Wang and Y. Wu, *Anisotropic Herz-type Morrey spaces with variable exponent*, Khayyam J. Math., **2(2)**(2016), 177–187.

[10] H. Zhao and J. Zhou, *Anisotropic Herz-type Hardy spaces with variable exponents and their applications*, Acta Math. Hungar., **156(2)**(2018), 309–335.