KYUNGPOOK Math. J. 64(2024), 197-204
https://doi.org/10.5666/KMJ.2024.64.2.197
pISSN 1225-6951 eISSN 0454-8124

Minimal Generators of Syzygy Modules Via Matrices

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Abstract. Let $R = \mathbb{K}[x]$ be a univariate polynomial ring over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $A \in M_{m,m}(R)$ be an $m \times m$ matrix over R with non-zero determinate $\det(A) \in R$. In this paper, utilizing linear-algebraic techniques, we investigate the relationship between a basis for the syzygy module of $f_1, \dots, f_m$ and a basis for the syzygy module of $g_1, \dots, g_m$, where $[g_1, \dots, g_m] = [f_1, \dots, f_m]A$.

1. Introduction

Systems of linear equations, vector spaces, and bases for vector spaces are widely studied in linear algebra. Modules are a generalization of vector spaces from linear algebra in which the "scalars" are allowed to be from an arbitrary ring, rather than a field. Many linear algebraic methods remain effective over principal ideal domains where the extended Euclidean algorithm may be utilized for computing with unimodular matrices. This paper is devoted to the some applications of linear algebra in studying modules over a univariate polynomial ring $R = \mathbb{K}[x]$ where $\mathbb{K}$ is an algebraically closed field of characteristic zero.

Over the ring R, given $a_1, \dots, a_k, b \in R$, the problem to determine if a single equation

$$a_1x_1 + \cdots a_kx_k = b$$

has a solution $x_1, \dots, x_k \in R$ amounts to determining if b is in the ideal generated by $a_1, \dots, a_k$; this is the ideal membership problem. In the case of a system of several equations, the ideal membership problem turns into the submodule membership problem. Furthermore, given $\mathbf{a}_1, \dots, \mathbf{a}_k \in R^n$, a solution $(x_1, \dots, x_k) \in R^k$ to the homogeneous equation

$$\mathbf{a}_1x_1 + \cdots + \mathbf{a}_kx_k = \mathbf{0}$$

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* Corresponding Author.
Received April 20, 2023; revised November 18, 2023; accepted March 10, 2024.
2020 Mathematics Subject Classification: 15A23, 15A54, 16E05.
Key words and phrases: Smith Normal Form, Cauchy-Binet Formula, Basis, Modules.

is called a syzygy of $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$. The solution set for this system of homogeneous equations provides a system of generators for the syzygy module of $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$. Hence, the computation of the null space of the matrix of a system of linear equations over R is essentially the syzygy problem, the goal is to find a generating set for the syzygy module.

A fundamental fact of linear algebra over a field that is a finitely generated vector space has a basis – the minimal generating (spanning) sets of a vector space are linearly independent and therefore form a basis. However, modules are more complicated than vector spaces; for instance, not all modules have a basis. It remains an interesting and an active research area to find a minimal generating set, or an upper bound for the size of a minimal generating set for different kinds of modules under various of conditions [2], [4], [6], and [8]. Hilbert's syzygy theorem [7] states that, if M is a finitely generated module over a multivariate polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$, then the n-th syzygy module of M is always a free module, i.e., a module that has a basis – a generating set consisting of linearly independent elements. In particular, over a univariate polynomial ring, Hilbert's syzygy theorem asserts that over a principal ideal ring, every submodule of a free module is itself free. Hence, it is an interesting problem to find the basis for syzygy modules.

This short paper investigates the minimal set of generators of a family of special modules. Our focus is centered on two finitely generated modules – the first module is $\mathrm{Syz}(\mathbf{f})$, the syzygy module of $\mathbf{f} = [f_1, \ldots, f_m]$, where $f_1, \ldots, f_m \in R = \mathbb{K}[x]$ with $\mathbb{K}$ an algebraically closed field of characteristic zero; the second module is $\mathrm{Syz}(\mathbf{g})$, the syzygy module of $\mathbf{g} = [g_1, \ldots, g_m]$, where $[g_1, \ldots, g_m] = [f_1, \ldots, f_m]A$ with $A \in M_{m,m}(R)$ and $0 \neq \det(A) = d(x) \in R$. Geometrically, if $\gcd(\mathbf{f}) = 1$, then $\mathbf{f}$ may be viewed as a polynomial parametrization of a rational space curve in a m-dimensional affine space. Similarly, $\mathbf{g} = [g_1, \ldots, g_m] = \mathbf{f}A$ can be considered as another rational space curve constructed by applying a linear transformation A to the curve $\mathbf{f}$ over the ring R.

