KYUNGPOOK Math. J. 64(2024), 205-218
https://doi.org/10.5666/KMJ.2024.64.2.205
pISSN 1225-6951 eISSN 0454-8124

The G-Drazin Inverse of an Operator Matrix over Banach Spaces

Farzaneh Tayebi, Nahid Ashrafi and Rahman Bahmani
Department of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran
e-mail : ftayebis@gmail.com, nashrafi@semnan.ac.ir
and rbahmani@semnan.ac.ir

Marjan Sheibani Abdolyousefi*
Farzanegan Campmus, Semnan University, Semnan, Iran
e-mail : m.sheibani@semnan.ac.ir

Abstract. Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}.$$

New additive results for the generalized Drazin inverse of an operator over a Banach space are presented. we extend the main results of a paper of Shakoor, Yang and Ali from 2013 and of Wang, Huang and Chen from 2017. Appling these results to 2×2 operator matrices we also generalize results of a paper of Deng, Cvetković-Ilić and Wei from 2010.

1. Introduction

Throughout the paper, X is a Banach space and $\mathcal{A}$ denotes the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. An element a in $\mathcal{A}$ has generalized Drazin inverse, i.e., g-Drazin inverse, if and only if there exists $b \in comm(a)$ such that $b = bab$ and $a - a^2b \in \mathcal{A}^{qnil}$. Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in U(\mathcal{A})$ for every $x \in comm(a)\}$. For a Banach algebra $\mathcal{A}$ it is well known that $a \in \mathcal{A}^{qnil} \Leftrightarrow \lim\limits_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0$. The preceding b, if exists, is unique, and is denoted by a^d. We call a^d the g-Drazin inverse of a. We always use $\mathcal{A}^d$ to denote the set of all operators having g-Drazin

* Corresponding Author.
Received June 29, 2023; revised January 22, 2024; accepted January 22, 2024.
2020 Mathematics Subject Classification: 16E90, 15A09.
Key words and phrases: generalized Drazin inverse, additive property, operator matrix, spectral idempotent.

inverses in $\mathcal{A}$. It was proved that $a \in \mathcal{A}^d$ if and only if there exists an idempotent $p \in comm(a)$ such that $a+p$ is invertible and $ap \in \mathcal{A}^{qnil}$, i.e., $a \in \mathcal{A}$ is quasipolar (see [15, Theorem 4.2]). Let $a, b \in \mathcal{A}^d$. It is attractive to explore when the sum $a+b$ has g-Drazin inverse. In [12, Theorem 2.3], Djordjević and Wei proved that if $ab = 0$ then $a+b \in \mathcal{A}^d$. In [11, Theorem 1], Deng and Wei proved that if $ab = ba$ then $a+b \in \mathcal{A}^d$ if and only if $1 + a^d b$ is g-Drazin invertible. In [25], Zou et al. proved that if $a^2 b = aba$ and $b^2 a = bab$ then $a+b \in \mathcal{A}^d$. We refer the reader to [1, 4, 5, 7, 8, 10, 17, 18, 19] for further results.

In Section 2, we present some new additive results of g-Drazin inverses of the sum $a+b$ under a number of polynomial conditions. These generalize the main results of Shakoor et al. (see [21, Lemma 5]).

In Section 3, we consider the g-Drazin inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$. Here, M is a bounded operator on $X \oplus Y$. Such operator matrices have various applications in singular differential and difference equations, Markov chains, and iterative methods. The Drazin inverse of operator matrices has been well studied recently, e.g., [2, 3, 6, 9, 13, 22, 23, 24]. We generalize recent results of Deng et al. (see [10, Theorem 2, 3 and 5]), and of Yang and Liu (see [23, Theorem 3.3]).

If $a \in \mathcal{A}$ has g-Drazin inverse a^d, then the element $a^\pi = 1 - aa^d$ is called the spectral idempotent of a. In Section 4, we illustrate the g-Drazin inverse of a 2×2 operator matrix M under various conditions on spectral idempotents.

