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ON THE EXISTENCE OF REAL QUADRATIC FIELDS WITH ODD PERIOD OF MINIMAL TYPE

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Abstract. In this paper, under the ABC-conjecture, we show that there exist infinitely many real quadratic fields with odd period of minimal type.

1. Introduction

In [3], Kawamoto and Tomita defined the notion of *real quadratic fields with period ℓ of minimal type* by using the simple continued fraction expansions of certain quadratic irrationals (see Definition in Section 2 below). Following that, they showed that there exist exactly 51 real quadratic fields of class number 1 that are not of minimal type, with one more possible exception ([3, Proposition 4.4]). This is a very interesting result. On the other hand, as for the existence of real quadratic fields of minimal type, the following have been known:

- Only $\mathbb{Q}(\sqrt{5})$ is a real quadratic field with period 1 of minimal type ([3, Example 3.4]).
- There are no real quadratic fields with period 2, 3 of minimal type ([3, Example 3.5]).
- There exist infinitely many real quadratic fields with period ℓ of minimal type for any even $\ell \geq 4$ ([3, Theorem 1.1], [2, Theorems 2, 3]).

Thus, we shall prove the following theorem by considering certain quadratic irrationals.

Theorem 1. *Assume the ABC-conjecture. For each odd integer $\ell\,(\geq 5)$, there exist infinitely many real quadratic fields with period ℓ of minimal type.*

Throughout this paper, we denote the ring of rational integers by $\mathbb{Z}$ and the field of rational numbers by $\mathbb{Q}$, respectively. For any non-negative integer n, let F_n and L_n denote the Fibonacci and Lucas numbers, respectively, which are defined by

$$\begin{cases} F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \ (n \geq 2), \\ L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \ (n \geq 2). \end{cases}$$

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2. Real quadratic fields of minimal type

In this section, we recall the definition of real quadratic fields of minimal type ([3, Theorem 3.1, Definition 3.1]).

Let $a_1, a_2, \ldots, a_{\ell-1}$ be a symmetric sequence of $\ell - 1\,(\geq 1)$ positive integers. From this, we define nonnegative integers q_n and r_n $(0 \leq n \leq \ell)$ by

$$\begin{cases} q_0 = 0, \quad q_1 = 1, \quad q_n = a_{n-1}q_{n-1} + q_{n-2}, \\ r_0 = 1, \quad r_1 = 0, \quad r_n = a_{n-1}r_{n-1} + r_{n-2}, \end{cases} \tag{2.1}$$

inductively. For brevity, we put

$$A := q_\ell, \quad B := q_{\ell-1}, \quad C := r_{\ell-1}, \tag{2.2}$$

and define polynomials $g(x)$, $h(x)$, $f(x)$ by

$$g(x) = Ax - (-1)^\ell BC,\ h(x) = Bx - (-1)^\ell C^2,\ f(x) = g(x)^2 + 4h(x). \tag{2.3}$$

Furthermore, let s_0 be the least integer x for which $g(x) > 0$, that is, $x > (-1)^\ell BC/A$. We consider three cases separately:

(I) $A \equiv 1 \pmod 2$, (II) $(A, C) \equiv (0, 0) \pmod 2$, (III) $(A, C) \equiv (0, 1) \pmod 2$.

When Case (I) or Case (II) occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)/4$ and $a_0 := g(s)/2$. Here, we choose an even integer s in Case (I). Assume that

$$g(s) > a_1, \ldots, a_{\ell-1} \tag{2.4}$$

holds. Then, d and a_0 are positive integers, d is non-square, $a_0 = [\sqrt{d}]$ and the simple continued fraction expansion of $\sqrt{d}$ is

$$\sqrt{d} = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0}] \tag{2.5}$$

with minimal period ℓ. Also, in Case (III), there is no positive integer d such that (2.5) is the simple continued fraction expansion of $\sqrt{d}$.

When Case (I) or Case (III) occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)$ and $a_0 := (g(s) + 1)/2$. Here, we choose an odd integer s in Case (I). Assume that (2.4) holds. Then, d and a_0 are positive integers, d is non-square, $d \equiv 1 \pmod 4$, $a_0 = [(1+\sqrt{d})/2]$ and the simple continued fraction expansion of $(1 + \sqrt{d})/2$ is

$$\frac{1 + \sqrt{d}}{2} = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0 - 1}] \tag{2.6}$$

with minimal period ℓ. Also, in Case (II), there is no positive integer d such that $d \equiv 1 \pmod 4$ and (2.6) is the simple continued fraction expansion of $(1 + \sqrt{d})/2$.

Conversely, let d be a non-square positive integer and put $\omega_d = \sqrt{d}$ or $\omega_d = (1 + \sqrt{d})/2$. Here we assume $d \equiv 1 \pmod 4$ if $\omega_d = (1 + \sqrt{d})/2$. Then it is known that the simple continued fraction expansion is of the form

$$\omega_d = [a_0, \overline{a_1, a_2, \ldots, a_\ell}],$$

where ℓ is the minimal period. Moreover, the sequence $a_1, a_2, \ldots, a_{\ell-1}$ is symmetric. From this, we get the quadratic polynomial $f(x)$ and the integer s_0 as above. Then d becomes uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$, and (2.4) holds. If $d \equiv 1 \pmod 4$ in addition, then the same thing is true for $(1+\sqrt{d})/2$.

Definition ([3, Definition 3.1])**.** Let d be a non-square positive integer. As we stated above, d is uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \ldots, a_{\ell-1}$ of the simple continued fraction expansion of $\sqrt{d}$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)/4$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* $\sqrt{d}$. When $d \equiv 1 \pmod 4$ in addition, d is uniquely of the form $d = f(s)$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \ldots, a_{\ell-1}$ of the simple continued fraction expansion of $(1+\sqrt{d})/2$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* $(1+\sqrt{d})/2$.

Furthermore, for a square-free positive integer $d > 1$, we say that $\mathbb{Q}(\sqrt{d})$ is a *real quadratic field with period* ℓ *of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \pmod 4$, and if d is a positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \pmod 4$.

3. Properties of Fibonacci and Lucas numbers

There are many properties of Fibonacci and Lucas numbers (see, for example, [5]). We list them which we need in the proof of our theorems.

