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# ASYMPTOTIC BEHAVIOR FOR STRONGLY DAMPED WAVE EQUATIONS ON $\mathbb{R}^3$ WITH MEMORY

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Abstract. We consider the following strongly damped wave equation on $\mathbb{R}^3$ with memory

$$u_{tt} - \alpha\Delta u_t - \beta\Delta u + \lambda u - \int_0^\infty \kappa'(s)\Delta u(t-s)ds + f(x,u) + g(x,u_t) = h,$$

where a quite general memory kernel and the nonlinearity $f$ exhibit a critical growth. Existence, uniqueness and continuous dependence results are provided as well as the existence of regular global and exponential attractors of finite fractal dimension.

## 1. Introduction

The main goal of this paper is to discuss the long-time behavior of the weak solutions for the following strongly damped wave equation with memory on $\mathbb{R}^3$,

$$\text{(1.1)} \qquad \begin{cases} u_{tt} - \alpha\Delta u_t - \beta\Delta u + \lambda u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds & \\ + f(x,u) + g(x,u_t) = h(x), & x \in \mathbb{R}^3, \quad t > 0, \\ u(x,t) = u_0(x,t), & x \in \mathbb{R}^3, \quad t \le 0, \\ \displaystyle\lim_{|x|\to\infty} u(x,t) = 0, & t \ge 0, \end{cases}$$

where $\alpha$ and $\beta$ are positive constants, $\mu$ is a summable positive function, and

$$\text{(1.2)} \qquad \eta^t = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad s \in \mathbb{R}^+.$$

Now, we define the strictly positive non-increasing function

$$\kappa(s) = \beta + \int_s^\infty \mu(r)dr, \quad s \in [0,+\infty).$$

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The above equation reads

$$u_{tt} - \alpha\Delta u_t - \kappa(0)\Delta u + \lambda u - \int_0^\infty \kappa'(s)\Delta u(t-s)ds + f(x,u) + g(x,u_t) = h,$$

that is, a semilinear wave equation with a strong damping and convolution terms.

In (1.1), with $\mu \equiv 0$, we obtain the usual strongly damped wave equation

$$u_{tt} - \alpha\Delta u_t - \beta\Delta u + f(\cdot,u) + g(\cdot,u_t) = h. \tag{1.3}$$

Well-posedness and long time behavior (in terms of attractors) of solutions for equation (1.3) on bounded domains have been investigated by many authors (see, e.g., [7, 8, 20–22] and references therein). Besides, equation (1.3) on unbounded domain (on $\mathbb{R}^N$) has been also studied in [5, 9] and some references therein.

The problem (1.1) in the case of bounded domains, without $g(\cdot,u_t)$ and when the memory kernel $\mu$ does not vanish (which reduces to a strongly damped wave equation with memory effects), has been studied in [2, 10, 14], for a subcritical nonlinearity and the following assumptions imposed on the memory kernel

$$\mu'(s) + \delta\mu(s) \le 0, \ \forall s > 0,$$

for some $\delta > 0$. Besides, in [11], under the much weaker condition on the memory kernel,

$$\mu(r+s) \le Ne^{-\delta r}\mu(s)$$

for some $N \ge 1$, $\delta > 0$, every $r \ge 0$, and almost every $s > 0$, Plinio, Pata and Zelik pointed out the existence of global attractors of optimal regularity for both critical and supercritical nonlinearities.

Recently, [19] also considered equation (1.1) in the case of time-dependent memory and without $g(\cdot,u_t)$. In this situation, the well-posedness, the existence and the regularity of the time-dependent global attractor have been proved.

However, to the best of our knowledge, up to now, although there have been several results on attractors for a strongly damped wave equation with memory, hardly any of the previous studies deal with the equations on unbounded domains and memory kernel effects. More specifically, we consider this equation in the case of containing critical nonlinear term which makes the model more complex.

The novelty of this paper is that it overcomes the essential difficulties: “both the Sobolev embedding on $\mathbb{R}^3$ and the critical growth of $f$ cause the lack of compactness, as well as the complexity of the model caused by the memory term” and establishes the well-posedness, the existence of the global and exponential attractors for the equation with memory and critical nonlinearity.

To study problem (1.1), we assume that the nonlinearity $f, g$, the external force $h$, and the memory term satisfy the following conditions:

(**H1**) The convolution (or memory) kernel $\kappa$ is a nonnegative summable function having the explicit form

$$\kappa(s) = \int_s^\infty \mu(r)dr,$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing (hence nonnegative) piecewise absolutely continuous in each interval $[0, T]$ with $T > 0$. In particular, $\mu$ is allowed to exhibit (infinitely many) jumps. Moreover, we require that

$$\kappa(s) \leq \theta\mu(s) \tag{1.4}$$

for some $\theta > 0$ and every $s > 0$. As shown in [13], this is completely equivalent to the requirement that

$$\mu(r + s) \leq Ne^{-\delta r}\mu(s) \tag{1.5}$$

for some $N \geq 1$, $\delta > 0$, every $r \geq 0$ and almost every $s > 0$. As a consequence,

$$\kappa(s) \leq Ce^{-\delta s}.$$

(**H2**) The nonlinearity $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, with $f(\cdot, 0) = 0$, satisfy for some $C > 0$ the growth bound

$$|f'_u(x, u)| \leq C\left(1 + |u|^4\right), \quad |f'_x(x, u)| \leq C|u|^5, \tag{1.6}$$

$$\liminf_{|u|\to\infty} \frac{F(x, u)}{u^2} \geq 0, \quad \text{uniformly as } x \in \mathbb{R}^3,$$

(1.7)
$$\liminf_{|u|\to\infty} \frac{uf(x, u) - d_1F(x, u)}{u^2} \geq 0, \quad \text{uniformly as } x \in \mathbb{R}^3 \text{ and for some } d_1 > 0,$$

where $F(x, u) = \int_0^u f(x, s)ds$ is a primitive of $f$.

(**H3**) Let $g \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ with $g(\cdot, 0) = 0$, satisfy for some $C \geq 0$ the growth bounds

$$|g'_m(x, m)| \leq C(1 + |m|^4), \tag{1.8}$$

along with the dissipation conditions

$$\liminf_{|m|\to\infty} g'_m(x, m) > -\lambda. \tag{1.9}$$

(**H4**) The external force $h$ is in $L^2(\mathbb{R}^3)$.

*Remark* 1.1. The main difficulties when we study the asymptotic behavior of the problem are the lack of compactness caused by the unbounded domain, and the fact that the nonlinearities $f$ and $g$ exhibit critical growth.

It is noticed that the condition in (**H1**) of the memory term is weaker than the usual condition in [3, 4] in the sense that $\mu$ can be weakly singular at the origin. For instance, we can take $\mu(s) = \frac{ce^{-as}}{s^{1-b}}$ with $c \geq 0$ and $a, b > 0$.

We infer from (**H2**) that for every $\nu_i > 0$, $i = 1, 2, 3$, there exists $C_{\nu_i} \geq 0$ such that

$$\langle f(x,u), u\rangle - d_1\langle F(x,u), 1\rangle + \nu_1\|u\|^2 + C_{\nu_1} > 0, \tag{1.10}$$

and

$$\langle F(x,u), 1\rangle \geq -\nu_2\|u\|^2 - C_{\nu_2}. \tag{1.11}$$

It is obvious that (1.9) implies that there are $\lambda > 0$ and $C_\lambda > 0$ such that

$$\langle g(x,r) - \lambda r, r\rangle \geq \lambda\|r\|^2 - C_\lambda. \tag{1.12}$$

## 2. Notations and preliminaries

In this section, we recall some notations about function spaces and preliminary results.

We introduce the Hilbert spaces $H_0 = L^2(\mathbb{R}^3)$, $H_1 = H^1(\mathbb{R}^3)$, and $H_2 = H^2(\mathbb{R}^3)$. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the $L^2(\mathbb{R}^3)$-inner product and $L^2(\mathbb{R}^3)$-norm, respectively. Besides, $\langle\cdot,\cdot\rangle_b$, $b = 0, 1, 2$ and $\|\cdot\|_b$ denote the $H_b$-inner product and $H_b$-norm, respectively.

In view of (1.5), let $L^2_\mu(\mathbb{R}^+; H_b)$ be the Hilbert space of functions $\varphi\colon \mathbb{R}^+ \to H_b$ endowed with the inner product

$$\langle\varphi_1, \varphi_2\rangle_{b,\mu} = \int_0^\infty \mu(s)\,\langle\varphi_1(s), \varphi_2(s)\rangle_b\,ds,$$

and let $\|\varphi\|_{b,\mu}$ denote the corresponding norm. We introduce product Hilbert spaces

$$\mathcal{H}_1 = H_1 \times H_0 \times L^2_\mu(\mathbb{R}^+; H_1), \quad \mathcal{H}_2 = H_2 \times H_1 \times L^2_\mu(\mathbb{R}^+; H_2).$$

We begin with rephrasing (1.1) as an autonomous dynamical system on a suitable phase space. To this aim, as in [6], a new variable that reflects the history of equation (1.1) is introduced, that is to be,

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad s \in \mathbb{R}^+.$$

Notice that $\eta^t$ satisfies the boundary condition $\eta^t(0) := \lim_{s\to 0}\eta^t(s) = 0$ and formally fulfills the equation

$$\eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,t), \tag{2.1}$$

with $\eta^0(s) = \eta_0(s)$.

Taking for simplicity $\alpha = \beta = 1$, the first equation of (1.1) can be transformed into the following system

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + \lambda u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(x,u) + g(x,u_t) = h(x), \\ \eta^t_t = -\eta^t_s + u_t. \end{cases} \tag{2.2}$$

The associated initial-boundary conditions are

$$(2.3)\qquad \begin{cases} u(x,0) = u_0(x), & x \in \mathbb{R}^3, \\ u_t(x,0) = v_0(x), & x \in \mathbb{R}^3, \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u_0(x,-s), & (x,s) \in \mathbb{R}^3 \times \mathbb{R}^+. \end{cases}$$

Denote

$$z(t) = (u(t), u_t(t), \eta^t), \quad z_0 = (u_0, v_0, \eta_0).$$

To estimate the nonlinear term, we use the decomposition of $g$ as follows.

**Lemma 2.1.** *For every fixed $\lambda > 0$, the decomposition*

$$g(x,r) = \phi(x,r) - \lambda r + \phi_c(x,r)$$

*holds for some $\phi, \phi_c \in C^1(\mathbb{R})$ with the following properties*:

1. *$\phi_c$ is compactly supported with $\phi_c(x,0) = 0$;*
2. *$\phi$ vanishes inside $[-1,1]$ and fulfills for some $c \geq 0$ and every $r \in \mathbb{R}$ the bounds*

$$0 \leq \phi'(x,r) \leq c|r|^4.$$

*Proof.* By (1.9), we can see that $g'(x,m) \geq -\lambda$, for all $|r| \geq k$ for $k \geq 1$ large enough. Choosing then any smooth function $\vartheta : \mathbb{R} \to [0,1]$ satisfying

$$r\vartheta'(x,r) \geq 0, \quad \vartheta = \begin{cases} 0 & \text{if } |r| \leq k, \\ 1 & \text{if } |r| \geq k+1. \end{cases}$$

It is immediate to check that

$$\phi(x,r) = \vartheta(x,r)[g(x,r) + \lambda r],$$
$$\phi_c(x,r) = [1 - \vartheta(x,r)][g(x,r) + \lambda r]$$

comply with the requirements. ☐

Due to Lemma 2.1, the function on $H_1$ given by

$$\Phi_0(w) = 2\int_{\mathbb{R}^3}\int_0^w \phi(x,r)drdx$$

fulfills for every $w \in H_1$ the inequality

$$(2.4)\qquad 0 \leq \Phi_0(w) \leq 2\langle \phi(x,w), w\rangle.$$

Besides, since

$$|\phi(x,w)|^{\frac{6}{5}} = |\phi(x,w)|^{\frac{1}{5}}|\phi(x,w)| \leq c|w||\phi(x,w)|,$$

we can get that for all $C > 0$ sufficiently large

$$(2.5)\qquad \|\phi(x,w)\|_{L^{\frac{6}{5}}} \leq C\langle \phi(x,w), w\rangle^{\frac{5}{6}}, \quad \forall w \in H_1.$$

We conclude the section by recalling a Gronwall-type lemma needed in the sequel.

**Lemma 2.2** (see [7]). *Given $k \geq 1$ and $C \geq 0$, let $\Lambda_\varepsilon : [0,\infty) \to [0,\infty)$ be a family of absolutely continuous functions satisfying for every $\varepsilon > 0$ small, the following inequalities hold*

$$\frac{1}{k}\Lambda_0 \leq \Lambda_\varepsilon \leq k\Lambda_0 \quad \textit{and} \quad \frac{d}{dt}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \leq C\varepsilon^6\Lambda_\varepsilon^3 + C.$$

*Then there are constants $\delta > 0$, $R \geq 0$, and an increasing function $\mathcal{Q} \geq 0$ such that*

$$\Lambda_0 \leq \mathcal{Q}(\Lambda_0(0))e^{-\delta t} + R.$$

The plan of the paper is as follows: In Section 3, we discuss the well-posedness of the Cauchy problem (1.1). In Section 4, we establish the existence of a global attractor and its regularity. Finally, in Section 5, we study the exponential attractor.

## 3. Existence and uniqueness of weak solutions

We first define the solution for (2.2) with initial-boundary condition (2.3) as follows.

**Definition 3.1.** A triplet form $z = (u, u_t, \eta^t)$ is called a *weak solution* of problem (2.2) for $T > 0$ with the initial datum $z(0) = z_0 \in \mathcal{H}_1$ if $z \in C([0,T];\mathcal{H}_1)$ and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} u_{tt}\varphi dxdt + \int_0^T \int_{\mathbb{R}^3} \nabla u_t \nabla\varphi dxdt + \int_0^T \int_{\mathbb{R}^3} \nabla u \nabla\varphi dxdt \\
&+ \int_0^T \int_0^\infty \mu(s)\langle\nabla\eta(s)\nabla\varphi\rangle dsdt + \lambda\int_0^T \int_{\mathbb{R}^3} u\varphi dxdt \\
&+ \int_0^T \int_{\mathbb{R}^3} f(x,u)\varphi dxdt + \int_0^T \int_{\mathbb{R}^3} g(x,u_t)\varphi dxdt \\
= &\int_0^T \int_{\mathbb{R}^3} h\varphi dxdt, \\
&\int_0^T \int_0^\infty \mu(s)(\nabla\eta_t^t, \nabla\xi^t(s))dsdr + \int_0^T \int_0^\infty \mu(s)(\nabla\eta_s^t, \nabla\xi^t(s))dsdr \\
= &\int_0^T \int_0^\infty \mu(s)(\nabla u_t, \nabla\xi^t(s))dsdr
\end{aligned}$$

for every test functions $\varphi \in H_1$ and $\xi^t \in L^2_\mu(\mathbb{R}^+, H_1)$, and a.e. $t \in [0,T]$.

The following result on the existence and uniqueness of weak solutions to the model (1.1)-(1.2) (also (2.2)) was proved by a Faedo-Garlerkin.