2. G-Drazin inverses

The aim of this section is to establish new additive results for g-Drazin inverses and give the explicit formulas for the g-Drazin inverse of the sum $a+b$. We begin with

Lemma 2.1. *Let $a, b \in \mathcal{A}$ and $ab = 0$. If $a, b \in \mathcal{A}$ have g-Drazin inverses, then $a+b \in \mathcal{A}$ has g-Drazin inverse and*

$$(a+b)^d = (1 - bb^d)[\sum_{i=0}^{\infty} b^n (a^d)^n] a^d + b^d [\sum_{i=0}^{\infty} (b^d)^n a^n](1 - aa^d).$$

Proof. See [12, Theorem 2.3]. □

Lemma 2.2. *Let $a, b, c \in \mathcal{A}$. If $a, b \in \mathcal{A}$ have g-Drazin inverses, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})$ has g-Drazin inverse.*

Proof. See [12, Lemma 2.2]. □

Lemma 2.3. *Let $a, b \in \mathcal{A}^d$. If $a^2b = 0, b^2 = 0$ and $bab = 0$, then*

$$(a+b)^d = a^d + b(a^d)^2 + ab(a^d)^3.$$

Proof. Since $(a+b)^2 = a^2 + (ab+ba), (ab+ba)^2 = 0$ and $a^2(ab+ba) = 0$,

$$((a+b)^2)^d = (a^2 + (ab+ba))^d.$$

As $(ab+ba)^2 = 0$, then $(ab+ba)^d = 0$. By applying Lemma 2.1 and using $(a^2)^d = (a^d)^2$ we have,

$$((a+b)^2)^d = (a^d)^2 + (ab+ba)(a^d)^4.$$

Hence,

$$(a+b)^d = (a+b)((a+b)^2)^d = a^d + b(a^d)^2 + ab(a^d)^3,$$

as required. □

We are ready to prove:

Theorem 2.4. *Let $a, b \in \mathcal{A}^d$. If $a^3b = 0, bab = 0, ba^2b = 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = (1, b)M^d \begin{pmatrix} a \\ 1 \end{pmatrix}, M^d = A^d + B(A^d)^2 + AB(A^d)^3,$$

where $A = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}, B = \begin{pmatrix} ab & a^2b + ab^2 \\ 0 & ab \end{pmatrix}$, *and*

$$A^d = (I - KK^d)\big[\sum_{n=0}^{\infty} K^n (H^d)^n\big]H^d + K^d\big[\sum_{n=0}^{\infty}(K^d)^n H^n\big](I - HH^d);$$
$$H^d = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Proof. Set

$$M = \begin{pmatrix} a^2 + ab & a^2b + ab^2 \\ a+b & b^2 + ab \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} + \begin{pmatrix} ab & a^2b + ab^2 \\ 0 & ab \end{pmatrix} = A + B.$$

Since $a, b \in \mathcal{A}^d$, it follows by Lemma 2.2, that A has g-Drazin inverse. Clearly, $B^2 = 0$, and so B has g-Drazin inverse. By a direct computation, we see that $AB = \begin{pmatrix} 0 & 0 \\ a^2b & a^2b^2 \end{pmatrix}$, and so $A^2B = 0$ and $BAB = 0$. Moreover, $B^2 = 0$. Accordingly, M has g-Drazin inverse by Lemma 2.3.

Clearly, $M = (\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b))^2$. It follows by [14, Theorem 2.7] that $\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)$ has g-Drazin inverse. By using Cline's formula, $a+b = (1,b)\begin{pmatrix} a \\ 1 \end{pmatrix}$ has g-Drazin inverse.