A natural question to ask is "how to link a basis for $\mathrm{Syz}(\mathbf{f})$ with a basis for $\mathrm{Syz}(\mathbf{g})$"? To answer this question, we evoke the Smith normal form of the matrix A, and take advantage of Cauchy-Binet formula to relate a basis for $\mathrm{Syz}(\mathbf{f})$ with a basis for $\mathrm{Syz}(\mathbf{g})$ via a matrix $B \in M_{m-1,m-1}(R)$ such that $\det(B) = \lambda \det(A)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$.

This paper is structured as the following. We begin in Section 2 with a brief review of results concerning a matrix factorization over PIDs, and the concepts of syzygy modules. We then answer the question that "how to link a basis for $\mathrm{Syz}(\mathbf{f})$ with a basis for $\mathrm{Syz}(\mathbf{g})$" in Section 3. The main result of this paper is Theorem 3.1, where we provide an explicit equation to describe a relationship between a basis for $\mathrm{Syz}(\mathbf{f})$ and a basis for $\mathrm{Syz}(\mathbf{g})$. We flush out our theorem by a simple and illustrative example. We end our paper by a few conclusions in Section 4.

2. A Brief Review

In this section, we provide a brief review of results concerning a matrix factorization over a PID, the concepts of syzygy modules, and the Hilbert-Burch theorem. First, we recall the Smith normal form over a PID and the Cauchy-Binet formula.

Theorem 2.1. (Smith Normal Form over PID [1, Theorem 3.1]) *Let R be a PID and let $A \in M_{m,n}(R)$. Then there is a $U \in GL(m,R)$ and a $V \in GL(n,R)$ such that*

$$UAV = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank}(A)$, and D is a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_r$ with $d_i \neq 0$ for $i = 1, \ldots, r$, and $d_i \mid d_{i+1}$ for $i = 1, \ldots, r-1$. Furthermore, if R is a Euclidean domain, the matrices U, V can be taken to be a product of elementary matrices.

Theorem 2.2. (Cauchy-Binet formula [1, Theorem 2.34, Page 210]) *Suppose that A is an $a \times b$ matrix, B is an $b \times c$ matrix, I is a subset of $\{1, \ldots, a\}$ with k elements, and J is a subset of $\{1, \ldots, c\}$ with k elements. Let $[A]_{I,J}$ be the minor of A associated to the ordered sequences of indexes I and J. Then the Cauchy-Binet formula is*

$$[AB]_{I,J} = \sum_K [A]_{I,K}[B]_{K,J}, \tag{2.1}$$

where the sum extends over all subsets K of $\{1, \ldots, b\}$ with k elements.

Next, we review a few concepts and results concerning syzygies, please refer [3, Chapters 4, 5] and [5, Chapter 20] for details. Recall that given a generating set $m_1, \ldots, m_k$ of an ideal (or a module) over a ring R, a relation or first *syzygy* between the generators is a k-tuple $(a_1, \ldots, a_k) \in R^k$ such that

$$a_1 m_1 + a_2 m_2 + \cdots a_k m_k \equiv 0.$$

The set of syzygies form a module $\operatorname{Syz}(m_1, \ldots, m_k)$.