Lemma 1. *For any* $n, m \in \mathbb{Z}$ *with* $n > m > 0$*, we have*

$$F_n = \tfrac{\alpha^n - \beta^n}{\sqrt{5}}, \tag{3.1}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{3.2}$$

$$L_{n+m} - (-1)^m L_{n-m} = 5F_nF_m, \tag{3.3}$$

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \tag{3.4}$$

$$L_n^2 = 5F_n^2 + (-1)^n 4 \tag{3.5}$$

and

$$F_n \equiv 0 \pmod 2 \iff n \equiv 0 \pmod 3, \tag{3.6}$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$*. Moreover, if* m *is even* (*resp.* m *is odd*)*, then we have*

$$\frac{F_{n+1}}{F_n} < \frac{F_{m+1}}{F_m} \quad \left(\textit{resp.}\ \frac{F_{m+1}}{F_m} < \frac{F_{n+1}}{F_n}\right). \tag{3.7}$$

4. Positive integers with odd period of minimal type

Let L be a positive integer with $L \geq 2$ and put $\ell = 2L+1$. The goal of this section is to construct positive integers d with period ℓ of minimal type such that the symmetric part of the simple continued fraction expansion of $\sqrt{d}$ or $(1+\sqrt{d})/2$ is

$$\underbrace{1,\dots,1}_{L-1}, F_L^2 u, F_L^2 u, \underbrace{1,\dots,1}_{L-1}$$

with $u \in \mathbb{Z},\ u > 0$. From this sequence, we get

$$q_n = F_n \ (0 \leq n \leq L), \quad q_{L+1} = F_L^3 u + F_{L-1},$$
$$r_n = F_{n-1} \ (1 \leq n \leq L), \quad r_{L+1} = F_L^2 F_{L-1} u + F_{L-2}$$

by using (2.1). Then by [4, Lemma 2.3], the integers A, B, C defined by (2.2) are given as

$$A = F_L^6 u^2 + 2F_L^3 F_{L-1} u + F_L^2 + F_{L-1}^2,$$
$$B = F_L^5 F_{L-1} u^2 + F_L^2 (F_{L-1}^2 + F_L F_{L-2}) u + F_{L-1}(F_L + F_{L-2}),$$
$$C = F_L^4 F_{L-1}^2 u^2 + 2F_L^2 F_{L-1} F_{L-2} u + F_{L-1}^2 + F_{L-2}^2.$$

Define the polynomials $g(x), h(x)$ and $f(x)$ as (2.3). Then the integer s_0 and the value of $f(s_0)$ are given as follows:

Proposition 1. *Let the notation be as above.*

(1) *If $L = 2$, then*

$$s_0 = -u^2 + u - 1,$$
$$f(s_0) = u^4 + 2u^3 + 3u^2 - 2u + 1.$$

(2) *If $L \geq 3$, then*

$$s_0 = -F_L^3 F_{L-1}^3 u^2 - F_{L-1}^2 (3F_L F_{L-2} - F_{L-1}^2) u - F_{L-2}(2F_{L-1} - F_{L-2}),$$
$$f(s_0) = F_{L+1}^2 F_L^6 u^4 + 2F_{L+1} F_L^3 (F_{L+1}^2 - F_{L+1} F_L + F_L^2) u^3$$
$$+ (8F_{L+1}^2 F_L^2 - 6F_{L+1} F_L^3 + 1) u^2 - 2(F_{L+1}^2 - 5F_{L+1} F_L + F_L^2) u + 5.$$

Proof. (1) Let $L = 2$. Then we have

$$A = u^2 + 2u + 2, \quad B = u^2 + u + 1, \quad C = u^2 + 1.$$

Thus, s_0 is the least integer x for which

$$x > -\frac{(u^2+u+1)(u^2+1)}{u^2+2u+2}.$$

Hence by

$$-\frac{(u^2+u+1)(u^2+1)}{u^2+2u+2} = -\frac{u^4+u^3+2u^2+u+1}{u^2+2u+2} = -u^2+u-2+\frac{u+3}{u^2+2u+2}$$

and

$$0 < \frac{u+3}{u^2+2u+2} < 1,$$

we get $s_0 = -u^2 + u - 1$. From this, moreover, we have

$$g(s_0) = As_0 + BC = u^2 + u - 1,$$
$$h(s_0) = Bs_0 + C^2 = u^2,$$

and

$$f(s_0) = g(s_0)^2 + 4h(s_0) = u^4 + 2u^3 + 3u^2 - 2u + 1.$$

(2) Let $L \geq 3$ and put

$$S := -F_L^3 F_{L-1}^3 u^2 - F_{L-1}^2(3F_L F_{L-2} - F_{L-1}^2)u - F_{L-2}(2F_{L-1} - F_{L-2}).$$

First, we calculate $g(S)$. By straightforward calculations, we obtain

$$g(S) = AS + BC = c_2 u^2 + c_1 u + c_0,$$

where

$$\begin{aligned} c_2 &= F_L^8 - F_L^7 F_{L-1} - 3F_L^6 F_{L-1}^2 + F_L^5 F_{L-1}^3 + 3F_L^4 F_{L-1}^4 + F_L^3 F_{L-1}^5 \\ &= F_L^3(F_L + F_{L-1})(F_L^2 - F_L F_{L-1} - F_{L-1}^2)^2, \\ c_1 &= F_L^6 - F_L^5 F_{L-1} - 2F_L^4 F_{L-1}^2 - F_L^3 F_{L-1}^3 + 2F_L^2 F_{L-1}^4 + 3F_L F_{L-1}^5 + F_{L-1}^6 \\ &= (F_L^2 + F_L F_{L-1} + F_{L-1}^2)(F_L^2 - F_L F_{L-1} - F_{L-1}^2)^2, \\ c_0 &= F_L^4 - 2F_L^3 F_{L-1} - F_L^2 F_{L-1}^2 + 2F_L F_{L-1}^3 + F_{L-1}^4 \\ &= (F_L^2 - F_L F_{L-1} - F_{L-1}^2)^2. \end{aligned}$$

Here we remove F_{L-2} by substituting $F_{L-2} = F_L - F_{L-1}$. Now it follows from (3.2) that

$$\begin{aligned} F_L^2 - F_L F_{L-1} - F_{L-1}^2 &= (F_L - F_{L-1})F_L - F_{L-1}^2 \\ &= F_{L-2}F_L - F_{L-1}^2 = (-1)^{L-1}. \end{aligned} \tag{4.1}$$

Thus, we obtain

$$\begin{aligned} c_2 &= F_L^3(F_L + F_{L-1}), \\ c_1 &= F_L^2 + F_L F_{L-1} + F_{L-1}^2, \\ c_0 &= 1, \end{aligned}$$

and hence,

$$\begin{aligned} g(S) &= F_L^3(F_L + F_{L-1})u^2 + (F_L^2 + F_L F_{L-1} + F_{L-1}^2)u + 1 \\ &> F_L^3(F_L + F_{L-1})u^2 > F_L^2 u. \end{aligned} \tag{4.2}$$