By virtue of [16, Theorem 2.1],

$$(a+b)^d = \big((1,b)\begin{pmatrix} a \\ 1 \end{pmatrix}\big)^d = (1,b)M^d\begin{pmatrix} a \\ 1 \end{pmatrix}.$$

In light of Lemma 2.3, $M^d = A^d + B(A^d)^2 + AB(A^d)^3$. Moreover, we have

$$\begin{aligned} A &= \begin{pmatrix} a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & b^2 \end{pmatrix} \\ &:= H+K. \end{aligned}$$

One easily checks that

$$H = \begin{pmatrix} a^2 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix}(a,0).$$

Since $(a,0)\begin{pmatrix} a \\ 1 \end{pmatrix} = a^2 \in \mathcal{A}^d$, it follows by Cline's formula, we see that

$$\begin{aligned} H^d &= \begin{pmatrix} a \\ 1 \end{pmatrix}((a^2)^d)^2(a,0) = \begin{pmatrix} a \\ 1 \end{pmatrix}(a^d)^4(a,0) \\ &= \begin{pmatrix} a(a^d)^4a & 0 \\ (a^d)^4a & 0 \end{pmatrix} = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}. \end{aligned}$$

Likewise, we have

$$K^d = \begin{pmatrix} 0 \\ b \end{pmatrix}(b^d)^4(1,b) = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Clearly, $HK = 0$. By virtue of [12, Theorem 2.3],

$$A^d = (I-KK^d)\big[\sum_{n=0}^{\infty}K^n(H^d)^n\big]H^d + K^d\big[\sum_{n=0}^{\infty}(K^d)^nH^n\big](I-HH^d).$$

□

Corollary 2.5. *Let $a,b \in \mathcal{A}^d$. If $a^2b = 0, b^2a = 0$ and $(ab)^3 = 0$, then $a+b \in \mathcal{A}^d$.*

Proof. Since $(ab)^3 = 0$, we see that $ab \in \mathcal{A}^d$. By using Cline's formula, $ba \in \mathcal{A}^d$. Since $a^2b = 0, b^2a = 0$, it follows by Lemma 2.3, that $a^2+ab, ba+b^2 \in \mathcal{A}^d$. Let $p = a^2+ab$ and $q = ba+b^2$. Then $pq = ab^3$. Hence

$$qpq = (ba+b^2)ab^3 = 0, qp^2q = (qp)(pq) = (ba^3)(ab^3) = 0$$

and

$$p^3q = p(p^2q) = (a^2+ab)(abab^3) = abababab^3 = (ab)^3b^2 = 0.$$

In view of Theorem 2.4, $(a+b)^2 = p+q \in \mathcal{A}^d$. This completes the proof by [14, Theorem 2.7]. □

Let $a, b \in \mathcal{A}^d$. If $a^2b = 0$ and $bab = 0$, then $a+b \in \mathcal{A}^d$. This is a direct consequence of Theorem 2.4. Furthermore, we derive

Corollary 2.6. *Let $a, b \in \mathcal{A}^d$. If $a^2b = 0, bab^2 = 0$ and $(ab)^3 = 0$, then $a+b \in \mathcal{A}^d$.*

Proof. Let $p = a^2+ab$ and $q = ba+b^2$. As in the proof in Corollary 2.5, we see that $p, q \in \mathcal{A}^d$. We easily check that $pq = ab^3+ab^2a$, and so $qpq = (ba+b^2)(ab^3+ab^2a) = 0$, $p^2q = (a^2+ab)(ab^3+ab^2a) = 0$. Therefore $(a+b)^2 = p+q \in \mathcal{A}^d$. The proof is complete by Theorem 2.4. □

Wang et al. studied the Drazin inverse of the sum of two bounded linear operators (see [22]). We now generalize the main results in [22] as follows.

Theorem 2.7. *Let $a, b \in \mathcal{A}^d$. If $a^3b = 0, a^2b+bab = 0$, then $a+b \in \mathcal{A}^d$ and*

$$(a+b)^d = (1, b)M^d \begin{pmatrix} a \\ 1 \end{pmatrix}, M^d = A^d + B(A^d)^2 + B^2(A^d)^3,$$

where $A = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}, B = \begin{pmatrix} ab & a^2b+ab^2 \\ 0 & ab \end{pmatrix}$, *and*

$$A^d = (I-KK^d)\big[\sum_{n=0}^{\infty} K^n(H^d)^n\big]H^d + K^d\big[\sum_{n=0}^{\infty}(K^d)^nH^n\big](I-HH^d);$$
$$H^d = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Proof. Set

$$M = \begin{pmatrix} a^2+ab & a^2b+ab^2 \\ a+b & b^2+ab \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} + \begin{pmatrix} ab & a^2b+ab^2 \\ 0 & ab \end{pmatrix} := A+B.$$