**Theorem 3.2.** *Assume that hypotheses* **(H1)-(H4)** *hold. Then for any $z_0 = (u_0, v_0, \eta_0) \in \mathcal{H}_1$, problem (2.2)-(2.3) has a unique weak solution $z = (u, u_t, \eta^t)$ on the interval $[0,T]$ satisfying*

$$z \in C([0,T];\mathcal{H}_1).$$

*Moreover, the weak solution depends continuously on the initial data on $\mathcal{H}_1$.*

*Proof. Existence.* For each integer $n \geq 1$, we denote by $P_n$ and $Q_n$ the projections on the subspaces

$$\operatorname{span}(\varphi_1, \dots, \varphi_n) \subset H_1, \quad \operatorname{span}(\zeta_1, \dots, \zeta_n) \subset L^2_\mu(\mathbb{R}^+, H_1),$$

respectively. Consider the approximate solution $z_n(t) = (u_n(t), \partial_t u_n(t), \eta^t_n(s))$ in the form

$$u_n(t) = \sum_{j=1}^{n} a_{nj}(t)\varphi_j, \quad \partial_t u_n(t) = \sum_{j=1}^{n} a'_{nj}(t)\varphi_j, \quad \eta^t_n(s) = \sum_{j=1}^{n} b_{nj}(t)\zeta_j(s),$$

where $a_{nk}(t)$ and $b_{nj}(t)$ are determined by the system of second order ordinary differential equations

$$\begin{aligned} &\left\langle \sum_{k=1}^{n} a''_{nk}(t)\varphi_k, \varphi_j \right\rangle + \left\langle \sum_{k=1}^{n} (\nu_k + \lambda) a'_{nk}(t)\varphi_k, \varphi_j \right\rangle \\ &+ \left\langle \sum_{k=1}^{n} \nu_k a_{nk}(t)\varphi_k, \varphi_j \right\rangle + \left\langle \sum_{k=1}^{n} b_{nk}(t)\zeta_k, \zeta_j \right\rangle_{1,\mu} \\ &+ \left\langle f\left(\sum_{k=1}^{n} a_{nk}(t)\varphi_k\right), \varphi_j \right\rangle + \left\langle g\left(\sum_{k=1}^{n} a'_{nk}(t)\varphi_k\right), \varphi_j \right\rangle \\ &= \langle h, \varphi_j \rangle, \quad j, k = 1, 2, \dots, n \end{aligned} \tag{3.1}$$

with the initial data

$$\left(u_n, \partial_t u_n, \eta^t_n\right)\big|_{t=0} = (P_n u_0, P_n v_0, Q_n \eta_0). \tag{3.2}$$

Since $\det(\langle \varphi_j, \varphi_k \rangle) \neq 0$ and the nonlinear functions $f$ and $g$ are continuous, by the Peano existence theorem, there exists at least one local solution to (3.1)–(3.2) in the interval $[0, T_n)$. Thus this allows constructing the approximate solution $z_n(t)$. Multiplying the equation $(3.1)_j$ by the function $a'_{nj}(t)$, summing from $j = 1$ to $n$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\partial_t u_n\|^2 + \|\nabla u_n\|^2 + \lambda\|u_n\|^2 + \langle F(x, u_n), 1\rangle\right) \\ &+ \|\nabla \partial_t u_n\|^2 + \int_0^\infty \mu(s)\langle \nabla \eta^t_n(s), \nabla \partial_t u_n\rangle ds + \langle g(x, \partial_t u_n), \partial_t u_n\rangle \\ &= \langle h, \partial_t u_n\rangle. \end{aligned} \tag{3.3}$$

Using (2.1) and then integrating by parts, we have

$$\begin{aligned} &\int_0^\infty \mu(s)\langle \nabla \eta^t_n(s), \partial_t \nabla u_n\rangle ds \\ = &\int_0^\infty \mu(s)\langle \nabla \eta^t_n(s), \nabla \partial_t \eta^t_n(s)\rangle ds + \int_0^\infty \mu(s)\langle \nabla \eta^t_n(s), \nabla \partial_s \eta^t_n(s)\rangle ds \end{aligned}$$

$$= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty \mu(s)\|\nabla\eta_n^t(s)\|^2 ds\right) - \int_0^\infty \mu'(s)\|\nabla\eta_n^t(s)\|^2 ds.$$

Besides, from conditions (**H3**), (1.12) and the Cauchy inequality, we can see that

$$-\int_0^\infty \mu'(s)\|\nabla\eta_n^t(s)\|^2 ds \geq 0, \tag{3.4}$$

$$\langle g(x,\partial_t u_n), \partial_t u_n\rangle \geq 2\lambda\|\partial_t u_n\|^2 - C_\lambda, \tag{3.5}$$

$$2\langle h, \partial_t u_n\rangle \leq \frac{1}{\lambda}\|h\|^2 + \lambda\|\partial_t u_n\|^2. \tag{3.6}$$

On the other hand, by multiplying the second equation of (2.2) by $\eta_n^t$ in $L^2_\mu(\mathbb{R}^+, H_0)$, we get

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds - 2\int_0^\infty \mu'(s)\|\eta_n^t\|^2 ds &= 2\int_0^\infty \mu(s)\langle\eta_n^t(s), \partial_t u_n\rangle ds \\ &\leq \frac{\kappa(0)}{\lambda}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds + \lambda\|\partial_t u_n\|^2. \end{aligned}\tag{3.7}$$

Therefore, summation of (3.3) and (3.7) and combining all the above estimates, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t u_n\|^2 + \|\nabla u_n\|^2 + \lambda\|u_n\|^2 + \|\eta_n^t\|^2_{1,\mu} + \langle F(x,u_n), 1\rangle\right) \\ &\quad + \|\nabla\partial_t u_n\|^2 + \lambda\|\partial_t u_n\|^2 ds \\ &\leq \frac{\kappa(0)}{\lambda}\int_0^\infty \mu(s)\|\eta_n^t\|^2 ds + C\|h\|^2 + C.\end{aligned}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}y(t) + \|\nabla\partial_t u_n\|^2 + \lambda\|\partial_t u_n\|^2 ds \leq Cy(t) + C(\|h\|^2 + 1),$$

where

$$y(t) = \|\partial_t u_n\|^2 + \|\nabla u_n\|^2 + \lambda\|u_n\|^2 + \|\eta_n^t\|^2_{1,\mu} + \langle F(x,u_n), 1\rangle,$$

and $\|z_n\|^2_{\mathcal{H}_1} \leq C_1 y(t)$.

Applying Gronwall's lemma, we deduce that

$$y(t) \leq e^{CT}y(0) + Ce^{CT}\left(\|h\|^2 + 1\right),$$

where $y(0) \leq C_2(\|z_0\|^2_{\mathcal{H}_1} + \|u_0\|_1^6)$. This inequality implies that

$$\{u_n\} \text{ is bounded in } L^\infty(0,T;H_1), \tag{3.8}$$

$$\{\eta_n^t\} \text{ is bounded in } L^\infty(0,T;L^2_\mu(\mathbb{R}^+,H_1)). \tag{3.9}$$

Integrating from 0 to $t$, we obtain

$$\{\partial_t u_n\} \text{ is bounded in } L^2(0,T;H_1). \tag{3.10}$$

Now, multiplying the equation (3.1) by the function $a''_{nj}(t)$, summing from $j = 1$ to $n$, we get

$$\begin{aligned} 2\|\partial_{tt}u_n\|^2 + \frac{d}{dt}Q(t) &= 2\langle f'_u(x,u_n)\partial_t u_n, \partial_t u_n\rangle + 2\|\nabla\partial_t u_n\|^2 + 2\lambda\|\partial_t u_n\|^2 \\ &\quad + 2\int_0^\infty \mu(s)\langle\nabla\partial_t\eta_n^t, \nabla\partial_t u_n\rangle ds, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} Q(t) = \|\nabla\partial_t u_n\|^2 &+ \langle\nabla u_n, \nabla\partial_t u_n\rangle + \lambda\langle u_n, \partial_t u_n\rangle \\ &+ 2\int_0^\infty \mu(s)\langle\nabla\partial_t\eta_n^t, \nabla\partial_t u_n\rangle ds + \langle f(x,u_n), \partial_t u_n\rangle \\ &+ \langle G(x,\partial_t u_n), 1\rangle - \langle h(x), \partial_t u_n\rangle. \end{aligned}$$

Using (3.8), (3.9), and (1.6), we obtain

$$\begin{aligned} \langle f'_u(x,u_n)\partial_t u_n, \partial_t u_n\rangle + 2\|\nabla u_t\|^2 &\le 2\|f'_u(x,u_n)\|_{L^{3/2}}\|\partial_t u_n\|_{L^6}^2 + 2\|\nabla\partial_t u_n\|^2 \\ &\le C(1+\|u_n\|_1^4)\|\partial_t u_n\|_1^2 \le C\|\partial_t u_n\|_1^2, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &2\int_0^\infty \mu(s)\langle\nabla\eta_t^t(s), \nabla\partial_t u_n\rangle ds \\ &= 2\int_0^\infty \mu(s)\langle\nabla\partial_s\eta_n^t - \nabla\partial_t u_n, \nabla\partial_t u_n\rangle ds \\ &\le 2\int_0^\infty \mu(s)\|\nabla\partial_s\eta_n^t(s)\|\|\nabla\partial_t u_n\| ds + 2\kappa(0)\|\nabla\partial_t u_n\|^2 \\ &\le 2\int_0^\infty \mu(s)\|\nabla\partial_s\eta_n^t(s)\|^2 ds + C\|\nabla\partial_t u_n\|^2 \\ &\le -\int_0^\infty \mu'(s)\|\nabla\eta_n^t(s)\|^2 ds + C\|\partial_t u_n\|_1^2. \end{aligned} \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), then integrating over $(0,T)$, we get

$$\int_0^T \|\partial_{tt}u_n(r)\|^2 dr + Q(T) \le Q(0) + \int_0^T \|\partial_t u_n(r)\|_1^2 dr,$$

where $Q(0) \le C(\|z_0\|_{\mathcal{H}_1})$. This inequality implies that

$$\{\partial_{tt}u_n\} \text{ is bounded in } L^2(0,T;H_0). \tag{3.14}$$

Combining (3.8), (3.9), (3.10) and (3.14), we deduce that there exists a subsequence of $\{u_n\}$ and $\{\partial_t u_n\}$, $\{\eta_n^t\}$ (still denoted by $\{u_n\}$, $\{\partial_t u_n\}$ and $\{\eta_n^t\}$) such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly-star in } L^\infty(0,T;H_1), \\ &\partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;H_1), \\ &\partial_{tt}u_n \rightharpoonup \partial_{tt}u \text{ weakly in } L^2(0,T;H_0), \end{aligned}$$

$$\eta_n^t \rightharpoonup \eta^t \text{ weakly-star in } L^\infty(0,T;L^2_\mu(\mathbb{R}^+,H^1_0(\Omega))), \tag{3.15}$$

and

$$\begin{aligned}
&\Delta u_n \rightharpoonup \Delta u \text{ weakly in } L^2(0,T;H^{-1}(\mathbb{R}^3)),\\
&\Delta\partial_t u_n \rightharpoonup \Delta\partial_t u \text{ weakly in } L^2(0,T;H^{-1}(\mathbb{R}^3)),\\
&\Delta\eta_n^t \rightharpoonup \Delta\eta^t \text{ weakly in } L^2(0,T;L^2_{\mu_t}(\mathbb{R}^+,H^{-1}(\mathbb{R}^3))).
\end{aligned} \tag{3.16}$$

Using (**H1**), we have

$$\|f(x,u_n)\|^{6/5}_{L^{6/5}} \le C\left(\|u_n\| + \|u_n\|^5_{L^6}\right) \le C\left(1+\|u_n\|^5_1\right),$$

and

$$\|g(x,\partial_t u_n)\|^{6/5}_{L^{6/5}} \le C\left(\|\partial_t u_n\| + \|\partial_t u_n\|^5_{L^6}\right) \le C\left(1+\|\partial_t u_n\|^5_1\right).$$

Using (3.8), (3.9), and (3.10) once again, we have

$$\begin{aligned}
&\{f(x,u_n)\} \text{ is bounded in } L^{6/5}(0,T;L^{6/5}(\mathbb{R}^3)),\\
&\{g(x,\partial_t u_n)\} \text{ is bounded in } L^{6/5}(0,T;L^{6/5}(\mathbb{R}^3)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&f(x,u_n) \rightharpoonup \chi_1 \text{ weakly in } L^{6/5}(0,T;L^{6/5}(\mathbb{R}^3)),\\
&g(x,\partial_t u_n) \rightharpoonup \chi_2 \text{ weakly in } L^{6/5}(0,T;L^{6/5}(\mathbb{R}^3)).
\end{aligned} \tag{3.17}$$

In addition, for each $m \ge 1$, we denote $B_m = \{x \in \mathbb{R}^N : |x| \le m\}$. Let $\phi \in C^1([0,+\infty))$ be a function such that

$$0 \le \phi \le 1, \quad \phi|_{[0,1]} = 1, \quad \phi(s) = 0 \quad \text{for all } s \ge 2.$$

For each $n$ and $m$ we define

$$v_{n,m}(x,t) = \phi\left(\frac{|x|^2}{m^2}\right)u_n(x,t), \quad \partial_t v_{n,m}(x,t) = \phi\left(\frac{|x|^2}{m^2}\right)\partial_t u_n(x,t).$$

From (3.8), (3.9), and (3.10), for all $m \ge 1$, we have the sequences $\{v_{n,m}\}_{n\ge1}$ and $\{\partial_t v_{n,m}\}_{n\ge1}$ are bounded $L^2\left(0,T;H^1_0(B_{2m})\right)$. Since $B_{2m}$ is a bounded set, then $H^1_0(B_{2m}) \hookrightarrow L^2(B_{2m})$ compactly. Then, by in [17, Theorem 13.3 and Remark 13.1] we can deduce that

$$\{\partial_t v_{n,m}\} \text{ and } \{v_{n,m}\} \text{ are precompact in } L^2(0,T;L^2(B_{2m})),$$

and thus

$$\{\partial_t u_n|_{B_m}\} \text{ and } \{u_n|_{B_m}\} \text{ are precompact in } L^2\left(0,T;L^2(B_m)\right).$$

By a diagonal procedure, using (3.15), we deduce that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$(u_n,\partial_t u_n) \to (u,u_t) \text{ a.e. in } B_m \times (0,T) \text{ as } n \to +\infty \quad \text{for all } m \ge 1.$$

Then, since $f(\cdot,\cdot)$ are continuous,

$$f(x,u_n) \to f(x,u) \text{ and } g(x,\partial_t u_n) \to g(x,u_t) \quad \text{a.e. in } B_m \times (0,T),$$

and since $\{f(x,u_n)\}$ and $\{g(x,\partial_t u_n)\}$ are bounded in $L^{6/5}(0,T;L^{6/5}(B_m))$, by [15, Chapter 1, Lemma 3.1], we get

$$f(\cdot,u_n) \rightharpoonup f(\cdot,u) \text{ and } g(\cdot,\partial_t u_n) \rightharpoonup g(\cdot,u_t) \text{ in } L^{6/5}(0,T;L^{6/5}(B_m)).$$

From (3.17),

$$f(\cdot,u_n) \rightharpoonup \chi_1|_{B_m\times(0,T)} \text{ and } g(\cdot,\partial_t u_n) \rightharpoonup \chi_2|_{B_m\times(0,T)} \text{ in } L^{6/5}(0,T;L^{6/5}(B_m)).$$

Therefore,

$$\chi_1 = f(x,u), \quad \chi_2 = g(x,u_t) \quad \text{a.e. in } B_m\times(0,T) \quad \text{for all } m\geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^3$, we obtain

$$\chi_1 = f(x,u) \quad \chi_2 = g(x,u_t) \quad \text{a.e. in } \mathbb{R}^3\times(0,T). \tag{3.18}$$

Now combining (3.15), (3.16), (3.17), and (3.19), we see that $z_n = (u_n, \partial_t u_n, \eta_n^t)$ satisfies

$$u_{tt} - \Delta u_t - \Delta u + \lambda u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(x,u) + g(x,u_t) = h,$$

in $H^{-1}(\mathbb{R}^3) + L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^3)$ for a.e. $t\in[0,T]$. By standard arguments, we can check that $z$ satisfies the initial condition $z(0)=z_0$, and this implies that $z$ is a weak solution of problem (2.2).