In particular, we have $g(S) > 0$. Next, we calculate $g(S-1)$. Also, straightforward calculations give

$$g(S-1) = A(S-1) + BC = g(S) - A = c_2' u^2 + c_1' u + c_0',$$

where

$$\begin{aligned} c_2' &= F_L^3(F_L + F_{L-1} - F_L^3), \\ c_1' &= F_L^2 + F_L F_{L-1} + F_{L-1}^2 - 2F_L^3 F_{L-1}, \end{aligned}$$

$$c'_0 = 1 - F_L^2 - F_{L-1}^2.$$

Noting that $L \geq 3$, we can easily verify that all c'_i are negative. Let us explain that $c'_1 < 0$ holds for example. Since $L \geq 3$, we have $F_L > F_{L-1} \geq 1$ and $F_L \geq 2$. Then we have

$$\begin{aligned} c'_1 &= F_L^2 + F_L F_{L-1} + F_{L-1}^2 - 2F_L^3 F_{L-1} \\ &< F_L^2 + F_L^2 + F_L^2 - 2F_L^3 = F_L^2(3 - 2F_L) < 0. \end{aligned}$$

Thus, we have $g(S-1) < 0$. Therefore, we get

$$s_0 = S = -F_L^3 F_{L-1}^3 u^2 - F_{L-1}^2(3F_L F_{L-2} - F_{L-1}^2)u - F_{L-2}(2F_{L-1} - F_{L-2}).$$

Hence by $F_{L-2} = F_L - F_{L-1}$, $F_{L-1} = F_{L+1} - F_L$ and (4.1), we obtain

$$\begin{aligned} g(s_0) &= F_L^3 F_{L+1} u^2 + (F_{L+1}^2 - F_{L+1}F_L + F_L^2)u + 1, \\ h(s_0) &= F_L^2 F_{L+1}(F_{L+1} - F_L)u^2 + (-F_{L+1}^2 + 3F_{L+1}F_L - F_L^2)u + 1, \end{aligned}$$

and

$$\begin{aligned} f(s_0) = {} & F_{L+1}^2 F_L^6 u^4 + 2F_{L+1}F_L^3(F_{L+1}^2 - F_{L+1}F_L + F_L^2)u^3 \\ & + (8F_{L+1}^2 F_L^2 - 6F_{L+1}F_L^3 + 1)u^2 - 2(F_{L+1}^2 - 5F_{L+1}F_L + F_L^2)u + 5. \end{aligned}$$

Proposition 1 is now proved. □

First, we consider the case $L = 2$. In this case, we have

$$\begin{aligned} A &\equiv u \pmod 2, \\ C &\equiv u + 1 \pmod 2, \\ s_0 &\equiv 1 \pmod 2, \end{aligned}$$

and hence, Case (III) occurs (resp. Case (I) occurs and s_0 is odd) if u is even (resp. u is odd). Moreover, if $u \geq 2$, then we have

$$g(s_0) = u^2 + u - 1 > u = F_2^2 u > 1.$$

Next, we consider the case $L \geq 3$. By using (3.6), we have

$$(F_L, F_{L-1}, F_{L-2}) \equiv \begin{cases} (0,1,1) \pmod 2 & \text{if } L \equiv 0 \pmod 3, \\ (1,0,1) \pmod 2 & \text{if } L \equiv 1 \pmod 3, \\ (1,1,0) \pmod 2 & \text{if } L \equiv 2 \pmod 3. \end{cases}$$

Then we see that

$$A \equiv 1 \pmod 2 \iff \begin{cases} L \equiv 0 \pmod 3 \\ \text{or “}L \equiv 1 \pmod 3,\ u : \text{even”} \\ \text{or “}L \equiv 2 \pmod 3,\ u : \text{odd”}, \end{cases}$$

$$(A, C) \equiv (0,0) \pmod 2 \quad \text{does not occur},$$

$$(A, C) \equiv (0,1) \pmod 2 \iff \begin{cases} \text{“}L \equiv 1 \pmod 3,\ u : \text{odd”} \\ \text{or “}L \equiv 2 \pmod 3,\ u : \text{even”} \end{cases}$$

and

$$s_0 \equiv \begin{cases} 0 \pmod 2 & \text{if "}L \equiv 0 \pmod 3,\ u:\text{odd" or "}L \equiv 2 \pmod 3,\ u:\text{odd"},\\ 1 \pmod 2 & \text{if "}L \equiv 0 \pmod 3,\ u:\text{even" or "}L \equiv 1 \pmod 3,\ u:\text{even"}.\end{cases}$$

The following table summarizes the above:

	$L \equiv 0 \pmod 3$	$L \equiv 1 \pmod 3$	$L \equiv 2 \pmod 3$
u : even	Case (I), s_0 : odd	Case (I), s_0 : odd	Case (III)
u : odd	Case (I), s_0 : even	Case (III)	Case (I), s_0 : even

Moreover, by (4.2), we have

$$g(s_0) > F_L^2 u > 1.$$

Thus, it follows from what has been stated in Section 2 that the following holds:

Theorem 2. (1) *For a positive integer u, put $d := u^4 + 2u^3 + 3u^2 - 2u + 1$. If $u \geq 2$, then d is a positive integer with period 5 of minimal type for $(1+\sqrt{d})/2$ and the continued fraction expansion of $(1+\sqrt{d})/2$ is of the form*

$$(1+\sqrt{d})/2 = [a_0, \overline{1, u, u, 1, 2a_0 - 1}],$$

where $a_0 = (u^2 + u)/2$.

(2) *Let $L \geq 3$. For a positive integer u, put*

$$\begin{aligned} d := {} & F_{L+1}^2 F_L^6 u^4 + 2F_{L+1}F_L^3(F_{L+1}^2 - F_{L+1}F_L + F_L^2)u^3 \\ & + (8F_{L+1}^2F_L^2 - 6F_{L+1}F_L^3 + 1)u^2 - 2(F_{L+1}^2 - 5F_{L+1}F_L + F_L^2)u + 5. \end{aligned}$$

If either $u \equiv 0 \pmod 2$ or "$u \equiv 1 \pmod 2$ and $L \equiv 1 \pmod 3$", then d is a positive integer with period $2L+1$ of minimal type for $(1+\sqrt{d})/2$ and the continued fraction expansion of $(1+\sqrt{d})/2$ is of the form

$$(1+\sqrt{d})/2 = [a_0, \overline{\underbrace{1, \ldots, 1}_{L-1}, F_L^2 u, F_L^2 u, \underbrace{1, \ldots, 1}_{L-1}, 2a_0 - 1}],$$

where $a_0 = \{F_L^3 F_{L+1} u^2 + (F_{L+1}^2 - F_{L+1}F_L + F_L^2)u + 2\}/2$. Similarly, for a positive integer u, put

$$\begin{aligned} d := {} & \{F_{L+1}^2 F_L^6 u^4 + 2F_{L+1}F_L^3(F_{L+1}^2 - F_{L+1}F_L + F_L^2)u^3 \\ & + (8F_{L+1}^2F_L^2 - 6F_{L+1}F_L^3 + 1)u^2 - 2(F_{L+1}^2 - 5F_{L+1}F_L + F_L^2)u + 5\}/4. \end{aligned}$$

If $u \equiv 1 \pmod 2$ and $L \equiv 0, 2 \pmod 3$, then d is a positive integer with period $2L+1$ of minimal type for $\sqrt{d}$ and the continued fraction expansion of $\sqrt{d}$ is of the form

$$\sqrt{d} = [a_0, \overline{\underbrace{1, \ldots, 1}_{L-1}, F_L^2 u, F_L^2 u, \underbrace{1, \ldots, 1}_{L-1}, 2a_0}],$$

where $a_0 = \{F_L^3 F_{L+1} u^2 + (F_{L+1}^2 - F_{L+1}F_L + F_L^2)u + 1\}/2$.