Since $a^3b = 0, a^2b+bab = 0$, we see that

$$\begin{aligned} & (a+b)(a^2b+ab^2)+b^2(ab) \\ = \ & a^2b^2+ba^2b+bab^2+b^2(ab) \\ = \ & a^2b^2+ba^2b+(-a^2b)b+b(-a^2b) \\ = \ & 0. \end{aligned}$$

Thus, we have $AB = 0$. Since $(ab)^2 = a(bab) = -a^3b = 0$, we easily check that $B^3 = 0$; hence, B has g-Drazin inverse. Since $a, b \in \mathcal{A}$ have g-Drazin inverses, it follows by Lemma 2.2, that A has g-Drazin inverse. In light of Lemma 2.3, $M = A + B$ has g-Drazin inverse.

By using Lemma 2.3 again, $M^d = A^d + B(A^d)^2 + B^2(A^d)^3$. Obviously, we have

$$\begin{aligned} A &= \begin{pmatrix} a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & b^2 \end{pmatrix} \\ &:= H + K. \end{aligned}$$

As in the proof of Theorem 2.4, one easily checks that

$$H^d = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Moreover,

$$A^d = (I - KK^d)\big[\sum_{n=0}^{\infty} K^n (H^d)^n\big] H^d + K^d \big[\sum_{n=0}^{\infty} (K^d)^n H^n\big](I - HH^d),$$

as required. □

Corollary 2.8. *Let $a, b \in \mathcal{A}^d$. If $(ab)^2 = 0, a^2 b + bab = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Clearly, $a^3 b = -(ab)^2 = 0$, and therefore we obtain the result by Theorem 2.4. □

We note that Corollary 2.8, is a nontrivial generalization of [12, Theorem 2.3], as the following shows.

Example 2.9. *Let A and B be operators, acting on separable Hilbert space $l_2(N)$, defined as follows respectively:*

$$\begin{aligned} A(x_1, x_2, x_3, x_4, \cdots) &= (x_1, x_4, 0, 0, 0, \cdots), \\ B(x_1, x_2, x_3, x_4, \cdots) &= (0, x_3, x_3, x_1, 0, \cdots). \end{aligned}$$

Then we easily check that $(AB)^2 = 0, A^2 B + BAB = 0$ and $A, B \in \mathcal{L}(l_2(N))^d$. Hence $A + B$ has g-Drazin inverse by Corollary 2.8, in this case, $AB \neq 0$.

3. Splitting Approach

To illustrate the preceding results, we are concerned with the g-Drazin inverse for an operator matrix. Throughout this section, the operator matrix M is given by (1.1), i.e.,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.1}$$

where $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$. Using different splitting approaches, we will obtain various conditions for the g-Drazin inverse of M.

Theorem 3.1. *If $A^2BC = 0, A^2BD = 0, CBC = 0, CBD = 0, CABC = 0$ and $CABD = 0$. Then M has g-Drazin inverse.*

Proof. Let $p = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$, then $M = p + q$. By applying [10, Theorem 3], it is obvious that p, q have g-Dazin inverses. Now we have

$$p^3q = \begin{pmatrix} A^2BC & A^2BD \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ CBC & CBD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ CABC & CABD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.4, M has g-Drazine inverse. □

Corollary 3.2. *If $ABC = 0, ABD = 0, CBC = 0$ and $CBD = 0$. Then M has g-Drazin inverse.*

Proof. This is obvious by Theorem 3.1. □

Regarding a complex matrix as the operator matrix on $C \times C$, we now show that corollary 3.2 is a non-trivial generalization of [10, Theorem 2]

Example 3.3. *Let*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

be complex matrices and set

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$ABC = 0, ABD = 0, CBC = 0, CBD = 0.$$

In view of Corollary 3.2 M has g-Drazin inverse but $BC \neq 0$ and $BD \neq 0$.