Let $\mathbf{f} = [f_1, \ldots, f_m]$ with $f_i \in R$. The syzygy module, $\operatorname{Syz}(\mathbf{f}) = \operatorname{Syz}(f_1, \ldots, f_m)$, is the kernel of the map $R^m \to \langle f_1, \ldots, f_m \rangle$ that takes the standard basis elements of R^m to the given set of generators. Hilbert–Burch theorem yields that $\operatorname{Syz}(\mathbf{f})$ is a free module – a module that has a basis; and is minimally generated by $m-1$ elements – $\operatorname{rank}(\operatorname{Syz}(\mathbf{f})) = m-1$. In addition, if the minimal set of generators of $\operatorname{Syz}(\mathbf{f})$, namely a μ-basis, are expressed as the columns of a matrix $S \in M_{m,m-1}(R)$, then the ideal $\mathbf{f}$ is generated by the minors of size $m-1$ of the matrix of S. That is, if $\{1, \ldots, \hat{i}, \ldots, m\}$ are subsets of $\{1, 2, \ldots, m\}$ without the element i, and $[S]_{\{1,\ldots,\hat{i},\ldots,m\},\{1,\ldots,m-1\}}$ denotes the minor of size $m-1$ of the matrix of S without the i-th row, then the columns of S form a μ-basis if and only if

$$f_i = \alpha(-1)^{i+1}[S]_{\{1,\ldots,\hat{i},\ldots,m\},\{1,\ldots,m-1\}} \tag{2.2}$$

for some $\alpha \in \mathbb{K} \setminus \{0\}$, $\quad i = 1, \dots, m$.

Equation (2.2) is formulated from the following Hilbert–Burch theorem by applying $M = S$ and $N = \mathbf{f}$.

Theorem 2.3.(Hilbert–Burch Theorem [3, Proposition 2.6, Chater 6]) *Suppose that an ideal I in $R = \mathbb{K}[x_1, \dots, x_n]$ has a free resolution of the form*

$$0 \longrightarrow R^{m-1} \xrightarrow{M} R^m \xrightarrow{N} I \longrightarrow 0, \text{ for some } m.$$

Then there exists a non-zero element $g \in R$ such that $N = [gh_1, \dots, gh_m]$, where h_i is the determinant of the $(m-1) \times (m-1)$ submatrix of M obtained by deleting row i. If $\mathbb{K}$ is algebraically closed and the variety of I has dimension $n-2$, then we can take $g = 1$.

3. Main results

In this section, we investigate the relationship between a μ-basis for $\mathrm{Syz}(\mathbf{f})$ and a μ-basis for $\mathrm{Syz}(\mathbf{g})$, and prove Theorem 3.1, our main result.

We continue our notation – set $R = \mathbb{K}[x]$ with $\mathbb{K}$ an algebraically closed field of characteristic zero, and let $A \in M_{m,m}(R)$ where $0 \neq \det(A) = d(x) \in R$. In addition, we define $\mathbf{f} = [f_1, \dots, f_m]$ where $f_1, \dots, f_m \in R$, and $\mathbf{g} = [g_1, \dots, g_m] = [f_1, \dots, f_m]A$.

Theorem 3.1. *Let $S, T \in M_{m,m-1}(R)$ such that the columns of S and T are μ-bases for $\mathrm{Syz}(\mathbf{f})$ and $\mathrm{Syz}(\mathbf{g})$ respectively. Then there exists a $B \in M_{m-1,m-1}(R)$ such that $AT = SB$ with $\det(B) = \lambda \det(A)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$.*

Proof. By Theorem 2.1,

$$A = U \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \cdots & d_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & d_m \end{bmatrix} V, \tag{3.1}$$

where $U, V \in \mathrm{GL}(m, R)$, $d_i \mid d_{i+1}$, $\det(A) = \prod_{i=1}^m d_i = d(x) \neq 0$.

Set $\mathbf{f}' = \mathbf{f}U = [f'_1, \dots, f'_m]$, $\mathbf{g}' = \mathbf{g}V^{-1} = [g'_1, \dots, g'_m]$, and let the columns of $S', T' \in M_{m,m-1}(R)$ be μ-bases for $\mathrm{Syz}(\mathbf{f}')$ and $\mathrm{Syz}(\mathbf{g}')$ respectively.