5. Proof of the main theorem

In this section, we shall give a proof of Theorem 1. This is obtained as a consequence of a theorem of Granville.

ABC-conjecture. *Let a, b, c be coprime positive integers satisfying $a + b = c$. Then for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$c < C_\varepsilon N(a, b, c)^{1+\varepsilon},$$

where $N(a, b, c)$ is the product of the distinct primes dividing abc.

Theorem 3 ([1, Theorem 1]). *Suppose that $\varphi(X) \in \mathbb{Z}[X]$, without any repeated roots. Let κ be the largest integer which divides $\varphi(n)$ for all integers n, and select κ' to be the smallest divisor of κ for which κ/κ' is square-free. If the ABC-conjecture is true, then there are $\sim c_\varphi N$ positive integers $n \leq N$ for which $\varphi(n)/\kappa'$ is square-free, where c_φ is a certain positive constant.*

5.1. The case $\ell = 5$

From Theorem 2(1), it is sufficient to show that there are infinitely many integers $u\,(\geq 2)$ for which $u^4 + 2u^3 + 3u^2 - 2u + 1$ is square-free. To prove this, let us apply Theorem 3 to

$$\varphi(X) := X^4 + 2X^3 + 3X^2 - 2X + 1.$$

Since the discriminant of $\varphi(X)$ is 4352, $\varphi(X)$ does not have repeated roots. Since $\varphi(0) = 1$, it holds $\kappa = 1$. Thus, we can take $\kappa' = 1$ and hence, there are infinitely many positive integers n for which $\varphi(n)$ is square-free, as desired.

5.2. The case $\ell \geq 7$

Let $L \geq 3$. It follows from Theorem 2(2) that for a positive integer u',

$$\begin{aligned} d := {} & F_{L+1}^2 F_L^6 (2u')^4 + 2F_{L+1}F_L^3(F_{L+1}^2 - F_{L+1}F_L + F_L^2)(2u')^3 \\ & + (8F_{L+1}^2F_L^2 - 6F_{L+1}F_L^3 + 1)(2u')^2 \\ & - 2(F_{L+1}^2 - 5F_{L+1}F_L + F_L^2)(2u') + 5 \end{aligned}$$

is a positive integer with period $\ell = 2L + 1$ of minimal type for $(1 + \sqrt{d})/2$. Thus, as in Subsection 5.1, let us apply Theorem 3 to

$$\begin{aligned} \varphi(X) := {} & 2^4 F_{L+1}^2 F_L^6 X^4 + 2^4 F_{L+1}F_L^3(F_{L+1}^2 - F_{L+1}F_L + F_L^2)X^3 \\ & + 2^2(8F_{L+1}^2F_L^2 - 6F_{L+1}F_L^3 + 1)X^2 \\ & - 2^2(F_{L+1}^2 - 5F_{L+1}F_L + F_L^2)X + 5. \end{aligned}$$

Now we put $a := F_{L+1}$, $b := F_L$ for brevity. Then the discriminant $disc(\varphi)$ of $\varphi(X)$ is

$$\begin{aligned} disc(\varphi) = {} & 2^{16}a^2b^6\{16b^3a^{13} - 224b^4a^{12} + 336b^5a^{11} + (1520b^6 + 16b^2)a^{10} \\ & + (-1696b^7 - 312b^3)a^9 + (-4672b^8 + 800b^4 + 1)a^8 \end{aligned}$$

$$
\begin{aligned}
&+ (1872b^9 + 1360b^5 - 12b)a^7 + (6864b^{10} - 2768b^6 - 94b^2)a^6 \\
&+ (1312b^{11} - 2984b^7 + 592b^3)a^5 \\
&+ (-3456b^{12} + 2208b^8 + 75b^4 - 6)a^4 \\
&+ (-2160b^{13} + 2704b^9 - 1040b^5 + 20b)a^3 \\
&+ (-336b^{14} + 400b^{10} - 174b^6 + 118b^2)a^2 \\
&+ (16b^{15} - 120b^{11} + 196b^7 - 100b^3)a + (b^8 - 6b^4 + 5)\}.
\end{aligned}
$$

We have $disc(\varphi) > 0$, which will be proved in the next subsection. Therefore, $\varphi(X)$ does not have repeated roots. Moreover, we have $\varphi(2) \not\equiv 0 \pmod 5$ for any L. Indeed, this follows from the following table:

$L \pmod{20}$	0	1	2	3	4	5	6
$(F_L, F_{L+1}) \pmod 5$	$(0,1)$	$(1,1)$	$(1,2)$	$(2,3)$	$(3,0)$	$(0,3)$	$(3,3)$
$\varphi(2) \pmod 5$	3	1	3	1	4	4	2
$L \pmod{20}$	7	8	9	10	11	12	13
$(F_L, F_{L+1}) \pmod 5$	$(3,1)$	$(1,4)$	$(4,0)$	$(0,4)$	$(4,4)$	$(4,3)$	$(3,2)$
$\varphi(2) \pmod 5$	2	1	3	3	1	3	1
$L \pmod{20}$	14	15	16	17	18	19	
$(F_L, F_{L+1}) \pmod 5$	$(2,0)$	$(0,2)$	$(2,2)$	$(2,4)$	$(4,1)$	$(1,0)$	
$\varphi(2) \pmod 5$	4	4	2	2	1	3	

From this, together with $\varphi(0) = 5$, it also holds $\kappa = 1$, and hence, we can take $\kappa' = 1$. Thus, there are infinitely many positive integers u' for which $d = \varphi(u')$ is square-free, as desired. The proof of Theorem 1 is complete provided $disc(\varphi) > 0$ for any $L\,(\geq 3)$.