Theorem 3.4. *If $A^3B = 0, CA^2B = 0, BCB = 0, DCB = 0$, $BCAB = 0$ and $DCAB = 0$, then M has g-Drazin inverse.*

Proof. Let $p = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}$, then $M = p+q$. In view of Lemma 2.2, p and q have g-Drazin inverses. Now we have

$$p^3q = \begin{pmatrix} 0 & A^3B \\ 0 & CA^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & BCB \\ 0 & DCB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & BCAB \\ 0 & DCAB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.4, M has g-Drazine inverse. □

Corollary 3.5. *If $A^2B = 0, BCB = 0, DCB = 0$ and $CAB = 0$, then M has g-Drazin inverse.*

Proof. It is a special case of Theorem 3.3. □

Theorem 3.6. *If $(A^2 + BC)BD = 0, CABD = 0, DCBD = 0$, $ABC = 0$ and $CBC = 0$, then M has g-Drazin inverse.*

Proof. Let $p = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$, then $M = p + q$. Since $ABC = 0, CBC = 0$, it follows by Corollary 3.2 that p has g-Drazin inverse. By Lemma 2.2, q has g-Dazin inverse. Now we have

$$p^3q = \begin{pmatrix} 0 & A^2BD + BCBD \\ 0 & CABD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ 0 & DCBD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 2.4, M has g-Drazine inverse. □

Corollary 3.7. *If $BC = 0, BD = 0$, then M has g-Drazin inverse.*

Proof. This is clear from Theorem 3.6. □

We are now ready to prove:

Theorem 3.8. *If $A^3B = 0, CAB = 0, CA^2B = 0, BCB = 0$ and $DCB = 0$, then M has g-Drazin inverse.*

Proof. Let $p = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, then $M = p + q$. Clearly, p has g-Drazin inverse. Since $BCB = 0$ and $DCB = 0$, it follows by Corollary 3.5 that q has g-Dazin inverse. We check that

$$p^3q = \begin{pmatrix} 0 & A^3B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$qpq = \begin{pmatrix} 0 & 0 \\ 0 & CAB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$qp^2q = \begin{pmatrix} 0 & 0 \\ 0 & CA^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 2.4, M has g-Drazine inverse. □

As an immediate consequence of Theorem 3.8, we now derive

Corollary 3.9. *If $AB = 0$ and $CB = 0$, then M has g-Drazin inverse.*

4. Spectral Conditions

Let M be an operator matrix M given by (1.1). It is of interest to consider the g-Drazin inverse of M under generalized Schur condition $D = CA^dB$ (see [20]). The goal of this section is to consider another splitting of the block matrix M under such condition and present alternative theorems on spectral idempotents.

Theorem 4.1. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1). *If $I + A^dBCA^d$ is g-Drazin invertible,*

$$BCA^{\pi}A = 0, BCA^{\pi}B = 0, A^2BCA = ABCA^2, D = CA^dB,$$

then M has g-Drazin inverse.

Proof.

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

By assumption, we verify that $P^3Q = 0, QPQ = 0, QP^2Q = 0$. Clearly, Q is quasinilpotent, and so it has g-Drazin inverse. Furthermore, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ P_2 = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and $P_2P_1 = 0, P_2^2 = 0$. Clearly, P_2 has g-Drazin inverse. Moreover, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d.$$

It is obvious that, $A^2A^d = A(AA^d)$ has g-Drazin inverse. Since $(CA^d)\ (AA^dB) = CA^dB = D$ has g-Drazin inverse, it follows by Cline's formula that AA^dBCA^d has g-Drazin inverse.

Since $A^2BCA = ABCA^2$, we easily check that

$$\begin{array}{rcl} (A^2A^d)(AA^dBCA^d) & = & A^d(A^2BCA)(A^d)^2 \\ & = & A^d(ABCA^2)(A^d)^2 \\ & = & A^dABCAA^d \\ & = & (AA^dBCA^d)(A^2A^d). \end{array}$$

In light of [11, Theorem 1], $A^2A^d + AA^dBCA^d$ has g-Drazin inverse. By using Cline's formula again, P_1 has g-Drazin inverse. Thus, by [12, Theorem 2.3], P g-Drazin inverse. By using Theorem 2.4, M has g-Drazin inverse, as asserted. □

Corollary 4.2. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1)*. If $I + A^dBCA^d$ is g-Drazin invertible,*

$$BCA = 0, BCB = 0, D = CA^dB,$$

then M has g-Drazin inverse.