*Uniqueness and continuous dependence.* We assume that $z_1$ and $z_2$ are two solutions subject to initial data $z_1(0)$ and $z_2(0)$, respectively. Denote $(w,\bar\eta^t) = (u_1-u_2, \eta_1^t-\eta_2^t)$, we have

$$\begin{aligned} &w_{tt} - \Delta w_t - \Delta w + \lambda w - \int_0^\infty \mu(s)\Delta\bar\eta^t(s)ds \\ &\qquad + f(x,u_1) - f(x,u_2) + g(x,\partial_t u_1) - g(x,\partial_t u_2) = 0. \end{aligned} \tag{3.19}$$

Taking the inner product of (3.19) in $H_0$ with $w_t$, then using assumptions (2.1) and (1.9), we see that

$$\begin{aligned} &\frac{d}{dt}\left(\|w_t\|^2 + \lambda\|w\|^2 + \|\nabla w\|^2 + \int_0^\infty \mu(s)\|\nabla\bar\eta^t(s)\|^2 ds\right) \\ &+ 2\|\nabla w_t\|^2 + \int_0^\infty \mu'(s)\|\nabla\bar\eta^t(s)\|^2 ds \\ &\leq 2\lambda\|w_t\|^2 + 2C\left(1+\|u_1\|^4_{L^6} + \|u_2\|^4_{L^6}\right)\|w\|_{L^6}\|w_t\|_{L^6}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\left(\|w_t\|^2 + \lambda\|w\|^2 + \|\nabla w\|^2 + \int_0^\infty \mu(s)\|\nabla\bar\eta^t(s)\|^2 ds\right) \\ &\leq 2(1+\lambda)\|w_t\|^2 + C\|w\|_1^2, \end{aligned} \tag{3.20}$$

where

$$2C\left(1+\|u_1\|^4_{L^6} + \|u_2\|^4_{L^6}\right)\|w\|_{L^6}\|w_t\|_{L^6} \leq C\|w\|_1^2 + \|w_t\|^2 + \|\nabla w_t\|^2,$$

and

$$-\int_0^\infty \mu'(s)\|\nabla\bar{\eta}^t(s)\|^2 ds \geq 0.$$

On the other hand, as in (3.7), multiplying the second equation of (2.2) by $\bar{\eta}^t$ in $L^2_\mu(\mathbb{R}^+, L^2(\mathbb{R}^3))$, we get

$$\frac{d}{dt}\int_0^\infty \mu(s)\|\bar{\eta}^t\|^2 ds - 2\int_0^\infty \mu'(s)\|\bar{\eta}^t\|^2 ds \leq \frac{\kappa(0)}{\lambda}\int_0^\infty \mu(s)\|\bar{\eta}^t\|^2 ds + \lambda\|\partial_t w\|^2. \tag{3.21}$$

Summation of (3.20) and (3.21), we get

$$\begin{aligned}&\frac{d}{dt}(\|w_t\|^2 + \lambda\|w\|^2 + \|\nabla w\|^2 + \|\bar{\eta}^t(s)\|^2_{1,\mu})\\ &\leq C(\|w_t\|^2 + \lambda\|w\|^2 + \|\nabla w\|^2 + \|\bar{\eta}^t(s)\|^2_{1,\mu}).\end{aligned}$$

By the Gronwall inequality, we obtain

$$\begin{aligned}&\|w_t\|^2 + \lambda\|w\|^2 + \|\nabla w\|^2 + \|\bar{\eta}^t(s)\|^2_{1,\mu}\\ &\leq e^{CT}(\|w_t(0)\|^2 + \lambda\|w(0)\|^2 + \|\nabla w(0)\|^2 + \|\bar{\eta}^0(s)\|^2_{1,\mu}).\end{aligned} \tag{3.22}$$

This proves the uniqueness (when $z_1(0) = z_2(0)$) and the continuous dependence on the initial data of the weak solution. This completes the proof. □

## 4. The global attractor and its regularity

Theorem 3.2 allows us to define a continuous semigroup $S(t) : \mathcal{H}_1 \to \mathcal{H}_1$ associated to problem (2.2) by the formula

$$S(t)z_0 := z(t),$$

where $z(\cdot)$ is the unique global weak solution of (2.2) with the initial datum $z_0 \in \mathcal{H}_1$. The aim of this section is to prove the existence of a global attractor for $S(t)$ on $\mathcal{H}_1$, namely, to prove the following theorem.

**Theorem 4.1.** *Assume that* **(H1)**–**(H4)** *hold. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *possesses a compact global attractor in* $\mathcal{H}_1$.

To prove this theorem, by the classical abstract results on the existence of global attractors (see e.g. [18, Theorem 1.1]), we need to show that the semigroup $S(t)$ has a bounded absorbing set $B_0$ in $\mathcal{H}_1$ and $S(t)$ is asymptotically compact in $\mathcal{H}_1$.

### 4.1. Existence of an absorbing set

**Lemma 4.2.** *The following inequality holds*

$$\frac{d}{dt}\Psi(t) + \|\eta^t\|^2_{1,\mu} = 2\int_0^\infty \mu(s)\langle \eta^t(s), u(t)\rangle_1 ds,$$

*where* $\Psi(t) = \int_0^\infty \kappa(s)\|\eta^t(s) - u(t)\|_1^2 ds > 0$. *Moreover,*

$$\Psi(t) \leq C_0\left(\|\eta^t\|^2_{1,\mu} + \|u(t)\|_1^2\right).$$

*Proof.* By direct calculations and using the equations $\partial_t\eta^t - u_t = -\partial_s\eta^t$ and $\kappa'(s) = -\mu(s)$, we have the equalities

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= \frac{d}{dt}\left(\int_0^\infty \kappa(s)\|\eta^t(s)-u(t)\|_1^2 ds\right)\\ &= 2\int_0^\infty \kappa(s)\langle\partial_t\eta^t(s)-\partial_t u(t),\, \eta^t(s)-u(t)\rangle_1 ds\\ &= -2\int_0^\infty \kappa(s)\langle\partial_s\eta^t(s),\, \eta^t(s)-u(t)\rangle_1 ds\\ &= -2\int_0^\infty \kappa(s)\langle\partial_s\eta^t(s),\, \eta^t(s)\rangle_1 ds + 2\int_0^\infty \kappa(s)\langle\partial_s\eta^t(s),\, u(t)\rangle_1 ds\\ &= -\int_0^\infty \kappa(s)\frac{d}{ds}\|\eta^t\|_1^2 ds + 2\int_0^\infty \kappa(s)\frac{d}{ds}\langle\eta^t(s),\, u(t)\rangle_1 ds\\ &= \int_0^\infty \kappa'(s)\|\eta^t\|_1^2 ds - 2\int_0^\infty \kappa'(s)\langle\eta^t(s),\, u(t)\rangle_1 ds\\ &= -\|\eta^t\|_{1,\mu}^2 + 2\int_0^\infty \mu(s)\langle\eta^t(s),\, u(t)\rangle_1 ds.\end{aligned}$$

On the other hand, from (1.4), we learn that

$$\Psi(t) \le C_0\left(\|\eta^t\|_{1,\mu}^2 + \|u(t)\|_1^2\right).$$

The proof is complete. □

**Lemma 4.3.** *Let the hypotheses* **(H1)**–**(H4)** *hold. Then there exists a bounded absorbing set in* $\mathcal{H}_1$ *for the semigroup* $S(t)$.

$$\|z(t)\|_{\mathcal{H}_1}^2 \le \mathcal{Q}(\|z_0\|_{\mathcal{H}_1})e^{-\gamma t} + R_1 \tag{4.1}$$

*for every* $z_0 \in \mathcal{H}_1$. *Moreover,*

$$\sup_{z\in B}\int_t^T\left(\|u_t(r)\|_1^2 + \langle\phi(x,u_t),u_t\rangle - \int_0^\infty \mu'(s)\|\eta^r\|_1^2 ds\right)dr \le C + C(T-t) \tag{4.2}$$

*for all* $T > t \ge 0$.

*Proof.* For $a \in [0,1)$ to be fixed later, multiplying the first equation of (2.2) by $u_t(t) + au(t)$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Bigg(\|u_t\|^2 + \lambda(1-a)\|u\|^2 + (1+a)\|\nabla u\|^2 + \int_0^\infty \mu(s)\|\nabla\eta^t\|^2 ds\\ &\qquad + \langle F(x,u),1\rangle + 2a\langle u_t,u\rangle\Bigg)\\ &+ a\lambda\|u\|^2 + a\|\nabla u\|^2 - (\lambda+a)\|u_t\|^2 + \|\nabla u_t\|^2 + \langle\phi(x,u_t),u_t\rangle\\ &- \int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds + a\langle f(x,u),u\rangle\end{aligned}$$

$$(4.3)\qquad = -a\langle\phi(u_t),u\rangle - a\int_0^\infty \mu(s)\langle\nabla\eta^t(s),\nabla u\rangle ds + \langle q, u_t+au\rangle,$$

where $g(x,u_t)=\phi(x,u_t)-\lambda u_t+\phi_c(x,u_t),\ q=h-\phi_c(\cdot,u_t)$, and

$$\int_0^\infty \mu(s)\langle\nabla\eta^t,\nabla u_t\rangle ds = \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty \mu(s)\|\nabla\eta^t\|^2 ds\right) - \int_0^\infty \mu'(s)\|\nabla\eta^t\|^2 ds.$$

Using (1.10), we have and

$$a\langle f(x,u),u\rangle \geq d_1 a\langle F(x,u),1\rangle - \nu_1 a\|u\|^2 - C_{\nu_1}.$$

Besides, using Lemma 2.1 and Young inequality, we get

$$\begin{aligned}2\langle q,u_t+au\rangle &\leq 2\left(\|h\|+\|\phi_c(\cdot,u_t)\|\right)\left(a\|u\|+\|u_t\|\right)\\ &\leq \nu_1\left(a\|u\|^2+\|u_t\|^2\right)+C_0,\end{aligned}$$

where $q\in L^\infty(\mathbb{R}^+;H_0)$.

Multiplying the second equation of (2.2) by $j\eta^t$ in $L^2_\mu(\mathbb{R}^+,L^2(\mathbb{R}^3))$, we get

$$\begin{aligned}\frac{d}{dt}j\int_0^\infty \mu(s)\|\eta^t\|^2 ds - 2j\int_0^\infty \mu'(s)\|\eta^t\|^2 ds &= 2j\int_0^\infty \mu(s)\langle\eta^t(s),u_t\rangle ds\\ &\leq jk\|u_t\|^2 + j\int_0^\infty \mu(s)\|\eta^t\|^2 ds. \qquad (4.4)\end{aligned}$$

Putting

$$\begin{aligned}E_{ja}(t) &= \|u_t\|^2+\lambda(1-a)\|u\|^2+(1+a)\|\nabla u\|^2\\ &\quad+\int_0^\infty \mu(s)(j\|\eta^t\|^2+\|\nabla\eta^t\|^2)ds + 2a\langle u_t,u\rangle + \langle F(x,u),1\rangle + C_{\nu_2}.\end{aligned}$$

From (1.11) and the estimation

$$2a\langle u_t,u\rangle \leq \lambda a\|u\|^2 + \frac{a}{\lambda}\|u_t\|^2,$$

there exist positive constants $\delta_0$ small enough such that

$$E_{ja}(t) \geq \delta_0\left(\|u_t\|^2+\|u\|_1^2+\int_0^\infty \mu(s)\left(j\|\eta^t\|^2+\|\nabla\eta^t\|^2\right)ds\right)$$

and

$$(4.5)\qquad E_{j0}(t) \leq 2E_{ja}(t) \leq 4E_{j0}(t).$$

Summation of (4.3) and (4.4) and plugging all the above inequalities into (4.3), it follows that

$$\begin{aligned}&\frac{d}{dt}E_{ja} + 2a(\lambda-\nu_1)\|u\|^2 + 2a\|\nabla u\|^2 + (\lambda-\nu_1)\|u_t\|^2 + 2\|\nabla u_t\|^2\\ &+ 2d_1a\langle F(u),1\rangle + \frac{1}{2}\langle\phi(x,u_t),u_t\rangle - 2\int_0^\infty \mu'(s)\left(j\|\eta^t\|^2+\|\nabla\eta^t\|^2\right)ds\\ &+ 2a\int_0^\infty \mu(s)\langle\nabla\eta^t(s),\nabla u\rangle ds\end{aligned}$$

$$\leq -2a\langle\phi(x,u_t),u\rangle + jk\|u_t\|^2 + j\int_0^\infty \mu(s)\|\eta^t\|^2 ds + K,$$

where

$$K = \frac{C_\lambda}{2} + C_0 + 2d_1 a C_{\nu_2}, \quad \langle\phi(x,u_t),u_t\rangle \geq 2\lambda\|u_t\|^2 - 2C_\lambda.$$

Now we define the functional

$$\Lambda_{ja}(t) = E_{ja}(t) + a\Psi_j(t).$$

Using (4.5), (2.5) and Young inequality, we have

$$E_{j0}(t) \leq \Lambda_{j0}(t) \leq 2\Lambda_{ja}(t) \leq 4\Lambda_{j0}(t),$$

and

$$\begin{aligned}-2a\langle\phi(x,u_t),u\rangle \leq 2a\|\phi(x,u_t)\|_{L^{6/5}}\|u\|_{L^6} &\leq Ca\langle\phi(x,u_t),u_t\rangle^{5/6}\|u\|_1 \\ &\leq \frac{1}{4}\langle\phi(x,u_t),u_t\rangle + Ca^6\Lambda_j^3.\end{aligned}$$

From Lemma 4.2, by choosing $\gamma > 0$ which is small enough, we obtain

$$\begin{aligned}&\frac{d}{dt}\Lambda_{ja} + 2a\gamma\Lambda_{ja} + \frac{1}{2}\|u_t\|_1^2 + \frac{1}{4}\langle\phi(x,u_t),u_t\rangle \\ &-\int_0^\infty \mu'(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right)ds \\ \leq\ & Ca^6\Lambda_{ja}^3 - 2aj\int_0^\infty \langle\eta^t(s),u\rangle ds + jk\|u_t\|^2 + j\int_0^\infty \mu(s)\|\eta^t\|^2 ds + C.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\Lambda_{ja} + 2a\gamma\Lambda_{ja} + \frac{1}{2}\|u_t\|_1^2 - \int_0^\infty \mu'(s)\left(j\|\eta^t\|^2 + \|\nabla\eta^t\|^2\right)ds \\ &+ \frac{1}{4}\langle\phi(x,u_t),u_t\rangle \\ &\leq Ca^6\Lambda_{ja}^3 + jk\left(\|u_t\|^2 + a\|u\|^2\right) + j(a+1)\int_0^\infty \mu(s)\|\eta^t\|^2 ds + C \\ &\leq Ca^6\Lambda_{ja}^3 + jk\Lambda_{j0} + C, \end{aligned} \tag{4.6}$$

where

$$-2aj\int_0^\infty \langle\eta^t(s),u\rangle ds \leq jak\|u\|^2 + ja\int_0^\infty \mu(s)\|\eta^t\|^2 ds.$$