First, we claim that there exists a $B \in M_{m-1,m-1}(R)$ such that $DT' = S'B$ and $\det(B) = \frac{\beta}{\alpha}\det(A)$, where $\alpha, \beta \in \mathbb{K} \setminus \{0\}$ are such that for $i = 1, \dots, m-1$

$$f'_i = \alpha(-1)^{i+1}[S']_{\{1,\dots,\hat{i},\dots,m\},\{1,\dots,m-1\}}, \; g'_i = \beta(-1)^{i+1}[T']_{\{1,\dots,\hat{i},\dots,m\},\{1,\dots,m-1\}}.$$

To prove our claim, we note that the columns of T' form a μ-basis for $\mathrm{Syz}(\mathbf{g}')$ implies

$$\mathbf{g}'T' = \mathbf{g}V^{-1}T' = \mathbf{f}AV^{-1}T' = \mathbf{f}(UDV)V^{-1}T' = (\mathbf{f}U)DT' = \mathbf{f}'(DT') = \mathbf{0}.$$

By the definition of Syz($\mathbf{f}'$), the columns of the matrix DT' are elements of Syz($\mathbf{f}'$). Hence, they are generated by the columns of S', i.e.,

$$DT' = S'B \quad \text{for some } B \in M_{m-1,m-1}(R) \quad \Longleftrightarrow \tag{3.2}$$
$$(T')^t D^t = (DT')^t = (S'B)^t = B^t(S')^t, \tag{3.3}$$

where $(T')^t, (S')^t \in M_{m-1,m}(R)$, $D^t, B^t \in M_{m,m}(R)$.

Now, we apply Cauchy-Binet formula in Theorem 2.2 to both sides of Equation (3.3). First, we compute all $(m-1)\times(m-1)$ minors of $B^t(S')^t$ obtained by deleting the i-th row for $i = 1, 2, \ldots, m$:

$$\begin{aligned}[B^t(S')^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i}\ldots,m\}} &= \det(B^t)[(S')^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i}\ldots,m\}} \\ &= \det(B)\alpha(-1)^{i+1} f'_i.\end{aligned} \tag{3.4}$$

The last equality holds since the columns of S' form a μ-basis for Syz($\mathbf{f}'$), and by Equation (2.2),

$$[(S')^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i},\ldots,m\}} = \alpha(-1)^{i+1} f'_i, \text{ form some } \alpha \in \mathbb{K}\setminus\{0\},\ i = 1, 2, \ldots, m.$$

Again, applying Equation (2.1), we compute all $(m-1)\times(m-1)$ minors of $(T')^t D^t$ obtained by deleting the i-th row for $i = 1, 2, \ldots, m$:

$$\begin{aligned} & [(T')^t D^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i},\ldots,m\}} \\ = & \sum_{j=1}^{m} [(T')^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{j},\ldots,m\}} [D^t]_{\{1,\ldots,\hat{j},\ldots,m\},\{1,\ldots,\hat{i},\ldots,m\}} \\ = & \sum_{j=1}^{m} \beta(-1)^{j+1} g'_j [D^t]_{\{1,\ldots,\hat{j},\ldots,m\},\{1,\ldots,\hat{i},\ldots,m\}} \text{ for some } \beta \in \mathbb{K}\setminus\{0\},\end{aligned}$$

where the last equality is due to Equation (2.2). It is easy to observe that the diagonal matrix D in Equation (3.1) has the property that

$$[D^t]_{\{1,\ldots,\hat{j},\ldots,m\},\{1,\ldots,\hat{i},\ldots,m\}} = 0, \text{ if } j \neq i.$$

