J. Korean Math. Soc. **61** (2024), No. 4, pp. 761–778
https://doi.org/10.4134/JKMS.j230510
pISSN: 0304-9914 / eISSN: 2234-3008

SEPARABLE MINIMAL SURFACES AND THEIR LIMIT BEHAVIOR

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Abstract. A separable minimal surface is represented by the form of $f(x) + g(y) + h(z) = 0$, where f, g and h are real-valued functions of x, y and z, respectively. We provide exact equations for separable minimal surfaces with elliptic functions that are singly, doubly and triply periodic minimal surfaces and completely classify all them. In particular, parameters in the separable minimal surfaces change the shape of the surfaces, such as fundamental periods and its limit behavior, within the form $f(x) + g(y) + h(z) = 0$.

1. Introduction

Minimal surface theory in $\mathbb{R}^3$ is a classical subject in differential geometry and natural science. The theory is rooted in the calculus of variations developed by Euler and Lagrange, and various techniques in this theory have played key roles in differential geometry. Minimal surfaces can be distinguished according to the rank of lattices of translational invariance, namely, a minimal surface is called singly, doubly or triply periodic if it is invariant by an isometry group of $\mathbb{R}^3$ of rank 1, 2 or 3, respectively, acting properly and discontinuously.

A regular surface Σ in $\mathbb{R}^3$ is said to be *separable* if it can be expressed as

$$\Sigma = \{(x, y, z) \mid F(x, y, z) = f(x) + g(y) + h(z) = 0\},$$

where f, g and h are real-valued functions and ∇F is a non-vanishing vector field for every point on Σ. There are various examples of separable surfaces that appear for certain choices of the functions f, g and h: a translation surface, a homothetical surface, a rotationally symmetric surface, etc. The study of separable surfaces with geometric conditions (minimal, constant mean curvature, constant Gauss curvature, etc.) has been ongoing as follows: Scherk [11] found examples of minimal surfaces. Weingarten [14] investigated the problem to determine all separable minimal surfaces, realizing that they form a rich and large family of surfaces (more references for this are [1, 4, 5, 12]). Hasanis and López [6] classified all separable surfaces with constant Gaussian curvature in

Received September 24, 2023; Accepted February 5, 2024.
2020 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42.
Key words and phrases. Minimal surfaces, elliptic functions.

$\mathbb{R}^3$. Sergienko and Tkachev [13] constructed doubly periodic maximal surfaces represented by an implicit equation $\zeta(z) = \phi(x)\psi(y)$ in the Lorentzian 3-space. Kaya and López [7] classified some separable surfaces with zero mean curvature in the Lorentzian 3-space.

There are several kinds of families between two minimal surfaces in $\mathbb{R}^3$. We list up on separable minimal surfaces in $\mathbb{R}^3$ and their relations as follows: In [9], Nitsche introduced and showed several examples of separable minimal surfaces including the planes, catenoids, helicoids, Scherk's surfaces, Schwartz's surfaces, etc (see [9, Section 5.2, equations (40), (41), (42), (43), (44), (45) and (47)]). Rodríguez [10] gave classifications of doubly periodic minimal surfaces for up to genus 1, which is called Rodríguez' standard examples. Meeks [8] found a 5-parameters family of triply periodic minimal surfaces of genus 3. Ejiri, Fujimori and Shoda [2] showed that continuous deformations between Rodríguez' standard examples and the 5-parameters family given by Meeks. They [3] also provided a one-parameter family between Karcher's saddle towers and Rodríguez' standard examples as generic limits.

The organization of this paper is as follows: In Section 2, we provide a minimality condition for a separable equation, which yields four cases according to the behavior of a constant κ obtained by the separation of variables. In particular, we analyze the parameters related to the solutions and find explicit equations for all separable minimal surfaces in each case in Sections 3, 4 and 5, via elliptic functions. In Section 3, we consider the case $\kappa > 0$ and obtain five types of two parameter families of triply periodic minimal surfaces with genus 3 and some of them has limits such as a helicoid, a generalized Scherk's surface, a Scherk's tower and doubly periodic minimal surfaces with genus 1 and parallel planar ends. In Section 4, we obtain only a two parameter family of triply periodic minimal surfaces, including the Schwarz's P-surface. In Section 5, we deal with the case of $\kappa = 0$, which implies cases of a generalized Scherk's tower that can converges to a catenoid. As a summary of Sections 3, 4 and 5, we obtain as follows:

Theorem 1. *Separable minimal surfaces in $\mathbb{R}^3$ with an implicit form $f(x) + g(y) + h(z) = 0$ can be classified as follows* (*see Figure 1*):

(1) *There are five types of two-parameters families of triply periodic minimal surfaces, namely, snsnsn, snscsc, nssnns, scnssn and nssnnd-types minimal surfaces* (*see Proposition 3.2*) *and their limits are as follows*:
   (a) *snsnsn, snscsc and nssnns-type minimal surfaces can converge to a generalized Scherk's surface* (*see Proposition 3.3*).
   (b) *nssnns, scnssn and nssnnd-type minimal surfaces can converge to doubly periodic minimal surfaces with genus* 1 *and parallel planar ends* (*see Proposition 3.4*). *In particular, a doubly periodic minimal surface from a scnssn-type minimal surface can converge to a helicoid.*

(c) *A doubly periodic minimal surface from a nssnns-type minimal surface in* (a) *can converge to Scherk's tower* (*see Proposition 3.5*).