From (4.6), let $j = 0$ and then applying Lemma 2.2, there are constants $\gamma > 0$, $R \geq 0$, and an increasing function $\mathcal{Q} \geq 0$ such that

$$\begin{aligned}\Lambda_{00}(t) &\leq \mathcal{Q}(\Lambda_{00}(0))e^{-\gamma t} + R \\ &\leq C\left(\|z_0\|_{\mathcal{H}_1}^2 + 2d_2\|u_0\|_{L^6}^6\right)e^{-\gamma t} + R \\ &\leq \rho_0.\end{aligned} \tag{4.7}$$

Besides, considering (4.6) for $j \neq 0$, then using (4.7) and Lemma 2.2, we obtain

$$\begin{aligned}\Lambda_{10}(t) &\leq \mathcal{Q}(\Lambda_{10}(0))e^{-\gamma t} + R_1 \\ &\leq \left(\|z_0\|^2_{\mathcal{H}_1} + 2d_2\|u_0\|^6_{L^6}\right)e^{-\gamma t} + R_1.\end{aligned}$$

Hence there exists $\rho_1 > 0$ such that

$$\|z(t)\|^2_{\mathcal{H}_1} \leq \rho_1 \tag{4.8}$$

for all $z_0 \in B$ and for all $t \geq T_B$, where $B$ is an arbitrary bounded subset of $\mathcal{H}_1$. Finally, integrating (4.6) on $(t, T)$ and using (4.8), the proof is completed. ☐

To prove the asymptotic compactness in the next section, we must use some of the following lemmas:

**Lemma 4.4** (see [7, Lemma 6.2])**.** *If $B_0$ is an invariant absorbing set, then*

$$B_1 = S(1)B_0 \subset B_0$$

*remains invariant and absorbing, and any (bounded) function $\Lambda : B_1 \to \mathbb{R}$ satisfies*

$$\sup_{t\geq 0}\sup_{z_0\in B_1} \Lambda(S(t)z_0) = \sup_{t\geq 0}\sup_{z_0\in B_0} \Lambda(S(t+1)z_0) \leq \sup_{z_0\in B_0} \Lambda(S(1)z_0).$$

**Lemma 4.5.** *There exist an invariant absorbing set $B_1$ and a constant $C = C(B_1) \geq 0$ such that, for all initial data in $B_1$,*

$$\sup_{t\geq 0}\|u_t(t)\|^2_1 \leq C, \qquad \int_0^1 \|u_{tt}(t)\|^2 \leq C.$$

*Proof.* Now, we consider the initial data $z_0 \in B_0$. Taking the inner product in $H_0$ of (2.2) and $u_{tt}$, and adding to both sides the term $2\langle u, u_t\rangle$, we get

$$\begin{aligned}&\frac{d}{dt}\Bigg(\|u_t\|^2 + \|\nabla u_t\|^2 + 2\Phi_0(u_t) + 2\langle f(x,u), u_t\rangle + 2\langle \nabla u, \nabla u_t\rangle \\ &\qquad + 2\int_0^\infty \mu(s)\langle \nabla\eta^t(s), \nabla u_t\rangle ds\Bigg) + 2\|u_{tt}\|^2 \\ &= 2\langle f'_u(x,u)u_t, u_t\rangle + 2\|\nabla u_t\|^2 \\ &\qquad + 2\int_0^\infty \mu(s)\langle \nabla\eta^t_t(s), \nabla u_t\rangle ds + 2\langle u, u_t\rangle + 2\langle q, u_{tt}\rangle,\end{aligned} \tag{4.9}$$

where $q = h + \lambda u_t + \phi_c(\cdot, u_t)$ and $\Phi_0(u_t)$ is defined as in (2.4).

Using (4.8), (1.6), and Lemma 2.1, we obtain

$$\begin{aligned}\langle f'_u(x,u)u_t, u_t\rangle + 2\|\nabla u_t\|^2 &\leq 2\|f'_u(x,u)\|_{L^{3/2}}\|u_t\|^2_{L^6} + 2\|\nabla u_t\|^2 \\ &\leq C(1 + \|u\|^2_1)\|u_t\|^2_1 \leq C\|u_t\|^2_1,\end{aligned}$$

$$2\langle u, u_t\rangle + 2\langle q, u_{tt}\rangle \leq 2\|u\|\|u_t\| + 2\|q\|\|u_{tt}\| \leq \|u_{tt}\|^2 + C,$$

and

$$
\begin{aligned}
2\int_0^\infty \mu(s)\langle\nabla\eta_t^t(s),\nabla u_t\rangle ds &= 2\int_0^\infty \mu(s)\langle\nabla\eta_s^t(s)-\nabla u_t,\nabla u_t\rangle ds\\
&\leq 2\int_0^\infty \mu(s)\|\nabla\eta_s^t(s)\|\|\nabla u_t\|ds + 2\kappa(0)\|\nabla u_t\|^2\\
&\leq \int_0^\infty \mu(s)\|\nabla\eta_s^t(s)\|^2 ds + C\|\nabla u_t\|^2\\
&= -\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds + C\|u_t\|_1^2. \qquad (4.10)
\end{aligned}
$$

Now we define the functional

$$
\begin{aligned}
\Lambda = \Lambda(S(t)z_0) = \|u_t\|^2 + \|\nabla u_t\|^2 + 2\Phi_0(u_t) + 2\langle f(x,u),u_t\rangle + 2\langle\nabla u,\nabla u_t\rangle\\
+ 2\int_0^\infty \mu(s)\langle\nabla\eta^t(s),\nabla u_t\rangle ds + K,
\end{aligned}
$$

fulfils for sufficiently large $K = K(B_0, C_{\nu_1}) > 0$ so that

$$
\|u_t\|_1^2 \leq 2\Lambda \leq C(1 + \|u_t\|_1^2 + 2\langle\phi(u_t),u_t\rangle).
$$

In particular, we deduce from (4.2) that

$$
\int_0^1 \Lambda(S(t)z_0)dt + \int_0^1\int_0^\infty -\mu'(s)\|\nabla\eta^t(s)\|^2 ds dt \leq C.
$$

Combining (4.9)-(4.10), we obtain

$$
\frac{d}{dt}\Lambda + \|u_{tt}\|^2 \leq -\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds + C\|u_t\|_1^2 + K.
$$

Thus,

$$
\frac{d}{dt}\Lambda + \|u_{tt}\|^2 \leq C\Lambda - \int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds + K. \qquad (4.11)
$$

Therefore, multiplying at every fixed time $t \in [0,1]$ both terms of (4.11), we get

$$
\frac{d}{dt}[t\Lambda] \leq C\Lambda - \int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds + K,
$$

and subsequent integration on $[0,1]$ gives

$$
\Lambda(S(1)z_0) \leq C\int_0^1 \Lambda(S(t)z_0)dt + C \leq C.
$$

Hence, we can choose

$$
B_1 = S(1)B_0 \subset B_0
$$

and applying Lemma 4.4, we have

$$
\sup_{t\geq 0}\sup_{z_0\in B_1}\Lambda(S(t)z_0) \leq \sup_{z_0\in B_0}\Lambda(S(1)z_0) \leq C,
$$

establishing the desired bound

$$\sup_{t\geq 0}\sup_{z_0\in B_1}\|u_t(t)\|_1\leq C.$$

On the other hand, for initial data $z_0\in B_1$, the inequality (4.11) improves to

$$\frac{d}{dt}\Lambda+\|u_{tt}\|^2\leq -\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2ds+C.$$

Integrating the above inequality over $[0,1]$, we provide the remaining integral control. ☐

**Lemma 4.6.** *There exists an invariant absorbing set* $B_0$ *satisfying*

$$\sup_{t\geq 0}\sup_{z_0\in B_0}\left(\|u_t(t)\|_1^2+\|u_{tt}\|^2+\int_t^{t+1}\|u_{tt}(r)\|_1^2dr\right)<\infty.$$

*Proof.* Taking initial data $z_0\in B_1$, with $B_1$ is the invariant absorbing set of the previous lemma.

Differentiating (2.2) with respect to time and then multiplying both terms by $2u_{tt}$, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|u_{tt}\|^2+\|\nabla u_t\|^2+\lambda\|u_t\|^2\right)+2\|\nabla u_{tt}\|^2+2\langle\phi'(x,u_t)u_{tt},u_{tt}\rangle\\ =&-2\int_0^\infty\mu(s)\langle\nabla\eta_t^t(s),\nabla u_{tt}\rangle ds-2\langle f_u'(x,u)u_t,u_{tt}\rangle+2\langle\lambda u_{tt}-\phi_c'(x,u_t)u_{tt},u_{tt}\rangle.\end{aligned}$$

Since $\phi'(x,u_t)\geq 0$,

$$2\langle\phi'(x,u_t)u_{tt},u_{tt}\rangle\geq 0.$$

Using Lemma 4.5 and (4.1), we can see that

$$\begin{aligned}-2\langle f_u'(x,u)u_t,u_{tt}\rangle&\leq\|f_u'(x,u)\|_{L^{3/2}}\|u_t\|_{L^6}\|u_{tt}\|_{L^6}\\&\leq\|u_{tt}\|_1^2+C.\end{aligned}$$

Besides,

$$2\langle\lambda u_{tt}-\phi_c'(x,u_t)u_{tt},u_{tt}\rangle\leq C\|u_{tt}\|^2+C,$$

and

$$\begin{aligned}&-2\int_0^\infty\mu(s)\langle\nabla\eta_t^t(s),\nabla u_{tt}\rangle ds\\ =&-2\int_0^\infty\mu(s)\langle\nabla u_t-\nabla\eta_s^t(s),\nabla u_{tt}\rangle ds\\ \leq&\frac{d}{dt}(-2\kappa(0)\|\nabla u_t\|^2)+2\int_0^\infty\mu(s)\|\nabla\eta_s^t(s)\|\|\nabla u_{tt}\|ds\\ \leq&-2\kappa(0)\frac{d}{dt}\|\nabla u_t\|^2-\int_0^\infty\mu'(s)\|\nabla\eta^t(s)\|^2ds+\|\nabla u_{tt}\|^2.\end{aligned}$$

Summarizing, we arrive at

$$\frac{d}{dt}\Lambda+(\|\nabla u_{tt}\|^2+\|u_{tt}\|^2)\leq C\Lambda-\int_0^\infty\mu'(s)\|\nabla\eta^t(s)\|^2ds+C,\tag{4.12}$$

where

$$\Lambda = \|u_{tt}\|^2 + (1 + 2\kappa(0))\|\nabla u_t\|^2 + \lambda\|u_t\|^2.$$

Using Lemma 4.5, we get

$$\int_0^1 \Lambda(S(t)z_0)dt \le C.$$

Therefore, multiplying by $t$ and integrating on $[0, 1]$, we obtain

$$\Lambda(S(1)z_0) \le C.$$

Putting

$$B = S(1)B_1 \subset B_1,$$

we deduce from Lemma 4.4 that

$$\sup_{t\ge 0}\sup_{z\in B}(\|u_t(t)\|_1^2 + \|u_{tt}\|^2) = \sup_{t\ge 0}\sup_{z\in B}\Lambda(S(t)z_0) \le C.$$

Now, choosing initial data $z_0 \in B$, we can rewrite (4.12) as follow:

$$\frac{d}{dt}\Lambda + \|u_{tt}\|_1^2 \le -C\int_0^\infty \mu'(s)\|\eta^t(s)\|^2 ds + C.$$

Integrating from $t$ to $t + 1$ and using (4.2) the proof is over. ☐

### 4.2. Asymptotic compactness

One of the main difficulties of the problem is, of course, that the Sobolev embeddings are no longer compact.

For any $r > 0$ introduce two smooth positive functions $\varphi_r^i : \mathbb{R}^3 \to \mathbb{R}^+$, for $i = 0, 1$, such that

$$\varphi_r^0(x) + \varphi_r^1(x) = 1 \quad \text{for all } x \in \mathbb{R}^3,$$

and

$$\begin{aligned} \varphi_r^0(x) &= 0 \quad \text{if } |x| \le r, \\ \varphi_r^1(x) &= 0 \quad \text{if } |x| \ge r + 1. \end{aligned}$$

To make the asymptotic regular estimates, we decompose $f$ and define $h_i$, $i = 0, 1$, as follows:

$$-f(x, m) + h(x) - \phi_c(x, u_t) + g(x, 0) = -f_0(x, m) + h_0 - f_1(x, m) + h_1,$$

where

$$\begin{aligned} h_0 &= (h(x) - \phi_c(x, u_t) + g(x, 0))\varphi_r^0(x), \\ h_1 &= (h(x) - \phi_c(x, u_t) + g(x, 0))\varphi_r^1(x), \end{aligned}$$

and $f_i \in C^1(\mathbb{R}, \mathbb{R})$, $f_0(x, 0) = 0$ such that

$$\begin{aligned} f_0(x, m) &= \left(f(x, m) + (\nu_1 + d_1\nu_2)m + \frac{C_{\nu_1} + d_1 C_{\nu_1}}{m}\right)\sigma(m), \\ f_1(x, m) &= f(x, m) - f_0(x, m), \end{aligned}$$

with $\sigma : \mathbb{R} \to [0,1]$ is a Lipschitz function where $\sigma(m) = 0$ if $|m| \leq 1$ and $\sigma(m) = 1$ if $|m| > 2$.

Therefore, for some $C > 0$, the nonlinearities $f_i$ satisfy

$$f_0(x,m)m \geq 0, \quad F_0(x,m) = \int_0^m f_0(x,y)dy \geq 0, \tag{4.13}$$

$$|f_0(x,m)| \leq C|m|^5, \tag{4.14}$$

$$|f_1(x,m)| \leq C(1+|m|), \tag{4.15}$$

and finally,

$$h_1 = 0 \qquad \text{for } m \in \mathbb{R},\ |x| \geq r+1, \quad \|h_0\| \to 0 \quad \text{as } r \to \infty. \tag{4.16}$$

Now, we decompose the solution $S(t)z_0 = z(t)$ of problem (2.2) as follows:

$$S(t)z_0 = S_1(t)z_0 + S_2(t)z_0,$$

where $S_1(t)z_0 = z_1(t)$ and $S_2(t)z_0 = z_2(t)$, that is, $z = (u, u_t, \eta^t) = z_1 + z_2$, with

$$\begin{aligned} &u = v + w, \quad \eta^t = \xi^t + \zeta^t, \\ &z_1 = (v, v_t, \xi^t), \quad z_2 = (w, w_t, \zeta^t), \end{aligned}$$

solve the following problems:

$$\begin{cases} \partial_{tt}v - \Delta\partial_t v + \lambda v_t - \Delta v + \lambda v - \displaystyle\int_0^\infty \mu(s)\Delta\xi^t(s)ds + f_0(x,v) \\ + \phi(x,u_t) - \phi(x,w_t) = h_0, \\ \partial_t\xi^t = -\partial_s\xi^t + v_t, \\ (v(0), v_t(0), \xi^0) = z_0, \end{cases} \tag{4.17}$$

and

$$\begin{cases} \partial_{tt}w - \Delta\partial_t w + \lambda w_t - \Delta w + \lambda w - \displaystyle\int_0^\infty \mu(s)\Delta\zeta^t(s)ds \\ + f_0(x,u) - f_0(x,v) + \phi(x,w_t) = h_1 + \lambda u_t - f_1(x,u), \\ \partial_t\zeta = -\partial_s\zeta + w_t, \\ (w(0), w_t(0), \zeta^0) = (0,0,0). \end{cases} \tag{4.18}$$

By the standard Galerkin method, problems (4.17)-(4.18) are easily seen to satisfy existence and continuous dependence results analogous to those of Theorem 3.2.