5.3. The positivity of the discriminant

The goal of this section is to prove the following:

Proposition 2. *Under the notations of Subsection 5.2, we have $disc(\varphi) > 0$ for any $L\,(\geq 3)$.*

Proof. Put $D(L) := disc(\varphi)/(2^{16}a^2b^6)$. In case of $3 \leq L \leq 7$, we can verify

$$
\begin{aligned}
D(3) &= 26920512 > 0, \\
D(4) &= 8102250000 > 0, \\
D(5) &= 2684459417600 > 0, \\
D(6) &= 855360599155712 > 0, \\
D(7) &= 276546455581228560 > 0
\end{aligned}
$$

by straightforward calculations.

In the following, we consider the case $L \geq 8$. Now let us split $D(L)$ into five polynomials:

$$D(L) = f_1(L) + f_2(L) + g_1(L) + g_2(L) + h(L),$$

where

$$\begin{aligned} f_1(L) &= 16b^3a^{13} + 336b^5a^{11} - 1696b^7a^9 + 1872b^9a^7 + 1312b^{11}a^5 \\ &\quad - 2160b^{13}a^3 + 16b^{15}a, \\ f_2(L) &= -224b^4a^{12} + 1520b^6a^{10} - 4672b^8a^8 + 6864b^{10}a^6 - 3456b^{12}a^4 \\ &\quad - 336b^{14}a^2, \\ g_1(L) &= 16b^2a^{10} + 800b^4a^8 - 2768b^6a^6 + 2208b^8a^4 + 400b^{10}a^2, \\ g_2(L) &= -312b^3a^9 + 1360b^5a^7 - 2984b^7a^5 + 2704b^9a^3 - 120b^{11}a, \\ h(L) &= a^8 - 12ba^7 - 94b^2a^6 + 592b^3a^5 + 75b^4a^4 - 1040b^5a^3 - 174b^6a^2 \\ &\quad + 196b^7a + b^8 - 6a^4 + 20ba^3 + 118b^2a^2 - 100b^3a - 6b^4 + 5. \end{aligned}$$

Since $L \geq 8$, it follows from (3.7) that

$$\frac{21}{13} = \frac{F_8}{F_7} < \frac{a}{b} \leq \frac{F_9}{F_8} = \frac{34}{21},$$

and hence,

$$\frac{21}{13}b < a \leq \frac{34}{21}b. \tag{5.1}$$

Therefore, we obtain

$$\begin{aligned} g_1(L) &> 16b^2\left(\frac{21}{13}b\right)^{10} + 800b^4\left(\frac{21}{13}b\right)^8 - 2768b^6\left(\frac{34}{21}b\right)^6 + 2208b^8\left(\frac{21}{13}b\right)^4 \\ &\quad + 400b^{10}\left(\frac{21}{13}b\right)^2 \\ &= \frac{6209030667413587 7152096}{11823588092798847729}b^{12} \\ &> 0, \\ h(L) &> \left(\frac{21}{13}b\right)^8 - 12b\left(\frac{34}{21}b\right)^7 - 94b^2\left(\frac{34}{21}b\right)^6 + 592b^3\left(\frac{21}{13}b\right)^5 \\ &\quad + 75b^4\left(\frac{21}{13}b\right)^4 - 1040b^5\left(\frac{34}{21}b\right)^3 - 174b^6\left(\frac{34}{21}b\right)^2 + 196b^7\left(\frac{21}{13}b\right) \\ &\quad + b^8 - 6\left(\frac{34}{21}b\right)^4 + 20b\left(\frac{21}{13}b\right)^3 + 118b^2\left(\frac{21}{13}b\right)^2 - 100b^3\left(\frac{34}{21}b\right) \\ &\quad - 6b^4 + 5 \\ &= \frac{231902351941769392279}{489734418044922687}b^8 + \frac{26076650980}{142424919}b^4 + 5 \\ &> 0. \end{aligned}$$

From now on, we shall prove $f_1(L)+f_2(L)+g_2(L)>0$. By taking $n=L+1$ and $m=L$ in (3.3), we have

$$5ab = 5F_{L+1}F_L = L_{2L+1}-(-1)^L L_1 = L_{2L+1}-(-1)^L. \tag{5.2}$$

Moreover, by taking $n=L$ in (3.4), we have

$$a^2+b^2 = F_{2L+1}. \tag{5.3}$$

Furthermore, by taking $n=2L+1$ in (3.5), we have

$$L_{2L+1}^2 = 5F_{2L+1}^2 + (-1)^{2L+1}4 = 5F_{2L+1}^2 - 4. \tag{5.4}$$

Here, it follows from (5.3) that

$$\begin{aligned}(5F_{2L+1}^2-4)-\left(\sqrt{5}F_{2L+1}-\frac{1}{b^2}\right)^2 &= \frac{2\sqrt{5}F_{2L+1}}{b^2}-\frac{1}{b^4}-4\\ &= \frac{2\sqrt{5}(a^2+b^2)b^2-1-4b^4}{b^4}\\ &= \frac{(2\sqrt{5}-4)b^4+(2\sqrt{5}a^2b^2-1)}{b^4}\\ &> 0.\end{aligned}$$

From this, together with (5.4), we have

$$L_{2L+1} > \sqrt{5}F_{2L+1}-\frac{1}{b^2}. \tag{5.5}$$

Hence by (5.2), (5.3), (5.5), we get

$$\begin{aligned}ab &= \frac{1}{5}(L_{2L+1}-(-1)^L) > \frac{1}{5}(\sqrt{5}F_{2L+1}-\frac{1}{b^2}-(-1)^L)\\ &= \frac{1}{5}(\sqrt{5}(a^2+b^2)-\frac{1}{b^2}-(-1)^L).\end{aligned} \tag{5.6}$$

By putting

$$t(L) := (f_1(L)+g_2(L))/(ab),$$

it holds from (5.1) and $b>2$ that

$$\begin{aligned}t(L) &= 16b^2a^{12}+336b^4a^{10}-1696b^6a^8+1872b^8a^6+1312b^{10}a^4-2160b^{12}a^2\\ &\quad +16b^{14}-312b^2a^8+1360b^4a^6-2984b^6a^4+2704b^8a^2-120b^{10}\\ &> 16b^2\left(\frac{21}{13}b\right)^{12}+336b^4\left(\frac{21}{13}b\right)^{10}-1696b^6\left(\frac{34}{21}b\right)^8+1872b^8\left(\frac{21}{13}b\right)^6\\ &\quad +1312b^{10}\left(\frac{21}{13}b\right)^4-2160b^{12}\left(\frac{34}{21}b\right)^2+16b^{14}-312b^2\left(\frac{34}{21}b\right)^8\\ &\quad +1360b^4\left(\frac{21}{13}b\right)^6-2984b^6\left(\frac{34}{21}b\right)^4+2704b^8\left(\frac{21}{13}b\right)^2-120b^{10}\\ &= \frac{1920935609759164060943190560}{881200196968205322394641}b^{14}-\frac{2515414670164125966680}{60854572656469683}b^{10}\end{aligned}$$

$$> 2179b^{14} - 4134b^{10}$$
$$> 0.$$

Then by (5.6), we obtain

$$\text{(5.7)} \quad f_1(L) + f_2(L) + g_2(L) = f_2(L) + abt(L)$$
$$> f_2(L) + \frac{1}{5}(\sqrt{5}(a^2 + b^2) - \frac{1}{b^2} - (-1)^L)t(L).$$