Proof. Since $BCA = 0$, we see that $BCA^\pi A = 0$ and $A^2BCA = ABCA^2 = 0$. Moreover, $BCA^\pi B = BC(I_2 - AA^d)B = BCB = 0$. This completes the proof by Theorem 4.1. □

Theorem 4.3. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1)*. If $I + A^dBCA^d$ is g-Drazin invertible,*

$$A^\pi BCA^\pi = 0, ABCA^\pi = 0, A^2BCA = ABCA, D = CA^dB,$$

then M has g-Drazin inverse.

Proof. We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A^2A^d & B \\ CAA^d & CA^dB \end{pmatrix}, Q = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}.$$

Then we check that $P^3Q = 0, QPQ = 0, QP^2Q = 0$. Clearly, Q has g-Drazin inverse. Furthermore, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ P_2 = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$$

and $P_1P_2 = 0$. Clearly, P_2 is nilpotent, and it has g-Drazin inverse. Obviously, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d.$$

As in the proof of Theorem 4.1, we easily check that $A^2A^d + AA^dBCA^d$ has g-Drazin inverse. Therefore P_1 has g-Drazin inverse. By using Lemma 2.2, again, M has g-Drazin inverse, as asserted. □

By using the other splitting approach of the block operator matrix, we now ready to prove:

Theorem 4.4. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1). *If $I + A^dBCA^d$ is g-Drazin invertible,*

$$AA^\pi BC = 0, CA^\pi BC = 0, A^2BCA = ABCA, D = CA^dB,$$

then M has g-Drazin inverse.

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}.$$

Clearly, P is nilpotent, and so it has g-Drazin inverse.

Furthermore, we have

$$Q = Q_1 + Q_2,\ Q_1 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}$$

and $Q_1Q_2 = 0$. We easily see that Q_1 is quasinilpotent, and it has g-Drazin inverse. Moreover,

$$Q_2 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A^2A^d & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A^2A^d & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d.$$

As in the proof of Theorem 4.1, we easily check that $A^2A^d + AA^dBCA^d$ has g-Drazin inverse. Therefore Q_2 has g-Drazin inverse. By hypothesis, we check that $P^2 = 0$ and $QPQ = 0$, and so $P^3Q = 0$ and $P^2Q + QPQ = 0$. By virtue of Theorem 2.7, M has g-Drazin inverse, as asserted. □

Analogously, we derive

Proposition 4.5. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1). *If $I + A^dBCA^d$ is g-Drazin invertible,*

$$ABCA^{\pi}A = 0, A^2BCA = ABCA, D = CA^dB,$$

then M has g-Drazin inverse.

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, Q = \begin{pmatrix} A & AB \\ CAA^d & CA^dB \end{pmatrix}.$$

As in the proof of Theorem 4.1, we easily check that P and Q have g-Drazin inverses. Since $A^dBCA^{\pi}A = (A^d)^2ABCA^{\pi}A = 0$, we easily check that and $P^2 = 0$ and $QPQ = 0$. Therefore we complete the proof by Theorem 2.4. □

Corollary 4.6. *Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have g-Drazin inverses and M be given by* (1.1). *If $I + A^dBCA^d$ is g-Drazin invertible,*

$$ABC = 0, D = CA^dB,$$

then M has g-Drazin inverse.

Proof. This is obvious by Proposition 4.5. □

References

[1] M. Boumazgour, *Generalized Drazin inverse of restrictions of bounded linear operators*, Linear Multilinear Algebra, **66**(2018), 894–901.

[2] N. Castro-González, E. Dopazo and M. F. Martinez-Serrano, *On the Drazin inverse of the sum of two operators and its application to operator matrices*, J. Math. Analysis Appl., **350**(2009), 207–215.

[3] H. Chen and M. Sheibani, *The G-Drazin inverse for special operator* matrices, Operators and Matrices, **15**(2021), 151–162.