We will establish some a priori estimates about the solutions of (4.17) and (4.18). Firstly, we have some preliminaries lemmas.

**Lemma 4.7.** *The uniform bound*

$$\|v\|_1^2 + \|v_t\|^2 + \int_0^\infty \mu(s)\|\nabla\xi^t\|^2 ds \leq C$$

*holds, along with the integral estimate*

$$\int_0^\infty \|v_t(t)\|_1^2 dt \le C. \tag{4.19}$$

*Proof.* Multiplying the first equation of (4.17) by $2v_t$ we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|v_t(t)\|^2 + \lambda\|v(t)\|^2 + \|\nabla v(t)\|^2 + \int_0^\infty \mu(s)\|\nabla\xi^t\|^2 ds + 2\langle F_0(x,v),1\rangle\right)\\
&\quad + 2\lambda\|v_t(t)\|^2 + 2\|\nabla v_t(t)\|^2 - 2\int_0^\infty \mu'(s)\|\nabla\xi^t\|^2 ds\\
&\quad + 2\langle \phi(x,u_t) - \phi(x,w_t), v_t\rangle\\
&= 2\langle h_0, v_t\rangle.
\end{aligned}$$

From (2.4), **(H1)** and applying the Young inequality, we get

$$2\langle \phi(x,u_t) - \phi(x,w_t), v_t\rangle = 2\langle \phi'(x,u_t+\theta w_t)v_t, v_t\rangle \ge 0, \quad 0<\theta<1, \tag{4.20}$$

$$-2\int_0^\infty \mu'(s)\|\nabla\xi^t\|^2 ds > 0, \tag{4.21}$$

$$2\langle h_0, v_t\rangle \le C\|h_0\|^2 + \lambda\|v_t(t)\|^2. \tag{4.22}$$

Thus, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|v_t(t)\|^2 + \lambda\|v(t)\|^2 + \|\nabla v(t)\|^2 + \int_0^\infty \mu(s)\|\nabla\xi^t\|^2 ds + 2\langle F_0(x,v),1\rangle\right)\\
&\quad + a\|v_t(t)\|_1^2\\
&\le C\|h_0\|^2,
\end{aligned}$$

implying that

$$\begin{aligned}
&\|v_t(t)\|^2 + \|v(t)\|_1^2 + \int_0^\infty \mu(s)\|\nabla\xi^t\|^2 ds + \int_0^t \|v_t(r)\|_1^2 dr\\
&\le C\left(\|v_t(t)\|^2 + \lambda\|v(t)\|^2 + \|\nabla v(t)\|^2 + \int_0^\infty \mu(s)\|\nabla\xi^t\|^2 ds + 2\langle F_0(x,v),1\rangle\right)\\
&\quad + \int_0^t \|v_t(r)\|_1^2 dr \le C.
\end{aligned}$$

Since $t \ge 0$ is arbitrary, we are finished. ☐

Collecting Lemma 4.6 and (4.19) we draw an immediate corollary.

**Corollary 4.8.** *There is $M = M(\rho_2) > 0$ such that, for any time $T \ge 1$, the estimate*

$$\|w_t(t_T)\|_1 \le M$$

*occurs for some $t_T = t_T(z_0) \in [T-1, T]$.*

**Lemma 4.9.** *The uniform bound $\|w_t\|_1 \le C$ holds.*

*Proof.* Multiplying the first equation of (4.18) by $2w_{tt}$ we get

$$\begin{aligned}\frac{d}{dt}\Lambda + 2\|w_{tt}\|^2 \le\ & 2\langle h_1 + \lambda u_t - f_1(x,u), w_{tt}\rangle + 2\|w_t\|^2 \\ & + 2\langle f_0'(x,u)u_t - f_0'(x,v)v_t, w_t\rangle \\ & + 2\int_0^\infty \mu(s)\|\nabla\zeta^t\|\|\nabla w_{tt}\|ds,\end{aligned}$$

where

$$\begin{aligned}\Lambda = \ & \lambda\|w_t\|^2 + \|\nabla w_t\|^2 + \Phi_0(w_t) + 2\lambda\langle w_t, w\rangle \\ & + 2\langle\nabla w_t, \nabla w\rangle + 2\langle f_0(x,u) - f_0(x,v), w_t\rangle + K\end{aligned}$$

and $K = K(\rho_1) > 0$ large enough in order to have

$$\|w_t\|_1^2 \le \Lambda \le C(1 + \|w_t\|_1^6).$$

Indeed, thanks to Lemmas 4.6 and 4.7,

$$\begin{aligned}2|\langle f_0(x,u) - f_0(x,v), w_t\rangle| &\le 2\|f_0(x,u) - f_0(x,v)\|_{L^{6/5}}\|w_t\|_{L^6} \\ &\le \frac{1}{4}\|w_t\|_1^2 + C,\end{aligned}$$

and

$$\|w_t\|_1^2 \le \|v_t\|_1^2 + \|u_t\|_1^2 \le \|v_t\|_1^2 + C,$$

the right-hand side is controlled by

$$\begin{aligned}& 2(\|h_1\| + \lambda\|u_t\| + \|f_1(x,u)\|)\|w_{tt}\| + 2\|w_t\|^2 \\ & + 2\left(\|f_0'(x,u)\|_{L^{3/2}}\|u_t\|_{L^6} + \|f_0'(x,v)\|_{L^{3/2}}\|v_t\|_{L^6}\right)\|w_t\|_{L^6} \\ & + 2\int_0^\infty \mu(s)\|\nabla\zeta^t\|\|\nabla w_{tt}\|ds \\ \le\ & 2\|w_{tt}\|^2 + C\|w_t\|_1^2 + C\|v_t\|_1\|w_t\|_1 + C \\ \le\ & 2\|w_{tt}\|^2 + C\|v_t\|_1^2 + C.\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}\Lambda \le C\|v_t\|_1^2 + C. \tag{4.23}$$

Integrating (4.23) over $[t, T]$, $T > 0$, for some positive $t \ge T - 1$, and using (4.19), we get

$$\|w_t(T)\|_1^2 \le 2\Lambda(T) \le C + 2\Lambda(t) \le C(1 + \|w_t\|_1^6).$$

If $T \le 1$ we choose $t = 0$, otherwise we choose $t = t_T$ as in Corollary 4.8. In either case, the desired bound follows. ☐

Combining Lemmas 4.3, 4.6 and 4.7, we get

$$\|u\|_1^2 + \|v\|_1^2 + \|w\|_1^2 + \|u_t\|_1^2 + \|v_t\|_1^2 + \|w_t\|_1^2 + \|\eta^t(s)\|_{1,\mu}^2 \le C. \tag{4.24}$$

Firstly, we prove that the solution $v$ becomes small as $r \to \infty$ and $t \to \infty$.

**Lemma 4.10.** *Assume that hypotheses of $f_0$, $\phi$ and $h_0$ hold. Then the solutions of equation* (4.17) *satisfy the following estimate*: *for every $\omega > 0$ there exist $T_\omega > 0$, $r_\omega > r_0$ and a constant $\gamma_2 > 0$, such that the solution $v$ to* (4.17), *corresponding to $r = r_\omega$, fulfills the inequality*

$$\|S_1(t)z_0\|^2_{\mathcal{H}_1} \leq \|z_0\|_{\mathcal{H}_1} e^{-\gamma_2 t} + \omega \quad \textit{for all } t \geq 0.$$

*Proof.* Multiplying the first equation of (4.17) by $v_t + av$ and adding to both sides the term

$$\begin{aligned} &\frac{d}{dt} j \int_0^\infty \mu(s)\|\xi^t\|^2 ds - 2j\int_0^\infty \mu'(s)\|\xi^t\|^2 ds \\ &= 2j\int_0^\infty \mu(s)\langle \xi^t(s), v_t\rangle ds \\ &\leq jk\|v_t(t)\|^2 + j\int_0^\infty \mu(s)\|\xi^t(s)\|^2 ds, \end{aligned}$$

we get

$$\begin{aligned} &\frac{d}{dt}E_{ja} + a\lambda\|v(t)\|^2 + 2a\|\nabla v(t)\|^2 + \lambda\|v_t(t)\|^2 + 2\|\nabla v_t(t)\|^2 \\ &\quad + 2a\langle f_0(x,v), v\rangle + 2\langle \phi(x,u_t) - \phi(x,w_t), v_t\rangle \\ &\leq C\|h_0\|^2 + jk\|v_t(t)\|^2 + j\int_0^\infty \mu(s)\|\xi^t(s)\|^2 ds - 2a\langle \phi(x,u_t) - \phi(x,w_t), v\rangle, \end{aligned}$$

where

$$\begin{aligned} E_{ja} &= \|v_t(t)\|^2 + \lambda(1+a)\|v(t)\|^2 + (1+a)\|\nabla v(t)\|^2 \\ &\quad + \int_0^\infty \mu(s)(j\|\xi^t(s)\|^2 + \|\nabla\xi^t(s)\|^2)ds + 2\langle F_0(x,v), 1\rangle + 2a\langle u_t, u\rangle. \end{aligned}$$

Using (4.13), (4.14) and (4.24), we get

$$\|z_{1j}\|^2_{\mathcal{H}_1} \leq 2E_{j0} \leq 4E_{ja} \leq 8E_{j0} \leq C\|z_{1j}\|^2_{\mathcal{H}_1}. \tag{4.25}$$

From Lemma 2.1 and (4.24), we get

$$2\langle \phi(x,u_t) - \phi(x,w_t), v_t\rangle \geq 0,$$

and

$$\begin{aligned} 2a\langle \phi(x,u_t) - \phi(x,w_t), v\rangle &\leq 2a\|\phi(x,u_t) - \phi(x,w_t)\|_{L^{6/5}}\|v\|_{L^6} \\ &\leq Ca\|v_t\|_1\|v\|_1 \\ &\leq Ca^{1/2}\|v_t\|_1^2 + Ca^{3/2}\|v\|_1^2. \end{aligned}$$

Now we also define the functional

$$\Lambda_{ja}(t) = E_{ja}(t) + a\Psi_j(t),$$

where

$$\Psi(t) = \int_0^\infty \kappa(s)(j\|\xi^t(s) - v(t)\|^2 + \|\nabla(\xi^t(s) - v(t))\|^2 ds > 0.$$

Using (4.25), Lemma 4.2, and Young inequality, we have

$$\|z_{1j}\|^2_{\mathcal{H}_1} \leq \Lambda_{j0}(t) \leq 2\Lambda_{ja}(t) \leq 4\Lambda_{j0}(t) \leq C\|z_{1j}\|^2_{\mathcal{H}_1}$$

and the inequality

$$\begin{aligned}
&\frac{d}{dt}\Psi(t) + \int_0^\infty \mu(s)(j\|\xi^t\|^2 + \|\nabla\xi^t\|^2)ds \\
&= 2\int_0^\infty \mu(s)j\langle\xi^t, v\rangle + \langle\nabla\xi^t, \nabla v\rangle ds \\
&\leq \frac{1}{2}\int_0^\infty \mu(s)(j\|\xi^t\|^2 + \|\nabla\xi^t\|^2)ds + 2k(j\|v\|^2 + \|\nabla v\|^2).
\end{aligned}$$

Therefore, there exists a positive constant $\gamma$ such that

$$\frac{d}{dt}\Lambda_{ja} + 2\gamma\Lambda_{ja} \leq 4kj\Lambda_{j0} + C\|h_0\|^2. \tag{4.26}$$

Putting $j = 0$ in (4.26) and subsequently substituting the result into (4.26) with $j = 1$, we obtain

$$\|v(t)\|_1^2 + \|v_t(t)\|^2 + \|\xi^t(s)\|^2_{1,\mu} \leq \|z_0\|_{\mathcal{H}_1}e^{-\gamma_2 t} + \omega,$$

where the constant $\omega$ depends on $\|h_0\|$ with $\|h_0\| \to 0$ as $r \to \infty$. This completes the proof. ☐

Given $R > 0$, we shall denote $B(R) = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Based on Lemma 4.10, any solution $(w, w_t, \zeta^t)$ to (4.18) solves the Dirichlet problem on the bounded domain $B(R)$, in the time interval $[0, T_\omega]$. Namely, for every $t \in [0, T_\omega]$,

$$(w(t), w_t(t), \zeta^t(s))|_{\partial B(R)} = 0, \quad \forall s > 0.$$

Next, we prove that the solution $(w, w_t, \zeta^t)$ to (4.18) identically vanishes outside the set $B(R) \times [0, T_\omega]$. As in [1], given $\rho > 0$, we introduce the function $\psi_\rho : \mathbb{R}^3 \to [0, 1]$ as

$$\psi_\rho(x) = \begin{cases} 0, & |x| < \rho + 1, \\ \sin^2\left[\frac{\pi}{2}\left(\frac{|x|}{\rho+1} - 1\right)\right], & \rho + 1 \leq |x| \leq 2\rho + 2, \\ 1 & |x| > 2\rho + 2. \end{cases}$$

Therefore, we can easily obtain the following estimates hold for all $x \in \mathbb{R}$

$$|\nabla\psi_\rho(x)| \leq \frac{\pi}{2(\rho+1)}, \tag{4.27}$$

$$|\nabla\psi_\rho^2(x)| \leq \frac{\pi}{\rho+1}\psi_\rho(x), \tag{4.28}$$

$$|\Delta\psi_\rho(x)| \leq \frac{3\pi^2}{2(\rho+1)^2}. \tag{4.29}$$

**Lemma 4.11.** *There exist $R > 0$ and $T_\omega > 0$ such that the solution $(w, w_t, \zeta^t)$ to (4.18) identically vanishes outside the set $B(R) \times [0, T_\omega]$, in the sense that fulfills the inequality*

$$\|\psi_\rho w\|_1^2 + \|\psi_\rho w_t\|^2 + \|\psi_\rho \zeta^t\|_{1,\mu}^2 \leq \omega \quad \text{for all } t \geq T_\omega.$$

*Proof.* Taking the product in $H_0$ of (4.18) and $\psi_\rho^2 w_t$, and adding to both sides the term

$$\frac{d}{dt}\int_0^\infty \mu(s)\|\psi_\rho\zeta^t\|^2 ds - 2\int_0^\infty \mu'(s)\|\psi_\rho\zeta^t\|^2 ds = 2\int_0^\infty \mu(s)\langle\psi_\rho^2\zeta^t(s), w_t\rangle ds$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx + \lambda\int_{\mathbb{R}^3}\psi_\rho^2|w|^2dx + \int_0^\infty \mu(s)\|\psi_\rho\zeta^t\|^2ds\right)\\
&-\int_0^\infty \mu'(s)\|\psi_\rho\zeta^t\|^2ds + \lambda\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx - \int_{\mathbb{R}^3}\psi_\rho^2 w_t\Delta w dx\\
&-\int_{\mathbb{R}^3}\psi_\rho^2 w_t\Delta w_t dx - \int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2 w_t\Delta\zeta^t(s)dxds + \int_{\mathbb{R}^3}\psi_\rho^2\phi(x,w_t)w_t dx\\
=&\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\zeta^t(s)\psi_\rho^2 w_t dxds - \int_{\mathbb{R}^3}\psi_\rho^2(f_0(x,u)-f_0(x,v))w_t dx\\
&+\int_{\mathbb{R}^3}\psi_\rho^2(h_1+\lambda u_t - f_1(x,u))w_t dx.
\end{aligned}$$

Applying the Hölder, Young inequalities, and (4.24), we obtain

$$\begin{aligned}
\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2\zeta^t(s)w_t dxds &\leq \int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\zeta^t(s)||w_t|dxds\\
&\leq \int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\zeta^t(s)|^2dxds + k\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx,\\
\int_{\mathbb{R}^3}\psi_\rho^2 w_t\Delta w dx &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w|^2dx - \int_{\mathbb{R}^3}\nabla\psi_\rho^2 w_t\nabla w dx\\
&\leq -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w|^2dx + \frac{\pi}{\rho+1}\|w_t\|\|\nabla w\|\\
&\leq -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w|^2dx + \frac{C}{\rho+1},
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3}\psi_\rho^2 w_t\Delta w_t dx &= -\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx - \int_{\mathbb{R}^3}\nabla\psi_\rho^2 w_t\nabla w_t dx\\
&\leq -\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx + \frac{\pi}{\rho+1}\int_{\mathbb{R}^3}\psi_\rho|w_t||\nabla w_t|dx\\
&\leq -\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx + \frac{\pi^2}{4(\rho+1)^2}\|w_t\|^2
\end{aligned}$$

$$\leq -\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx+\frac{C}{\rho+1}.$$

Note that $h_1(x,t)=0$ for $m\in\mathbb{R}$, $|x|\geq r+1$, so we get $\int_{\mathbb{R}^3}\psi_\rho^2 h_1 w_t dx=0$.