Now we consider the value of $a^2 - \alpha^2 b^2$ by using (3.1). Noting $\alpha\beta = -1$, $1 + \alpha^2 = (5 + \sqrt{5})/2$ and $\alpha^2 - \beta^2 = \sqrt{5}$, we have

$$\begin{aligned}
a^2 - \alpha^2 b^2 &= \left(\frac{\alpha^{L+1} - \beta^{L+1}}{\sqrt{5}}\right)^2 - \alpha^2 \left(\frac{\alpha^L - \beta^L}{\sqrt{5}}\right)^2 \\
&= \frac{-2(-1)^{L+1} + \beta^{2L+2} + 2(-1)^L\alpha^2 - \alpha^2\beta^{2L}}{5} \\
&= (-1)^L \frac{2}{5}(1 + \alpha^2) - \frac{\beta^{2L}}{5}(\alpha^2 - \beta^2) \\
&= (-1)^L \frac{2}{5} \cdot \frac{5 + \sqrt{5}}{2} - \frac{\beta^{2L}}{5}\sqrt{5} \\
&= (-1)^L \left(1 + \frac{1}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}\alpha^{2L}}.
\end{aligned}$$

Therefore, by putting $e := 1/(\sqrt{5}\alpha^{2L}) > 0$, it holds that

$$a^2 = \alpha^2 b^2 + (-1)^L(1 + \frac{1}{\sqrt{5}}) - e.$$

By substituting this into the right hand side of (5.7) and arranging the terms in descending powers of b, we get

$$\begin{aligned}
&f_2(L) + \frac{1}{5}(\sqrt{5}(a^2 + b^2) - \frac{1}{b^2} - (-1)^L)t(L) \\
= &- 224b^4a^{12} + 1520b^6a^{10} - 4672b^8a^8 + 6864b^{10}a^6 - 3456b^{12}a^4 - 336b^{14}a^2 \\
&+ \frac{1}{5}(\sqrt{5}(a^2 + b^2) - \frac{1}{b^2} - (-1)^L)(16b^2a^{12} + 336b^4a^{10} - 1696b^6a^8 \\
&+ 1872b^8a^6 + 1312b^{10}a^4 - 2160b^{12}a^2 + 16b^{14} - 312b^2a^8 + 1360b^4a^6 \\
&- 2984b^6a^4 + 2704b^8a^2 - 120b^{10}) \\
= &\ c''_{14}b^{14} + c''_{12}b^{12} + c''_{10}b^{10} + c''_8b^8 + c''_6b^6 + c''_4b^4 + c''_2b^2 + c''_0,
\end{aligned}$$

where

$$\begin{aligned}
c''_{14} &= (-160\sqrt{5} - 160)e, \\
c''_{12} &= (1672\sqrt{5} + 3800)e^2 + (-1)^L(-4288\sqrt{5} - 9792)e + (1248\sqrt{5} + 2656), \\
c''_{10} &= (-2168\sqrt{5} - 5704)e^3 + (-1)^L(46128/5\sqrt{5} + 21616)e^2 \\
&\quad + (-9024\sqrt{5} - 98944/5)e + (-1)^L(576\sqrt{5} + 4032/5),
\end{aligned}$$

$$\begin{aligned}
c_8'' &= (7296\sqrt{5}/5 + 1728)e^4 + (-1)^L(-33168\sqrt{5}/5 - 57408/5)e^3 \\
&\quad + (217588\sqrt{5}/25 + 84548/5)e^2 \\
&\quad + (-1)^L(-16144\sqrt{5}/25 - 11696/25)e \\
&\quad + (-81056\sqrt{5}/25 - 188544/25), \\
c_6'' &= (376\sqrt{5}/5 - 1064)e^5 + (-1)^L(448\sqrt{5} + 23056/5)e^4 \\
&\quad + (-7264\sqrt{5}/25 - 25608/5)e^3 \\
&\quad + (-1)^L(-114696\sqrt{5}/25 - 175984/25)e^2 \\
&\quad + (1050656\sqrt{5}/125 + 441536/25)e \\
&\quad + (-1)^L(-505808\sqrt{5}/125 - 223088/25), \\
c_4'' &= (104\sqrt{5} - 168)e^6 + (-1)^L(-2064\sqrt{5}/5 + 480)e^5 \\
&\quad + (1108\sqrt{5}/5 - 7724/5)e^4 \\
&\quad + (-1)^L(52016\sqrt{5}/25 + 32032/5)e^3 \\
&\quad + (-134064\sqrt{5}/25 - 315704/25)e^2 \\
&\quad + (-1)^L(609584\sqrt{5}/125 + 271216/25)e \\
&\quad + (-186576\sqrt{5}/125 - 81968/25), \\
c_2'' &= (-16\sqrt{5}/5)e^7 + (-1)^L(112\sqrt{5}/5 + 96/5)e^6 + (-24\sqrt{5}/5 - 96/5)e^5 \\
&\quad + (-1)^L(-296\sqrt{5} - 2384/5)e^4 + (21584\sqrt{5}/25 + 9024/5)e^3 \\
&\quad + (-1)^L(-25296\sqrt{5}/25 - 292944/125)e^2 \\
&\quad + (286944\sqrt{5}/625 + 139808/125)e \\
&\quad + (-1)^L(-16064\sqrt{5}/625 - 9792/125), \\
c_0'' &= (-16/5)e^6 + (-1)^L(96\sqrt{5}/25 + 96/5)e^5 + (-96\sqrt{5}/5 + 24/5)e^4 \\
&\quad + (-1)^L(-224\sqrt{5}/25 - 736/5)e^3 + (2592\sqrt{5}/25 + 8544/25)e^2 \\
&\quad + (-1)^L(-82944\sqrt{5}/625 - 8448/25)e + (33344\sqrt{5}/625 + 78144/625).
\end{aligned}$$

We remark that by $L \geq 8$ and $[\sqrt{5}\alpha^{16}] = 4935$, we have

$$0 < e = \frac{1}{\sqrt{5}\alpha^{2L}} < \frac{1}{\sqrt{5}\alpha^{16}} < \frac{1}{4935}. \tag{5.8}$$