Thus,

$$\begin{aligned} & \sum_{j=1}^{m} \beta(-1)^{j+1} g'_j [D^t]_{\{1,\ldots,\hat{j},\ldots,m\},\{1,\ldots,\hat{i},\ldots,m\}} \\ = & \beta(-1)^{i+1} g'_i [D^t]_{\{1,\ldots,\hat{i},\ldots,m\},\{1,\ldots,\hat{i},\ldots,m\}} = \beta(-1)^{i+1} g'_i \frac{d(x)}{d_i}.\end{aligned}$$

In addition, the equality

$$\begin{bmatrix} g_1' \\ g_2' \\ \vdots \\ g_m' \end{bmatrix}^t = \mathbf{g}V^{-1} = \mathbf{f}(UDV)V^{-1} = (\mathbf{f}U)D = \mathbf{f}'D$$

$$= \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_m' \end{bmatrix}^t \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & d_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & d_m(x) \end{bmatrix} = \begin{bmatrix} d_1 f_1' \\ d_2 f_2' \\ \vdots \\ d_m f_m' \end{bmatrix}^t$$

yields that $\beta(-1)^{i+1} g_i' \dfrac{d(x)}{d_i} = \beta(-1)^{i+1}(d_i f_i')\dfrac{d(x)}{d_i} = \beta(-1)^{i+1} d(x) f_i'$. Hence,

$$[(T')^t D^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i},\ldots,m\}} = \beta(-1)^{i+1} d(x) f_i'. \tag{3.5}$$

Therefore, the Equation (3.3) yields that for each $i \in \{1, \ldots, m\}$

$$[B^t(S')^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i}\ldots,m\}} = [(T')^t D^t]_{\{1,\ldots,m-1\},\{1,\ldots,\hat{i},\ldots,m\}},$$

and the Equations (3.4) and (3.5) show that

$$\det(B)\alpha(-1)^{i+1} f_i' = \beta(-1)^{i+1} d(x) f_i' \quad \Longrightarrow \quad \det(B) = \lambda d(x),$$

where $\lambda = \frac{\beta}{\alpha} \in \mathbb{K} \setminus \{0\}$.

Hence, we have proved that there exists a $B \in M_{m-1,m-1}(R)$ such that $DT' = S'B$, $\det(B) = \lambda d(x)$, where $\lambda = \frac{\beta}{\alpha} \in \mathbb{K} \setminus \{0\}$ and α, β are such that for each $i = 1, \ldots, m-1$

$$f_i' = \alpha(-1)^{i+1}[S']_{\{1,\ldots,\hat{i},\ldots,m\},\{1,\ldots,m-1\}}, \; g_i' = \beta(-1)^{i+1}[T']_{\{1,\ldots,\hat{i},\ldots,m\},\{1,\ldots,m-1\}}.$$

Now we are ready to prove the statement of the theorem. Note that on one hand,

$$\mathbf{g}'T' = \mathbf{g}(V^{-1}T') = 0 \quad \Longrightarrow \quad V^{-1}T' = TC \text{ for some } C \in M_{m,m}(R),$$

that is, columns of $V^{-1}T' \in \mathrm{Syz}(\mathbf{g})$ and are generated by the μ-basis for $\mathrm{Syz}(\mathbf{g})$. On the other hand,

$$\mathbf{g}T = \mathbf{g}V^{-1}VT = \mathbf{g}'VT = 0 \quad \Longrightarrow \quad VT = T'C' \text{ for some } C' \in M_{m,m}(R),$$

that is, columns of $VT \in \mathrm{Syz}(\mathbf{g}')$ and are generated by the μ-basis for $\mathrm{Syz}(\mathbf{g}')$. Thus,

$$VT = T'C' \text{ and } V^{-1}T' = TC \quad \Longrightarrow \quad T = V^{-1}T'C' = TCC' \quad \Longrightarrow \quad CC' = I_{n,n}$$

where $I_{n,n}$ is an $n \times n$ identity matrix. This implies that $C, C' \in \mathrm{GL}(m, R)$, that is, $T = V^{-1}T'C' \cong V^{-1}T'$, equivalently $T' = VTC \cong VT$. Since μ-bases are unique up-to isomorphism, without loss of generality, we may set $T = V^{-1}T'$, equivalently $T' = VT$. Applying a similar argument to $\mathrm{Syz}(\mathbf{f})$ and $\mathrm{Syz}(\mathbf{f}')$, we obtain an analogous result about S and S'. We conclude that μ-bases for $\mathrm{Syz}(\mathbf{g})$ and $\mathrm{Syz}(\mathbf{f})$ can be expressed as the columns of the matrices T and S respectively where

$$T = V^{-1}T', \quad S = US' \quad \Longleftrightarrow \quad T' = VT, \quad S' = U^{-1}S.$$