Applying Lemma 4.10 and (4.15), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3}\psi_\rho^2 f_1(x,u)w_tdx &\leq C\int_{\mathbb{R}^3}\psi_\rho^2(|v|+|w|)|w_t|dx\\
&\leq C\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx+C\int_{\mathbb{R}^3}\psi_\rho^2|w|^2dx+a\omega,\\
\int_{\mathbb{R}^3}\psi_\rho^2\lambda u_tw_tdx &=\int_{\mathbb{R}^3}\psi_\rho^2\lambda v_tw_tdx+\int_{\mathbb{R}^3}\psi_\rho^2\lambda|w_t|^2dx\\
&\leq C\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx+a\int_{\mathbb{R}^3}\psi_\rho^2|v_t|^2dx\\
&\leq C\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^2dx+a\omega,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2w_t\Delta\zeta^t(s)dxds\\
&= -\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\nabla\psi_\rho^2w_t\nabla\zeta^t(s)dxds-\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2\nabla w_t\nabla\zeta^t(s)dxds\\
&\leq \frac{\pi}{\rho+1}\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho|w_t||\nabla\zeta^t(s)|dxds-\frac{1}{2}\frac{d}{dt}\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\\
&\quad+\int_0^\infty\mu'(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\\
&\leq \frac{\pi^2}{4(\rho+1)^2}\|w_t\|^2+\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\\
&\quad-\frac{1}{2}\frac{d}{dt}\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\\
&\leq \frac{C}{\rho+1}+\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\\
&\quad-\frac{1}{2}\frac{d}{dt}\int_0^\infty\mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds,
\end{aligned}$$

where

$$\int_0^\infty\mu'(s)\int_{\mathbb{R}^3}\psi_\rho^2|\nabla\zeta^t(s)|^2dxds\leq 0.$$

Using (1.6) and (4.24) and Lemma 2.1 we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}\psi_\rho^2(f_0(x,u)-f_0(x,v))w_tdx\\
&\leq C(1+\|u\|_1^4+\|v\|_1^4)\|\psi_\rho w\|_1\|\psi_\rho w_t\|_1
\end{aligned}$$

$$\leq \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + C \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w|^2 dx + \frac{C}{\rho+1},$$

and

$$\int_{\mathbb{R}^3} \psi_\rho^2 \phi(x, w_t) w_t dx \geq 0.$$

Summarizing, we arrive at

$$\frac{d}{dt} y(t) \leq C y(t) + \frac{C}{\rho+1} + 2a\omega,$$

where

$$y(t) = \int_{\mathbb{R}^3} \psi_\rho^2 |w_t|^2 dx + \int_{\mathbb{R}^3} \psi_\rho^2 (\lambda |w|^2 + |\nabla w|^2) dx + \int_0^\infty \mu(s) \psi_\rho^2 (\|\zeta^t\|^2 + \|\nabla \zeta^t\|^2) ds.$$

Applying the Gronwall lemma on $[0, T_\omega]$, recall that $y(0) = 0$, we obtain

$$y(T_\omega) \leq T_\omega e^{C T_\omega} \left( \frac{C}{\rho+1} + a\omega \right).$$

We can easily see that

$$\begin{aligned} &\|\psi_\rho w_t\|^2 + \|\psi_\rho w\|_1^2 + \|\psi_\rho \zeta^t(s)\|_{1,\mu}^2 \\ &\leq C y(T_\omega) + \int_{\mathbb{R}^3} |\nabla \psi_\rho|^2 |\nabla w(T_\omega)|^2 dx + \int_0^\infty \mu(s) \int_{\mathbb{R}^3} |\nabla \psi_\rho|^2 |\nabla \zeta^{T_\omega}(s)|^2 dx ds. \end{aligned}$$

On the other hand, using (4.27), we get

$$\int_{\mathbb{R}^3} |\nabla \psi_\rho|^2 |\nabla w(T_\omega)|^2 dx + \int_0^\infty \mu(s) \int_{\mathbb{R}^3} |\nabla \psi_\rho|^2 |\nabla \zeta^{T_\omega}(s)|^2 dx ds \leq \frac{C}{\rho+1}.$$

Thus, we conclude that

$$\|\psi_\rho w_t\|^2 + \|\psi_\rho w\|_1^2 + \|\psi_\rho \zeta^t(s)\|_{1,\mu}^2 \leq \frac{C}{\rho+1} + \frac{\omega}{2}$$

for fixed $C = C(\omega)$, independent of $\rho$, and $a$ small enough. Choosing $\rho \geq r_\omega$ large enough such that $\frac{C}{\rho+1} \leq \frac{\omega}{2}$ we are done. ☐

To state the next lemma, which provides the compact part in the decomposition of the solution, some definitions are needed. Let $B \subset \mathbb{R}^3$ be a smooth bounded domain. Define the linear operator

$$Aw = -\Delta w, \quad D(A) = H^2(B) \cap H_0^1(B).$$

Moreover, introduce the Hilbert spaces $V_\alpha = D(A^{\alpha/2})$, endowed with the inner products $\langle \cdot, \cdot \rangle = \langle A^{\alpha/2}, A^{\alpha/2} \rangle$ and norms $\| \cdot \|_\alpha$. Putting $\mathcal{H}_{\nu+1} = V_{\nu+1} \times V_1 \times$

$L^2_\mu(\mathbb{R}^+, V_{\nu+1})$ for $0 \le \nu$. By virtue of Lemma 4.11, any solution $w$ of (4.18) solves the Dirichlet problem on a fixed bounded domain

$$\begin{cases} w_{tt} + Aw_t + Aw + \displaystyle\int_0^\infty \mu(s) A\zeta^t(s)ds \\ + f_0(x,u) - f_0(x,v) + \phi(x,w_t) \\ = h_1 + \lambda v_t - \lambda w - f_1(x,u) \quad \text{on } B(R) \times [0, T_\omega], \\ \partial_t \zeta = -\partial_s \zeta + w_t, \\ (w, w_t, \zeta^t)|_{\partial B(R)} = 0, \\ (w(0), w_t(0), \zeta^0) = (0,0,0). \end{cases} \tag{4.30}$$

To prove the compactness of $S(t)$, we replace (1.8) with the more restrictive assumption as follows:

$$|g'_m(x,m)| \le C(1 + |m|^{p-1}), \quad 1 \le p < 5, \qquad |g'_x(x,m)| \le C|m|^p. \tag{4.31}$$

**Lemma 4.12.** *There exists a positive constant* $N_\omega > 0$ *such that the solution* $w$ *to* (4.30) *at time* $T_\omega$*, corresponding to* $r = r_\omega$*, fulfills the inequality*

$$\|(w(t), w_t(t), \zeta^t)\|^2_{\mathcal{H}_{\nu+1}} \le N_\omega \tag{4.32}$$

*for every* $z_0 \in \mathcal{H}_1$ *and* $0 < \nu < \frac{1}{2}$.

*Proof.* Multiplying the first equation of (4.30) by $A^\nu w_t(t)$, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|w_t\|^2_\nu + \|w\|^2_{\nu+1} + \|\zeta^t\|^2_{\nu+1,\mu}\right) - 2\int_0^\infty \mu'(s)\|\zeta^t(s)\|^2_{\nu+1} ds + 2\|w_t\|^2_{\nu+1} \\ \le\ & -2\langle f_0(x,u) - f_0(x,v), A^\nu w_t\rangle - 2\langle \phi(x,w_t), A^\nu w_t\rangle \\ & + 2\langle h_1 + \lambda v_t + \lambda w - f_1(x,u), A^\nu w_t\rangle. \end{aligned}$$

On the other hand, using (4.24) and the embedding $H^1_0(B(R)) \hookrightarrow L^6(B(R))$ and $D(A^{\frac{1-\nu}{2}}) \hookrightarrow L^{\frac{6}{3-2(1-\nu)}}(B(R))$, we have

$$\begin{aligned} 2\langle \phi(x,w_t), A^\nu w_t\rangle &\le C\|w_t\|^p_{L^{\frac{6p}{5-2\nu}}} \|A^\nu w_t\|_{L^{\frac{6}{3-2(1-\nu)}}} \\ &\le C\|w_t\|^p_1 \|w_t\|_{\nu+1} \\ &\le \frac{1}{4}\|w_t\|^2_{\nu+1} + C. \end{aligned}$$

Using (4.1), the condition (1.7) and $\nu < \frac{\nu+1}{2}$ as $0 < \nu < 1$, we get

$$\begin{aligned} &2\langle f_0(x,u) - f_0(x,v), A^\nu w_t\rangle \\ &\le C\int_{B(R)} (1 + |u|^4 + |v|^4)|w||A^\nu w_t| dx \\ &\le C\left(\int_{B(R)} (1 + |u|^4 + |v|^4)^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \left(\int_{B(R)} |w|^{\frac{6}{3-2(1+\nu)}} dx\right)^{\frac{3-2(1+\nu)}{6}} \end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{B(R)} |A^{\nu} w_t|^{\frac{6}{3-2(1-\nu)}} dx\right)^{\frac{3-2(1-\nu)}{6}}\\
&\leq C(1+\|u\|_{L^6}^4+\|v\|_{L^6}^4)\|w\|_{L^{\frac{6}{3-2(1+\nu)}}}\|A^{\nu} w_t\|_{L^{\frac{6}{3-2(1-\nu)}}}\\
&\leq C(1+\|u\|_1^4+\|v\|_1^4)\|w\|_{\nu+1}\|w_t\|_{\nu+1}\\
&\leq \frac{1}{4}\|w_t\|_{\nu+1}^2+C(\rho_1)\|w\|_{\nu+1}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
2\langle q, A^{\nu} w_t\rangle &\leq 2\|q\|\|A^{\nu} w_t\|\\
&\leq \frac{1}{2}\|w_t\|_{\nu+1}^2+C, \qquad \text{where } q=h_1+\lambda v_t+\lambda w-f_1(x,u).
\end{aligned}
$$

Notice that $-\int_0^{\infty}\mu'(s)\|\zeta^t(s)\|_{\nu+1}^2 ds \geq 0$, so we can omit this term in the above inequality. Thus,

$$
\frac{d}{dt}\left(\|w_t\|_{\nu}^2+2\|w\|_{\nu+1}^2+\|\zeta^t\|_{\nu+1,\mu}^2\right) \leq C\left(\|w_t\|_{\nu}^2+2\|w\|_{\nu+1}^2+\|\zeta^t\|_{\nu+1,\mu}^2\right)+C.
$$

Hence, the conclusion is drawn from the Gronwall lemma. ☐

In addition, for any $\zeta_0 \in L_{\mu}^2(\mathbb{R}^+, H_1)$, the Cauchy problem (see e.g. [2,16])

$$
\begin{cases}
\partial_t \zeta^t = -\partial_s \zeta^t + w_t, & t>0,\\
\zeta^0 = \zeta_0 = 0,
\end{cases}
$$

has a unique solution $\zeta^t \in C((0,\infty); L_{\mu}^2(\mathbb{R}^+, H_1))$, and

$$
\zeta^t(s) = \begin{cases}
w(t)-w(t-s), & 0<s\leq t,\\
\zeta_0(s-t)-\zeta_0(0)+w(t)-w(0), & s>t.
\end{cases}
$$

Thus, thanks to $\zeta^0(x,s)=0$, we have

$$
\zeta^t(s) = \begin{cases}
w(t)-w(t-s), & 0<s\leq t,\\
w(t), & s>t.
\end{cases} \tag{4.33}
$$

Let $B_0$ be the bounded absorbing set obtained in Lemma 4.3, we now prove the following result.

**Lemma 4.13.** *Setting*

$$
\mathcal{K}_T = PS_2(T)B_0
$$

*for* $T>0$ *large enough, where* $\{S_2(t)\}_{t\geq 0}$ *is the solution process of* (4.30), $P: H_0^1(B(R)) \times L^2(B(R)) \times L_{\mu}^2(\mathbb{R}^+, H_0^1(B(R))) \to L_{\mu}^2(\mathbb{R}^+, H_0^1(B(R)))$ *is the projection operator. Then there is a positive constant* $N_1 = N_1(\|B_0\|_{\mathcal{H}_1})$ *such that*

(1) $\mathcal{K}_T$ *is bounded in* $L_{\mu}^2(\mathbb{R}^+, V_{\nu+1}) \cap H_{\mu}^1(\mathbb{R}^+; H_0^1(B(R)))$,

(2) $\sup_{\xi\in\mathcal{K}_T}\|\xi(s)\|_{H_0^1(B(R))}^2 \leq N_1$.

*Moreover,* $\mathcal{K}_T$ *is relatively compact in* $L_{\mu}^2(\mathbb{R}^+, H_0^1(B(R)))$.

*Proof.* From (4.33) we have

$$\partial_s \xi^{t\varepsilon}(s) = \begin{cases} w(t-s), & 0 < s \le t, \\ 0, & s > t, \end{cases}$$

which, combining with Lemma 4.12, implies claim (1).

After that, using (4.33) once again, we can easily deduce that

$$\begin{aligned} &\|\xi^T(s)\|^2_{H^1_0(B(R_\omega))} \\ \le &\begin{cases} \int_0^s \|w(T-r)\|^2_{H^1_0(B(R_\omega))} dr \le \int_0^T \|w(T-r)\|^2_{H^1_0(B(R_\omega))} dr, & 0 < s \le T, \\ \int_0^T \|w(T-r)\|^2_{H^1_0(B(R_\omega))} dr, & s > T. \end{cases} \end{aligned}$$

By virtue of (4.32), we know that claim (2) holds. Because $V_{\nu+1} \hookrightarrow H^1_0(B(R_\omega))$ compactly, we conclude that $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H^1_0(B(R_\omega)))$ thanks to the following lemma. ☐

**Lemma 4.14** (see [16])**.** *Assume that $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function and satisfies the condition: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$ for all $s \ge s_0$. Moreover, let $X_0, X_1, X_2$ be Banach spaces, here $X_0, X_2$ are reflexive and satisfy*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2,$$

*where the embedding $X_0 \hookrightarrow X_1$ is compact. Let $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+, X_1)$ satisfy*

(1) *$\mathcal{C}$ is a subset in $L^2_\mu(\mathbb{R}^+, X_0) \cap H^1_\mu(\mathbb{R}^+, X_2)$;*
(2) *$\sup_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{X_1} \le h(x,s), \forall s \in \mathbb{R}^+,$ where $h \in L^1_\mu(\mathbb{R}^+)$.*

*Then $\mathcal{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+, X_1)$.*

*Proof of Theorem 4.1.* By Lemma 4.3, the family of semigroup $S(t)$ has a bounded absorbing $B_0$ in $\mathcal{H}_1$. Moreover, $S(t)$ is global asymptotically compact in $\mathcal{H}_1$ due to Lemmas 4.10, 4.12 and 4.13. Therefore, the family of semigroup $S(t)$ has the global attractor $\mathcal{A}$ in $\mathcal{H}_1$. ☐

In the next sections, we will prove the existence of exponential attractors of equation (1.1). This requires that the solutions of system (1.1) have higher-order regularity, on this account, we need to show that $u(t)$ and $\eta^t$ are bounded in $\mathcal{H}_2$.