(i) Suppose that L is even. Then by (5.8), we have

$$\begin{aligned}
c_{12}'' &= (1672\sqrt{5} + 3800)e^2 + (-4288\sqrt{5} - 9792)e + (1248\sqrt{5} + 2656) \\
&> 0 + (-4) + (1248\sqrt{5} + 2656) \\
&= 1248\sqrt{5} + 2652,
\end{aligned}$$

$$
\begin{aligned}
c''_{10} &= (-2168\sqrt{5} - 5704)e^3 + (46128/5\sqrt{5} + 21616)e^2 \\
&\quad + (-9024\sqrt{5} - 98944/5)e + (576\sqrt{5} + 4032/5) \\
&> (-1) + 0 + (-9) + (576\sqrt{5} + 4032/5) \\
&> 0, \\
c''_8 &= (7296\sqrt{5}/5 + 1728)e^4 + (-33168\sqrt{5}/5 - 57408/5)e^3 \\
&\quad + (217588\sqrt{5}/25 + 84548/5)e^2 + (-16144\sqrt{5}/25 - 11696/25)e \\
&\quad + (-81056\sqrt{5}/25 - 188544/25) \\
&> 0 + (-1) + 0 + (-1) + (-81056\sqrt{5}/25 - 188544/25) \\
&= -81056\sqrt{5}/25 - 188594/25, \\
c''_6 &= (376\sqrt{5}/5 - 1064)e^5 + (448\sqrt{5} + 23056/5)e^4 \\
&\quad + (-7264\sqrt{5}/25 - 25608/5)e^3 + (-114696\sqrt{5}/25 - 175984/25)e^2 \\
&\quad + (1050656\sqrt{5}/125 + 441536/25)e + (-505808\sqrt{5}/125 - 223088/25) \\
&> (-1) + 0 + (-1) + (-1) + 0 + (-505808\sqrt{5}/125 - 223088/25) \\
&= -505808\sqrt{5}/125 - 223163/25, \\
c''_4 &= (104\sqrt{5} - 168)e^6 + (-2064\sqrt{5}/5 + 480)e^5 + (1108\sqrt{5}/5 - 7724/5)e^4 \\
&\quad + (52016\sqrt{5}/25 + 32032/5)e^3 + (-134064\sqrt{5}/25 - 315704/25)e^2 \\
&\quad + (609584\sqrt{5}/125 + 271216/25)e + (-186576\sqrt{5}/125 - 81968/25) \\
&> 0 + (-1) + (-1) + 0 + (-1) + 0 + (-186576\sqrt{5}/125 - 81968/25) \\
&= -186576\sqrt{5}/125 - 82043/25, \\
c''_2 &= (-16\sqrt{5}/5)e^7 + (112\sqrt{5}/5 + 96/5)e^6 + (-24\sqrt{5}/5 - 96/5)e^5 \\
&\quad + (-296\sqrt{5} - 2384/5)e^4 + (21584\sqrt{5}/25 + 9024/5)e^3 \\
&\quad + (-25296\sqrt{5}/25 - 292944/125)e^2 + (286944\sqrt{5}/625 + 139808/125)e \\
&\quad + (-16064\sqrt{5}/625 - 9792/125) \\
&> (-1) + 0 + (-1) + (-1) + 0 + (-1) + 0 + (-16064\sqrt{5}/625 - 9792/125) \\
&= -16064\sqrt{5}/625 - 10292/125, \\
c''_0 &= (-16/5)e^6 + (96\sqrt{5}/25 + 96/5)e^5 + (-96\sqrt{5}/5 + 24/5)e^4 \\
&\quad + (-224\sqrt{5}/25 - 736/5)e^3 + (2592\sqrt{5}/25 + 8544/25)e^2 \\
&\quad + (-82944\sqrt{5}/625 - 8448/25)e + (33344\sqrt{5}/625 + 78144/625) \\
&> (-1) + 0 + (-1) + (-1) + 0 + (-1) + (33344\sqrt{5}/625 + 78144/625) \\
&> 0.
\end{aligned}
$$

Since $\beta^L > 0$ by $2 \mid L$, moreover, we have

$$eb^2 = \frac{1}{\sqrt{5}\alpha^{2L}} \left(\frac{\alpha^L - \beta^L}{\sqrt{5}} \right)^2 < \frac{1}{\sqrt{5}\alpha^{2L}} \left(\frac{\alpha^L}{\sqrt{5}} \right)^2 = \frac{1}{5\sqrt{5}}.$$

Therefore, by noting $b \geq 21$, we obtain

$$\begin{aligned}
(5.9)\quad & f_2(L) + \frac{1}{5}(\sqrt{5}(a^2+b^2) - \frac{1}{b^2} - 1)t(L) \\
&> (-160\sqrt{5} - 160)eb^{14} + (1248\sqrt{5} + 2652)b^{12} \\
&\quad + (-81056\sqrt{5}/25 - 188594/25)b^8 + (-505808\sqrt{5}/125 - 223163/25)b^6 \\
&\quad + (-186576\sqrt{5}/125 - 82043/25)b^4 + (-16064\sqrt{5}/625 - 10292/125)b^2 \\
&> (-160\sqrt{5} - 160)b^{12} \cdot \frac{1}{5\sqrt{5}} + (1248\sqrt{5} + 2652)b^{12} \\
&\quad + (-81056\sqrt{5}/25 - 188594/25)b^8 + (-505808\sqrt{5}/125 - 223163/25)b^6 \\
&\quad + (-186576\sqrt{5}/125 - 82043/25)b^4 + (-16064\sqrt{5}/625 - 10292/125)b^2 \\
&= (6208\sqrt{5}/5 + 2620)b^{12} \\
&\quad + (-81056\sqrt{5}/25 - 188594/25)b^8 + (-505808\sqrt{5}/125 - 223163/25)b^6 \\
&\quad + (-186576\sqrt{5}/125 - 82043/25)b^4 + (-16064\sqrt{5}/625 - 10292/125)b^2 \\
&> (6208\sqrt{5}/5 + 2620)b^8 \cdot 21^4 \\
&\quad + (-81056\sqrt{5}/25 - 188594/25)b^8 + (-505808\sqrt{5}/125 - 223163/25)b^8 \\
&\quad + (-186576\sqrt{5}/125 - 82043/25)b^8 + (-16064\sqrt{5}/625 - 10292/125)b^8 \\
&= (150911751616\sqrt{5}/625 + 63690048208/125)b^8 \\
&> 0.
\end{aligned}$$

Then by (5.7) and (5.9), we get $f_1(L) + f_2(L) + g_2(L) > 0$.