[4] H. Chen and M. Sheibani, *Block representations for the g-Drazin inverse in Banach algebras*, Miskolc Mathematical Notes, **24**(2023), 1259–1271.

[5] H. Chen and M. Sheibani, *Additive Properties of g-Drazin Invertible Linear Operators*, Filomat, **36**(2022), 3301–3309.

[6] A. S. Cvetković and G. V. Milovanović, *On the Drazin inverse of operator matrices*, J. Math. Analysis Appl., **375**(2011), 331–335.

[7] D. S. Cvetković-Ilić, D. S. Djordjević and Y. Wei, *Additive results for the generalized Drazin inverse in a Banach algebra*, Linear Algebra Appl., **418**(2006), 53–61.

[8] D. S. Cvetković-Ilić, X. Liu and Y. Wei, *Some additive results for the generalized Drazin inverse in a Banach algebra*, Electronic J. Linear Algebra, **22**(2011), 1049–1058.

[9] M. Dana and R. Yousefi, *Formulas for the Drazin inverse of matrices with new conditions and its applications*, Int. J. Appl. Comput. Math., **4**(2018), https://doi.org/10.1007/s40819-017-0459-5.

[10] C. Deng, D. S. Cvetcović-Ilić and Y. Wei, *Some results on the genrealized Derazin inverse of operator matrices*, Linear and Multilinear Algebra **58**(2010), 503–521.

[11] C. Deng and Y. Wei, *New additive results for the generalized Drazin inverse*, J. Math. Analysis Appl., **370**(2010), 313–321.

[12] D. S. Djordjević and Y. Wei, *Additive results for the generalized Drazin inverse*, J. Austral. Math. Soc., **73**(2002), 115–125.

[13] L. Guo, H. Zou and J. Chen, *The generalized Drazin inverse of operator matrices*, Hacet. J. Math. Stat., **49**(2020), 1134–1149.

[14] Y. Jiang. Y. Wen and Q. Zeng, *Generalizations of Cline's formula for three generalized inverses*, Revista. Un. Math. Argentina, **48**(2017), 127–134.

[15] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J., **38**(1996), 367–381.

[16] Y. Liao, J. Chen and J. Cui, *Cline's formula for the generalized Drazin inverse*, Bull. Malays. Math. Sci. Soc., **37**(2014), 37–42.

[17] X. Liu, X. Qin and J. Benitez, *New additive results for the generalized Drazin inverse in a Banach Algebra*, Filomat, **30**(2016), 2289-2294.

[18] D. Mosić and D.S, Djordjević, *Block representations of generalized Drazin inverse*, Appl. Math. Comput., **331**(2018), 200–209.

[19] D. Mosić, H. Zou and J. Chen, *The generalized Drazin inverse of the sum in a Banach algebra*, Ann. Funct. Anal., **8**(2017), 90–105.

[20] A. Shakoor, H. Yang and I. Ali, *The Drazin inverses of the sum of two matrices and block matrix*, J. Appl. Math. & Informatics, **31**(2013), 343–352.

[21] A. Shakoor, H. Yang and I. Ali, *Some results for the Drazin inverse of the sum of two matrices and some block matrices*, J. Appl. Math., **2013**(2013), 804313.

[22] H. Wang, J. Huang and A. Chen, *The Drazin inverse of the sum of two bounded linear operators and it's applications*, Filomat, **31**(2017), 2391–2402.

[23] H. Yu and X. Liu, *The Drazin inverse of the sum of two matrices and its applications*, J. Comput. Appl. Math., **235**(2011), 1412–1417.

[24] D. Zhang, Y. Jin and D. Mosić, *A note on formulae for the generalized Drazin inverse of anti-triangular block operator matrices in Banach spaces*, Banach J. Math. Anal., **16**(2022), DOI: 10.1007/s43037-022-00176-8.

[25] H. Zou, D. Mosić and J. Chen, *Generalized Drazin invertiblity of the product and sum of two elements in a Banach algebra and its applications*, Turkish. J. Math., **41**(2017), 548–563.