### 4.3. Higher-order regularity

From Theorem 4.1, we immediately obtain the following regularity result.

**Lemma 4.15.** *The attractor $\mathcal{A}$ is bounded in $\mathcal{H}_{\nu+1}$ for all $\frac{1}{4} \le \nu < \frac{1}{2}$.*

To prove $\mathcal{A}$ is bounded in $\mathcal{H}_2$, we argue as follows. For $z_0 \in \mathcal{A}$, we split the solution $S(t)z_0 = z(t)$ into the sum $S_1(t)z_0 + S_2(t)z_0$, where $S_1(t)z_0 = v(t)$ and

$S_2(t)z_0 = w(t)$, instead of (4.17) and (4.18) solving, respectively,

$$\begin{cases} \partial_{tt}v - \Delta\partial_t v + \lambda v_t - \Delta v + \lambda v - \displaystyle\int_0^\infty \mu(s)\Delta\xi^t(s)ds + \phi(x,u_t) - \phi(x,w_t) = h_0, \\ \partial_t\xi^t = -\partial_s\xi^t + v_t, \\ (v(0), v_t(0), \xi^0) = z_0, \end{cases}$$

and

$$\begin{cases} \partial_{tt}w - \Delta\partial_t w + \lambda w_t - \Delta w + \lambda w - \displaystyle\int_0^\infty \mu(s)\Delta\zeta^t(s)ds \\ + f(x,u) + \phi(x,w_t) = h_1 + \lambda u_t, \\ \partial_t\zeta = -\partial_s\zeta + w_t, \\ (w(0), w_t(0), \zeta^0) = (0,0,0). \end{cases} \tag{4.34}$$

As the particular case of Lemma 4.10, we know that

$$\|S_1(t)z_0\|^2_{\mathcal{H}_1} \le Ce^{-\gamma t} + \omega \quad \text{for all } t \ge 0. \tag{4.35}$$

Besides, as in Lemmas 4.3, 4.6 and 4.7, we also obtain

$$\|u\|_1^2 + \|v\|_1^2 + \|w\|_1^2 + \|u_t\|_1^2 + \|v_t\|_1^2 + \|w_t\|_1^2 + \|\eta^t(s)\|^2_{1,\mu} + \|w_{tt}\|^2 \le C. \tag{4.36}$$

**Lemma 4.16.** *There exist* $T_\omega > 0$ *and* $\rho \ge r_\omega$ *such that the solution* $w$ *to* (4.34) *at time* $T_\omega$*, corresponding to* $r = r_\omega$*, fulfills the inequality*

$$\|\psi_\rho w\|_2^2 + \|\psi_\rho w_t\|_1^2 + \|\psi_\rho \zeta^t\|^2_{2,\mu} \le \omega, \quad \forall t \ge T_\omega.$$

*Proof.* Taking the product in $H_0$ of (4.18) and $-\psi_\rho^2\Delta w_t$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx + \int_{\mathbb{R}^3}\psi_\rho^2|\Delta w|^2dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^3}\psi_\rho^2|\Delta\zeta^t(s)|^2dxds\right) \\ &+ \int_{\mathbb{R}^3}\psi_\rho^2|\Delta w_t|^2dx + \lambda\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx - \int_0^\infty \mu'(s)\int_{\mathbb{R}^3}\psi_\rho^2|\Delta\zeta^t(s)|^2dxds \\ &+ \int_{\mathbb{R}^3}\nabla\psi_\rho^2 w_{tt}\nabla w_t dx - \int_{\mathbb{R}^3}\psi_\rho^2\phi(x,w_t)\Delta w_t dx - \int_{\mathbb{R}^3}\psi_\rho^2 f(x,u)\Delta w_t dx \\ &= -\lambda\int_{\mathbb{R}^3}\nabla\psi_\rho^2 w_t\nabla w_t dx - \lambda\int_{\mathbb{R}^3}\nabla\psi_\rho^2 w\nabla w dx - \lambda\int_{\mathbb{R}^3}\psi_\rho^2\nabla w\nabla w_t dx \\ &+ \int_{\mathbb{R}^3}\psi_\rho^2(h_1 + \lambda u_t)\Delta w_t dx. \end{aligned} \tag{4.37}$$

Applying the Hölder, Young inequalities and (4.36), we obtain

$$\begin{aligned} \int_{\mathbb{R}^3}\nabla\psi_\rho^2 w_{tt}\nabla w_t dx &\le \frac{\pi}{\rho+1}\int_{\mathbb{R}^3}\psi_\rho|w_{tt}||\nabla w_t|dx \\ &\le \int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx + \frac{C}{(\rho+1)^2}\|w_{tt}\|^2 \\ &\le \int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx + \frac{C}{\rho+1}, \end{aligned}$$

$$\begin{aligned}
-\lambda \int_{\mathbb{R}^3} \nabla\psi_\rho^2 w_t \nabla w_t dx &\leq \frac{\pi}{\rho+1} \int_{\mathbb{R}^3} \psi_\rho |w_t||\nabla w_t| dx \\
&\leq \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + \frac{C}{\rho+1}, \\
-\lambda \int_{\mathbb{R}^3} \nabla\psi_\rho^2 w \nabla w dx &\leq \frac{\pi}{\rho+1} \int_{\mathbb{R}^3} \psi_\rho |w||\nabla w| dx \\
&\leq \frac{C}{\rho+1}, \\
-2\lambda \int_{\mathbb{R}^3} \psi_\rho^2 \nabla w \nabla w_t dx &\leq \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + \lambda^2 \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w|^2 dx.
\end{aligned}$$

Note that $h_1(x,t) = 0$ for $m \in \mathbb{R}$, $|x| \geq r+1$, we get $\int_{\mathbb{R}^3} \psi_\rho^2 h_1 w_t dx = 0$. Using (4.35) and (4.36), we obtain

$$\begin{aligned}
-2\lambda \int_{\mathbb{R}^3} \psi_\rho^2 u_t \Delta w_t dx &\leq 2\int_{\mathbb{R}^3} \nabla\psi_\rho^2 |u_t||\nabla w_t| dx + 2\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla u_t||\nabla w_t| dx \\
&\leq \frac{2\pi}{\rho+1} \int_{\mathbb{R}^3} \psi_\rho |u_t||\nabla w_t| dx + 2\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla(v_t + w_t)||\nabla w_t| dx \\
&\leq 2\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + \int_{\mathbb{R}^3} \psi_\rho^2 |\nabla v_t||\nabla w_t| dx + \frac{C}{\rho+1} \\
&\leq 3\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + \omega + \frac{C}{\rho+1}.
\end{aligned}$$

Applying (4.36), Lemma 4.15 and noting that $D(A^{\frac{\nu+1}{2}}) \hookrightarrow L^{12}$, $\frac{1}{4} \leq \nu < 1$, we deduce that $\|u\|_{L^{12}}^{12} \leq \|u\|_{\mathcal{H}_{\nu+1}}^{12} \leq C$.

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \psi_\rho^2 f(x,u) \Delta w_t dx \\
&\leq \int_{\mathbb{R}^3} \psi_\rho^2 |f_u'(x,u)||\nabla u||\nabla w_t| dx + \int_{\mathbb{R}^3} \psi_\rho^2 |f_x'(x,u)||\nabla w_t| dx \\
&\quad + \int_{\mathbb{R}^3} \nabla\psi_\rho^2 |f(x,u)||\nabla w_t| dx \\
&\leq C\int_{\mathbb{R}^3} \psi_\rho^2 (1+|u|^4)|\nabla(v+w)||\nabla w_t| dx + C\int_{\mathbb{R}^3} \psi_\rho^2 |(v+w)|^5 |\nabla w_t| dx \\
&\quad + C\int_{\mathbb{R}^3} \nabla\psi_\rho^2 (1+|u|^5)|\nabla w_t| dx \\
&\leq C\omega + \frac{1}{2}\int_{\mathbb{R}^3} \psi_\rho^2 |\Delta w_t|^2 dx + C\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w_t|^2 dx + C\int_{\mathbb{R}^3} \psi_\rho^2 |\nabla w|^2 dx + \frac{C}{\rho+1}.
\end{aligned}$$

Using (4.31), (4.36), and since $-\int_{\mathbb{R}^3} \psi_\rho^2 \phi_{w_t}'(x,w_t)|\nabla w_t|^2 dx \leq 0$, we have

$$-\int_{\mathbb{R}^3} \psi_\rho^2 \phi(x,w_t) \Delta w_t dx = \int_{\mathbb{R}^3} \psi_\rho^2 \phi_{w_t}'(x,w_t)|\nabla w_t|^2 dx + \int_{\mathbb{R}^3} \psi_\rho^2 \phi_x'(x,w_t)|\nabla w_t| dx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\nabla\psi_\rho^2\phi(x,w_t)\nabla w_t dx\\
&\le C\int_{\mathbb{R}^3}\psi_\rho^2|w_t|^p|\nabla w_t|dx+\frac{C}{\rho+1}\int_{\mathbb{R}^3}\psi_\rho|w_t|^4|\nabla w_t|dx\\
&\le \frac{1}{4}\int_{\mathbb{R}^3}\psi_\rho^2|\Delta w_t|^2dx+C\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx+\frac{C}{\rho+1}.
\end{aligned}
$$

Plugging all the above inequalities into (4.37), it follows that

$$\frac{d}{dt}y(t)\le Cy(t)+C\left(\frac{1}{\rho+1}+\omega\right),$$

where

$$y(t)=\int_{\mathbb{R}^3}\psi_\rho^2|\nabla w_t|^2dx+\int_{\mathbb{R}^3}\psi_\rho^2|\Delta w|^2dx+\int_0^\infty\mu(s)\psi_\rho^2\|\Delta\zeta^t\|^2ds.$$

Applying the Gronwall lemma on $[0,T_\omega]$, and recalling that $y(0)=0$, we obtain

$$y(T_\omega)\le CT_\omega e^{CT_\omega}\left(\frac{1}{\rho+1}+\omega\right). \tag{4.38}$$

Combining (4.38) and Lemma 4.11, we conclude that

$$\|\psi_\rho w_t\|_1^2+\|\psi_\rho w\|_2^2+\|\psi_\rho\zeta^t(s)\|_{2,\mu}^2\le C\omega$$

for fixed $C=C(R)$, independent of $\rho$. ☐

**Lemma 4.17.** *Under the assumptions* **(H1)**–**(H4)** (*in* (**H3**), (1.8) *is replaced by* (4.31)), *the following estimate holds*:

$$\|S_2(t)z_0\|_{\mathcal{H}_2}^2\le M_0 \tag{4.39}$$

*for some* $M_0>0$.

*Proof.* For $a\in[0,1)$ to be fixed later, multiplying the first equation of (4.34) by $w_t(t)-aw(t)$ in $L^2(\mathbb{R}^3)$, and adding to both sides the term

$$\frac{d}{dt}\int_0^\infty\mu(s)\|\zeta^t\|^2ds-2\int_0^\infty\mu'(s)\|\zeta^t\|^2ds=2\int_0^\infty\mu(s)\langle\zeta^t(s),w_t\rangle ds,$$

and as in the proof of Lemma 4.12, we get

$$\|(w,w_t,\zeta^t)\|_{\mathcal{H}_1}^2\le N\ \text{ for some } N>0. \tag{4.40}$$

Besides, multiplying the first equation of (4.34) by $-\Delta w_t(t)-a\Delta w(t)$ in $L^2(\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg(\|\nabla w_t\|^2+(1+a)\|\Delta w\|^2+\lambda(1+a)\|\nabla w\|^2\\
&+\int_0^\infty\mu(s)\|\Delta\zeta^t\|^2ds+2a\langle\nabla w_t,\nabla w\rangle\Bigg)\\
&+(\lambda-a)\|\nabla w_t\|^2+\|\Delta w_t\|^2+a\lambda\|\nabla w\|^2+a\|\Delta w\|^2
\end{aligned}
$$

$$\begin{aligned}
&+ a\int_0^\infty \mu(s)\langle \Delta\zeta^t, \Delta w\rangle ds - \int_0^\infty \mu'(s)\|\Delta\zeta^t(s)\|^2 ds\\
&+ \langle f(x,u), -\Delta w_t - a\Delta w\rangle + \langle \phi'_{w_t}(x,w_t)\nabla w_t, \nabla w_t\rangle\\
= \;& - a\langle \phi'_{w_t}(x,w_t)\nabla w_t, \nabla w\rangle - \langle \phi'_x(x,w_t), \nabla w_t + a\nabla w\rangle\\
&+ \langle h_1 + \lambda u_t, -\Delta w_t - a\Delta w\rangle. \qquad (4.41)
\end{aligned}$$

Applying Lemma 4.2, we have

$$\begin{aligned}
\frac{d}{dt}a\Psi(t) + a\int_0^\infty \mu(s)\|\Delta\zeta^t(s)\|^2 ds &= 2a\int_0^\infty \mu(s)\langle \Delta\zeta^t(s), \Delta w\rangle ds\\
&\le 2a\int_0^\infty \mu(s)\|\Delta\zeta^t\|\|\Delta w\| ds\\
&\le a^{\frac{1}{2}}\int_0^\infty \mu(s)\|\Delta\zeta^t\|^2 ds + a^{\frac{3}{2}}\|\Delta w\|^2. \qquad (4.42)
\end{aligned}$$

Using Lemma 4.15 and Sobolev embedding $D(A^{\frac{\nu+1}{2}}) \hookrightarrow L^{10}$, $\frac{1}{5} \le \nu < 1$, we deduce that

$$\|u\|_{L^{10}}^{10} \le \|u\|_{\mathcal{H}_{\nu+1}}^{10} \le C, \quad \frac{1}{5} \le \nu < 1.$$

Therefore

$$\begin{aligned}
a\langle f(x,u), -\Delta w_t - a\Delta w\rangle &\le C(1 + \|u\|_{L^{10}}^5)(\|\Delta w_t\| + a\|\Delta w\|)\\
&\le \frac{1}{2}\|\Delta w_t\|^2 + a^2\|\Delta w\|^2 + C.
\end{aligned}$$

Exploiting Lemma 2.1 and (4.31), (4.40), we get

$$\begin{aligned}
-a\langle \phi'_{w_t}(x,w_t)\nabla w_t, \nabla w\rangle &\le Ca\|\phi'_{w_t}(x,w_t)\|_{L^{3/2}}\|\nabla w_t\|_{L^6}\|\nabla w\|_{L^6}\\
&\le \frac{1}{4}(\|\Delta w_t\|^2 + a^2\|\Delta w\|^2) + C,\\
-\langle \phi'_x(x,w_t), \nabla w_t + a\nabla w\rangle &\le \|\phi'_x(x,w_t)\|_{L^{6/5}}^2 + \frac{1}{4}(\|\Delta w_t\|^2 + a^2\|\Delta w\|^2)\\
&\le \frac{1}{4}(\|\Delta w_t\|^2 + a^2\|\Delta w\|^2) + C,
\end{aligned}$$

and

$$\langle \phi'_{w_t}(x,w_t)\nabla w_t, \nabla w_t\rangle \ge 0.$$

Finally,

$$\langle h_1 + \lambda u_t, -\Delta w_t - a\Delta w\rangle \le \frac{1}{4}\|w_t\|_2^2 + a^2\|w\|_2^2 + C.$$

Putting

$$\begin{aligned}
\Lambda(t) = \|\nabla w_t\|^2 + (1+a)\|\Delta w\|^2 + \lambda(1+a)\|\nabla w\|^2 + \int_0^\infty \mu(s)\|\Delta\zeta^t\|^2 ds\\
+ 2a\langle \nabla w_t, \nabla w\rangle + a\Psi(t),
\end{aligned}$$

we get

$$\begin{aligned}
&\|\nabla w_t\|^2 + \|\Delta w\|^2 + \lambda\|\nabla w\|^2 + \int_0^\infty \mu(s)\|\Delta\zeta^t\|^2 ds \\
&\le \Lambda(t) \\
&\le 2\left(\|\nabla w_t\|^2 + \|\Delta w\|^2 + \lambda\|\nabla w\|^2 + \int_0^\infty \mu(s)\|\Delta\zeta^t\|^2 ds\right).
\end{aligned}$$

Summation of (4.41) and (4.42) and then combining all the above inequalities, we arrive at

$$\frac{d}{dt}\Lambda(t) + \alpha\Lambda(t) + \frac{1}{4}\|\Delta w_t\|^2 \le C. \tag{4.43}$$

By the Gronwall lemma, and using (4.36) and Lemma 4.2, we can get (4.35) immediately. This completes the proof. ☐

Now, we have the following lemma.