(ii) Suppose that L is odd. Then again by (5.8), we have

$$\begin{aligned}
c''_{12} &= (1672\sqrt{5} + 3800)e^2 + (4288\sqrt{5} + 9792)e + (1248\sqrt{5} + 2656) \\
&> 0 + 0 + (1248\sqrt{5} + 2656) \\
&= 1248\sqrt{5} + 2656, \\
c''_{10} &= (-2168\sqrt{5} - 5704)e^3 + (-46128/5\sqrt{5} - 21616)e^2 \\
&\quad + (-9024\sqrt{5} - 98944/5)e + (-576\sqrt{5} - 4032/5) \\
&> (-1) + (-1) + (-9) + (-576\sqrt{5} - 4032/5) \\
&= -576\sqrt{5} - 4087/5, \\
c''_8 &= (7296\sqrt{5}/5 + 1728)e^4 + (33168\sqrt{5}/5 + 57408/5)e^3 \\
&\quad + (217588\sqrt{5}/25 + 84548/5)e^2 + (16144\sqrt{5}/25 + 11696/25)e
\end{aligned}$$

$$
\begin{aligned}
&\quad + (-81056\sqrt{5}/25 - 188544/25) \\
&> 0 + 0 + 0 + 0 + (-81056\sqrt{5}/25 - 188544/25) \\
&> -81056\sqrt{5}/25 - 188544/25, \\
c_6'' &= (376\sqrt{5}/5 - 1064)e^5 + (-448\sqrt{5} - 23056/5)e^4 \\
&\quad + (-7264\sqrt{5}/25 - 25608/5)e^3 + (114696\sqrt{5}/25 + 175984/25)e^2 \\
&\quad + (1050656\sqrt{5}/125 + 441536/25)e + (505808\sqrt{5}/125 + 223088/25) \\
&> (-1) + (-1) + (-1) + 0 + 0 + (505808\sqrt{5}/125 + 223088/25) \\
&> 0, \\
c_4'' &= (104\sqrt{5} - 168)e^6 + (2064\sqrt{5}/5 - 480)e^5 + (1108\sqrt{5}/5 - 7724/5)e^4 \\
&\quad + (-52016\sqrt{5}/25 - 32032/5)e^3 + (-134064\sqrt{5}/25 - 315704/25)e^2 \\
&\quad + (-609584\sqrt{5}/125 - 271216/25)e + (-186576\sqrt{5}/125 - 81968/25) \\
&> (-1) + 0 + (-1) + (-1) + (-1) + (-5) + (-186576\sqrt{5}/125 - 81968/25) \\
&> -186576\sqrt{5}/125 - 82193/25, \\
c_2'' &= (-16\sqrt{5}/5)e^7 + (-112\sqrt{5}/5 - 96/5)e^6 + (-24\sqrt{5}/5 - 96/5)e^5 \\
&\quad + (296\sqrt{5} + 2384/5)e^4 + (21584\sqrt{5}/25 + 9024/5)e^3 \\
&\quad + (25296\sqrt{5}/25 + 292944/125)e^2 + (286944\sqrt{5}/625 + 139808/125)e \\
&\quad + (16064\sqrt{5}/625 + 9792/125) \\
&> (-1) + (-1) + (-1) + 0 + 0 + 0 + 0 + (16064\sqrt{5}/625 + 9792/125) \\
&> 0, \\
c_0'' &= (-16/5)e^6 + (-96\sqrt{5}/25 - 96/5)e^5 + (-96\sqrt{5}/5 + 24/5)e^4 \\
&\quad + (224\sqrt{5}/25 + 736/5)e^3 + (2592\sqrt{5}/25 + 8544/25)e^2 \\
&\quad + (82944\sqrt{5}/625 + 8448/25)e + (33344\sqrt{5}/625 + 78144/625) \\
&> (-1) + (-1) + (-1) + 0 + 0 + 0 + (33344\sqrt{5}/625 + 78144/625) \\
&> 0.
\end{aligned}
$$

Since $2 \nmid L$ and $2 + \beta^{2L} < 5$, moreover, it holds that

$$
\begin{aligned}
eb^2 &= \frac{1}{\sqrt{5}\alpha^{2L}} \left(\frac{\alpha^L - \beta^L}{\sqrt{5}} \right)^2 \\
&= \frac{1}{\sqrt{5}\alpha^{2L}} \cdot \frac{\alpha^{2L} - 2(-1)^L + \beta^{2L}}{5} \\
&= \frac{1}{5\sqrt{5}} + \frac{2 + \beta^{2L}}{5\sqrt{5}\alpha^{2L}} < \frac{1}{5\sqrt{5}} + \frac{1}{\sqrt{5}\alpha^{2L}} = \frac{1}{5\sqrt{5}} + e,
\end{aligned}
$$

and hence,

$$eb^4 < \left(\frac{1}{5\sqrt{5}} + e\right) b^2 = \frac{b^2}{5\sqrt{5}} + eb^2 < \frac{b^2}{5\sqrt{5}} + \frac{1}{5\sqrt{5}} + e$$
$$< \frac{b^2}{5\sqrt{5}} + \frac{1}{5\sqrt{5}} + \frac{1}{4935} < \frac{b^2}{5\sqrt{5}} + \frac{1}{10}.$$

Therefore, again by $b \geq 21$, we obtain

$$\begin{aligned}
(5.10) \quad & f_2(L) + \frac{1}{5}(\sqrt{5}(a^2+b^2) - \frac{1}{b^2} + 1)t(L) \\
& > (-160\sqrt{5} - 160)eb^{14} + (1248\sqrt{5} + 2656)b^{12} + (-576\sqrt{5} - 4087/5)b^{10} \\
& \quad + (-81056\sqrt{5}/25 - 188544/25)b^8 + (-186576\sqrt{5}/125 - 82193/25)b^4 \\
& > (-160\sqrt{5} - 160)b^{10} \cdot \left(\frac{b^2}{5\sqrt{5}} + \frac{1}{10}\right) + (1248\sqrt{5} + 2656)b^{12} \\
& \quad + (-576\sqrt{5} - 4087/5)b^{10} + (-81056\sqrt{5}/25 - 188544/25)b^8 \\
& \quad + (-186576\sqrt{5}/125 - 82193/25)b^4 \\
& = (6208\sqrt{5}/5 + 2624)b^{12} + (-592\sqrt{5} - 4167/5)b^{10} \\
& \quad + (-81056\sqrt{5}/25 - 188544/25)b^8 + (-186576\sqrt{5}/125 - 82093/25)b^4 \\
& > (6208\sqrt{5}/5 + 2624)b^{10} \cdot 21^2 + (-592\sqrt{5} - 4167/5)b^{10} \\
& \quad + (-81056\sqrt{5}/25 - 188544/25)b^{10} + (-186576\sqrt{5}/125 - 82093/25)b^{10} \\
& = (67777344\sqrt{5}/125 + 28638028/25)b^{10} \\
& > 0.
\end{aligned}$$

Then by (5.7) and (5.10), we get $f_1(L) + f_2(L) + g_2(L) > 0$. The proof is completed. □

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