Thus, replacing T' and S' by VT and $U^{-1}S$ in Equation (3.2) yields

$$DVT = DT' = S'B = U^{-1}SB \quad \Longleftrightarrow \quad UDVT = SB \quad \Longleftrightarrow \quad AT = SB.$$

Therefore, we conclude that there exists a $B \in M_{m-1,m-1}(R)$ such that $AT = SB$ with $\det(B) = \lambda \det(A)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$. □

Below, we will provide a simple example to illustrate Theorem 3.1.

Example 3.1. Consider $[f_1, f_2, f_3] = [1, x^2 - 1, x^3 + 1]$, and find a μ-basis of $\mathrm{Syz}(f_1, f_2, f_3)$ formed by the columns of the matrix $S = \begin{bmatrix} x^2-1 & x+1 \\ -1 & x \\ 0 & -1 \end{bmatrix}$, where the maximal minors of S, with an appropriate sign, give (f_1, f_2, f_3).

Let

$$\begin{aligned} A &= \begin{bmatrix} x+1 & x & 0 \\ x^2 & x^2 & x^2-x \\ x+1 & x+1 & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & x-1 \\ 0 & x+1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x+1 & x & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= UDV, \ \det(A) = x, \end{aligned}$$

and

$$\begin{aligned} & [g_1, g_2, g_3] = [f_1, f_2, f_3]A \\ = \ & [2x^4 + x^3 - x^2 + 2x + 2, 2x^4 + x^3 - x^2 + 2x + 1, 2x^4 - x^3 - x^2 + 2x]. \end{aligned}$$

A μ-basis for $\mathrm{Syz}(g_1, g_2, g_3)$ is given by the columns of the matrix $T = \begin{bmatrix} -2x^2+2x+1 & -2x2-x-1 \\ 2x^2-2 & 2x+2 \\ -2x-1 & 2x^2+x-1 \end{bmatrix}$, where the maximal minors of T, with an appropriate sign, give $2(g_1, g_2, g_3)$. We have that $AT = \begin{bmatrix} x+1 & -2x^3-x^2-1 \\ x & -x(x-1) \\ -1 & x+1 \end{bmatrix} = SB$, where $B = \begin{bmatrix} 0 & -2x \\ 1 & -1-x \end{bmatrix}$ and $\det(B) = 2x = 2\det(A)$.

4. Conclusions

Applying linear-algebraic techniques, we obtained a relationship on the minimal generators Syz($\mathbf{f}$) and Syz($\mathbf{g}$). We proved that if $S, T \in M_{m,m-1}(R)$ such that the columns of S and T are μ-bases for Syz($\mathbf{f}$) and Syz($\mathbf{g}$) respectively, then there exists a $B \in M_{m-1,m-1}(R)$ such that $AT = SB$ with $\det(B) = \lambda \det(A)$ for some $\lambda \in \mathbb{K} \setminus \{0\}$.

We took advantage the Smith normal form of the matrix A over $R = \mathbb{K}[x]$, a PID. However, $R = \mathbb{K}[x_1, \ldots, x_n]$, $n \geq 2$, is not a PID, and the Smith normal form will not apply in this case. It would be interesting to discover a relationship on the minimal generators Syz($\mathbf{f}$) and Syz($\mathbf{g}$) over $R = \mathbb{K}[x_1, \ldots, x_n]$ where $n \geq 2$. To our knowledge, this is still an open question.

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