**Lemma 4.18.** *For any bounded set $B$ in $\mathcal{H}_2$, the following estimate holds*:

$$\sup_{t\ge 0}\sup_{z_0\in B} \left\|(u(t), u_t(t), \eta^t(s))\right\|_{\mathcal{H}_2} \le C. \tag{4.44}$$

*Moreover, for every $t_1, t_2 > 0$, we have*

$$\int_{t_1}^{t_2} \|\Delta u_t(r)\|^2 dr \le C. \tag{4.45}$$

*Proof.* Let $z = (u, u_t, \eta^t)$ be a solution of (1.1) with initial data $z_0 \in B$. Now recasting the proof of Lemma 4.17, we end up with an inequality analogous to (4.43) and $(u, u_t, \eta^t)$ in place of $(w, w_t, \zeta^t)$. Since the initial data belong to $B \in \mathcal{H}_2$, applying the Gronwal lemma, we obtain (4.44). Besides, integrating (4.43) from $t_1$ to $t_2$ and using (4.44) we get (4.45). ☐

We have the following regularity result.

**Theorem 4.19** (Regularity of the global attractor)**.** *Under the assumptions of* **(H1)**–**(H4)** (*with* (1.8) *by* (4.31)) *for the memory term and the nonlinearity, and the assumption of* (4.35)*, the global attractor $\mathcal{A}$ is bounded in $\mathcal{H}_2$.*

Next, we can take a compact set $\mathbb{B}_1 \subset \mathcal{H}_2$, such that $\mathcal{B} = \overline{\cup_{t\ge T_\omega} S(t)\mathbb{B}_1}$ is a compact positive invariant set in $\mathcal{H}_2$ under $S(t)$.

## 5. Exponential attractors

Despite the existence of an exponentially attracting set, quantitative information on the attraction rate of the global attractor is usually very hard to find. To overcome this difficulty, it was introduced in [12] the concept of exponential attractor.

**Definition 5.1.** A compact set $\mathcal{E} \in \mathcal{H}_1$ is called an exponential attractor or inertial set for the semigroup $S(t)$ if the following conditions hold:

(1) $\mathcal{E}$ is positively invariant, i.e., $S(t)\mathcal{E} \subset \mathcal{E}$ for every $t \geq 0$;
(2) $\mathcal{E}$ has finite fractal dimension in $\mathcal{H}_1$;
(3) $\mathcal{E}$ is exponentially attracting for $S(t)$.

Recall that the *fractal dimension* of a compact set $K$ in a metric space $X$ is defined by

$$\dim_X K = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon, K)}{\log(1/\varepsilon)}, \tag{5.1}$$

where $N(\varepsilon, K)$ is the smallest number of balls of radius $\varepsilon$ necessary to cover $K$. The main result of this section is the following.

**Theorem 5.2.** *The semigroup $S(t)$ acting on $\mathcal{H}_1$ possesses an exponential attractor $\mathcal{E}$ contained and bounded in $\mathcal{H}_2$.*

As a byproduct, we have the following.

**Corollary 5.3.** *The global attractor $\mathcal{A}$ of $S(t)$ has a finite fractal dimension in $\mathcal{H}_1$.*

Now, we will make use of the projections $P_1$ and $P_2$ of $\mathcal{H}_1$ onto its components $H_1 \times H_0$ and $L^2_\mu(\mathbb{R}^+, H_1)$, namely

$$P_1(z) = P_1(u, u_t, \eta^t) = (u, u_t), \quad P_2(z) = P_2(u, u_t, \eta^t) = \eta^t.$$

**Lemma 5.4.** *Let the following assumptions hold*:

(1) *There exists $R_\star > 0$ such that the ball $\mathcal{B}_\star = \mathcal{B}_{\mathcal{H}_2}(R_\star)$ is exponentially attracting.*
(2) *There exists $R_1 > 0$ with the following property*: *for any given $R \geq 0$, there exists a nonnegative function $\psi$ vanishing at infinity such that*

$$\|S(t)z_0\|_{\mathcal{H}_2} \leq \psi(t) + R_1$$

*for all $z_0 \in \mathcal{B}(R)$.*

(3) *For every $R \geq 0$ and every $\theta > 0$ sufficiently large,*

$$\int_\theta^{2\theta} \|\partial_t(u(t), \partial_t u(t))\|^2_{H_1 \times H_0} dt \leq \mathcal{Q}(R + \theta)$$

*for all $(u, u_t) = P_1 S(t) z_0$.*

(4) *For every fixed $R \geq 0$, the semigroup $S(t) : \mathcal{B} \to \mathcal{B}$ admits a decomposition of the form $S(t) = S_1(t) + S_2(t)$ satisfying for all initial data $z_{0i} \in \mathcal{B}(R)$,*

$$\|S_1(z_{01}) - S_1(z_{02})\|_{\mathcal{H}_1} \leq \psi(t)\|z_{01} - z_{02}\|_{\mathcal{H}_1},$$

*and*

$$\|S_2(z_{01}) - S_2(z_{02})\|_{\mathcal{H}_2} \leq \mathcal{Q}(t)\|z_{01} - z_{02}\|_{\mathcal{H}_1}$$

*for both $\mathcal{Q}$ and the nonnegative function $\psi$ vanishing at infinity. Moreover, the function*

$$\bar{\eta}^t = P_2 S_2(t) z_{01} - P_2 S_2(t) z_{02}$$

*fulfills the Cauchy problem*

$$\partial_t \bar{\eta}^t = \partial_s \bar{\eta}^t + \bar{w}_t(t), \quad \bar{\eta}^0 = 0$$

*for some* $\bar{w}$ *satisfying the estimate*

$$\|\bar{w}(t)\|_1 \leq \mathcal{Q}(t)\|z_{01} - z_{02}\|_{\mathcal{H}_1}.$$

*Then,* $S(t)$ *possesses an exponential attractor* $\mathcal{E}$ *contained in the ball* $\mathcal{B}(R_1)$*.*

*Proof of Theorem 5.2.* The proof amounts to verifying the four points of the above Lemma 5.4. Indeed, combining (4.35), Lemma 4.16 and Lemma 4.17 we get (1) and (2). Besides, (3) is an immediate consequence of Lemma 4.6. Accordingly, we are left to show the validity of (4).

For every initial data $z_0 = (u_0, v_0, \eta_0) \in \mathcal{B}$, denote $S_1(t)z_0 = z_1(t)$ the solution at time $t$ to the linear homogeneous problem

$$\begin{cases} \partial_{tt} v - \Delta \partial_t v - \Delta v + \lambda v - \displaystyle\int_0^\infty \mu(s) \Delta \xi^t(s) ds = 0, \\ \hfill \partial_t \xi^t = -\partial_s \xi^t + v_t, \\ \hfill (v(0), v_t(0), \xi^0) = z_0, \end{cases}$$

and let

$$S_2 z_0 = S_1(t)z_0 - S(t)z_0 = z_2(t).$$

Let $R \geq 0$ be fixed, and let $z_{01}, z_{02} \in \mathcal{B}$. We decompose the difference

$$\left(\bar{u}(t), \bar{u}_t(t), \bar{\eta}^t\right) = S(t)z_{01} - S(t)z_{02} = \left(\bar{v}(t), \bar{v}_t(t), \bar{\xi}^t\right) + \left(\bar{w}(t), \bar{w}_t(t), \bar{\zeta}^t\right),$$

where

$$(\bar{v}(t), \bar{v}_t(t), \bar{\xi}^t) = S_1(t)z_{01} - S_1(t)z_{02}, \quad (\bar{w}(t), \bar{w}_t(t), \bar{\zeta}^t) = S_2(t)z_{01} - S_2(t)z_{02}$$

solve the problems

$$\begin{cases} \partial_{tt} \bar{v} - \Delta \partial_t \bar{v} - \Delta \bar{v} + \lambda \bar{v} - \displaystyle\int_0^\infty \mu(s) \Delta \bar{\xi}^t(s) ds = 0, \\ \hfill \partial_t \xi^t = \partial_s \xi^t + v_t, \\ \hfill (v(0), v_t(0), \xi^0) = z_{01} - z_{02}, \end{cases}$$

and

$$\begin{cases} \partial_{tt} \bar{w} - \Delta \partial_t \bar{w} - \Delta \bar{w} + \lambda \bar{w} - \displaystyle\int_0^\infty \mu(s) \Delta \bar{\zeta}^t(s) ds \\ + f(x, u_1) - f(x, u_2) + g(x, \partial_t u_1) - g(x, \partial_t u_2) = 0, \\ \partial_t \zeta = \partial_s \zeta + w, \\ (w(0), w_t(0), \zeta^0) = (0, 0, 0). \end{cases} \tag{5.2}$$

We first note that, on account of (2),

$$\|S(t)z_{0i}\|_{\mathcal{H}_2} \leq C.$$

On the other hand, as the particular case of Lemma 4.10, we get

$$\|S_1(t)z_{01} - S_1(t)z_{02}\|_{\mathcal{H}_1} \leq Ce^{-\gamma t}\|z_{01} - z_{02}\|_{\mathcal{H}_1}.$$

Now, for $a \in [0,1)$ to be fixed later, multiplying the first equation of (5.2) by $\bar{w}_t(t) - a\bar{w}(t)$ in $L^2(\mathbb{R}^3)$, and adding to both sides the term

$$\frac{d}{dt}\int_0^\infty \mu(s)\|\bar{\zeta}^t\|^2 ds - 2\int_0^\infty \mu'(s)\|\bar{\zeta}^t\|^2 ds = 2\int_0^\infty \mu(s)\langle\bar{\zeta}^t(s), \bar{w}_t\rangle ds,$$

and as in the proof of Lemma 4.12, we get

$$\|(\bar{w}, \bar{w}_t, \bar{\zeta}^t)\|^2_{\mathcal{H}_1} \leq N_0 \quad \text{for some } N_0 > 0.$$

Next, multiplying the first equation of (5.2) by $-\Delta\bar{w}_t(t) - a\Delta\bar{w}(t)$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\bar{w}_t\|^2 + (1+a)\|\Delta\bar{w}\|^2 + \lambda\|\nabla\bar{w}\|^2 \\
&+ \int_0^\infty \mu(s)\|\Delta\bar{\zeta}^t\|^2 ds + 2a\langle\nabla\bar{w}_t, \nabla\bar{w}\rangle\Big) \\
&- a\|\nabla\bar{w}_t\|^2 + \|\Delta\bar{w}_t\|^2 + a\lambda\|\nabla\bar{w}\|^2 + a\|\Delta\bar{w}\|^2 \\
&+ a\int_0^\infty \mu(s)\langle\Delta\bar{\zeta}^t(s), \Delta\bar{w}\rangle ds - \int_0^\infty \mu'(s)\|\Delta\bar{\zeta}^t(s)\|^2 ds \\
= \;& -\langle f(x,u_1) - f(x,u_2), -\Delta\bar{w}_t - a\Delta\bar{w}\rangle \\
&- \langle g(x,\partial_t u_1) - g(x,\partial_t u_2), -\Delta\bar{w}_t - a\Delta\bar{w}\rangle.
\end{aligned}$$

Due to (1.8) and the Agmon's inequality,

$$\|g(x,\partial_t u_1) - g(x,\partial_t u_2)\| \leq C\|\partial_t u_1 - \partial_t u_2\|.$$

Thus

$$-\langle g(x,\partial_t u_1) - g(x,\partial_t u_2), -\Delta\bar{w}_t - a\Delta\bar{w}\rangle \leq C\|\bar{u}_t\|\|\bar{w}_t + a\bar{w}\|_2.$$

Besides, by (1.6),

$$-\langle f(x,u_1) - f(x,u_2), -\Delta\bar{w}_t - a\Delta\bar{w}\rangle \leq C\|\bar{u}\|_1\|\bar{w}_t + a\bar{w}\|_2.$$

A final application of the Hölder inequality entails

$$\frac{d}{dt}\Lambda(t) \leq \alpha\Lambda(t) + C(\|\bar{u}\|_1^2 + \|\bar{u}_t\|^2),$$

where

$$\Lambda = \|\nabla\bar{w}_t\|^2 + (1+a)\|\Delta\bar{w}\|^2 + \lambda\|\nabla\bar{w}\|^2 + \int_0^\infty \mu(s)\|\Delta\bar{\zeta}^t\|^2 ds + 2a\langle\nabla\bar{w}_t, \nabla\bar{w}\rangle,$$

and $\|(\bar{w}, \bar{w}_t, \bar{\zeta}^t)\|^2_{\mathcal{H}_2} \leq \Lambda \leq 2\|(\bar{w}, \bar{w}_t, \bar{\zeta}^t)\|^2_{\mathcal{H}_2}$.

Arguing as in the proof of (3.22), we obtain

$$\|\bar{u}\|_1^2 + \|\bar{u}_t\|^2 \leq Ce^{Ct}\|z_{01} - z_{02}\|^2_{\mathcal{H}_1}.$$

Since $\Lambda(0) = 0$, an application of the Gronwall lemma provides the sought inequality

$$\begin{aligned}\Lambda(t) &\le C \int_0^t e^{C(t-r)} (\|\bar{u}(r)\|_1^2 + \|\bar{u}_t(r)\|^2) dr \\ &\le C e^{Ct} \|z_{01} - z_{02}\|_{\mathcal{H}_1}^2.\end{aligned}$$

In particular, we learn that

$$\|(\bar{w}, \bar{w}_t, \bar{\zeta}^t)\|_{\mathcal{H}_2}^2 \le C e^{Ct} \|z_{01} - z_{02}\|_{\mathcal{H}_1}^2,$$

which is exactly the last point of (4) to be verified. ☐

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