(2) *There is a Scherk's surface* (*see Proposition 3.6*).

(3) *There is a two-parameter family of triply periodic minimal surfaces containing the Schwarz's P-surface* (*see Proposition 4.1*).

(4) *There is a generalized Scherk's tower* (*see Proposition 5.1*) *and it can converge to a catenoid* (*see Proposition 5.2*).

In particular, any non-planar separable minimal surface can converge to a plane.

2. Preliminaries

We first summarize Jacobi's elliptic functions. The incomplete elliptic integral of the first kind F is defined by

$$\mathrm{F}(\varphi \mid k^2) = \mathrm{F}(\sin(\varphi); k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.$$

Substituting $t = \sin(\theta)$ and $x = \sin(\varphi)$, the integral can be represented as the Legendre normal form

$$\mathrm{F}(x; k) = \int_0^{x} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

The inverse function of $u = \mathrm{F}(\varphi \mid m)$ is called the *Jacobi amplitude* that depends on u and k, namely,

$$\varphi = \mathrm{am}(u, m).$$

Then, the elliptic sine sn, elliptic cosine cn and delta amplitude dn, which are called *Jacobi's elliptic functions*, are defined by

$$\begin{aligned} \mathrm{sn}(u, m) &= \sin(\mathrm{am}(u, m)), \\ \mathrm{cn}(u, m) &= \cos(\mathrm{am}(u, m)), \\ \mathrm{dn}(u, m) &= \tfrac{d}{du}\mathrm{am}(u, m). \end{aligned}$$

In particular, the parameter k is considered as $0 \le k \le 1$. Let us define $K(k) = u(\frac{\pi}{2}, k)$ that is a complete elliptic integral of the first kind. Therefore, $\mathrm{sn}(u) = \mathrm{sn}(u, k^2)$ is a monotonically increasing odd continuous function defined on $[-K(u), K(u)]$ and satisfies $\mathrm{sn}(0) = 0$, $\mathrm{sn}(\pm K(k)) = \pm 1$, $\mathrm{sn}(u + 2K(k), k^2) = -\mathrm{sn}(u, k^2)$ and $\mathrm{sn}(u + 4K(k), k^2) = \mathrm{sn}(u, k^2)$. For example, if $k = 0$, then $\mathrm{sn}(u, 0) = \sin(u)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Also, we have the following relations for $m = k^2$:

$$\mathrm{sc}(u, m) = \frac{\mathrm{sn}(u, m)}{\mathrm{cn}(u, m)}, \qquad \mathrm{ns}(u, m) = \frac{1}{\mathrm{sn}(u, m)}, \qquad \mathrm{nc}(u, m) = \frac{1}{\mathrm{cn}(u, m)},$$

$$\mathrm{nd}(u, m) = \frac{1}{\mathrm{dn}(u, m)}, \qquad \mathrm{dc}(u, m) = \mathrm{nc}(u, m)\mathrm{dn}(u, m),$$

$$\mathrm{ds}(u, m) = \mathrm{ns}(u, m)\mathrm{dn}(u, m).$$

Secondly, we consider the minimality condition for a separable surface Σ that is represented by the implicit form

$$F(x,y,z) = f(x) + g(y) + h(z). \tag{2-1}$$

The unit normal vector field ν and the mean curvature H are as follows:

$$\nu = \frac{\nabla F}{\|\nabla F\|},$$
$$H = \operatorname{div}\left(-\frac{\nabla F}{\|\nabla F\|}\right) = \frac{1}{\|\nabla F\|^2}(\nabla F(\|\nabla F\|) - \|\nabla F\|\triangle F).$$

In particular, the regularity of Σ is guaranteed by $\|\nabla F\| \neq 0$. Then, we have a minimal surface equation as follows:

$$((g')^2 + (h')^2)f'' + ((f')^2 + (h')^2)g'' + ((f')^2 + (g')^2)h'' = 0, \tag{2-2}$$

where the prime $'$ denotes a derivative of a function with respect to its variable. In the rest of this section, we begin with the classification of the separable minimal surfaces. Here we follow the same ideas with done by Nitsche [9, Section 5.2]. By completeness, we repeat the same arguments. Using the variables $u = f(x)$, $v = g(y)$ and $w = h(z)$ and the abbreviations $X(u) = (f'(x))^2$, $Y(u) = (g'(y))^2$ and $Z(w) = (h'(z))^2$, the equations (2-1) and (2-2) yield that

$$\begin{cases} u + v + w = 0, \\ (Y+Z)X' + (X+Z)Y' + (X+Y)Z' = 0. \end{cases} \tag{2-3}$$

In particular, if $X' = Y' = Z' = 0$, then f, g and h are linear functions, which means that Σ is a plane. Thus, three cases are appeared according to the value of $X'Y'Z'$.

Case 1. $X'Y'Z' \neq 0$.

We obtain a constant κ such that

$$\frac{X'''}{X'} = \frac{Y'''}{Y'} = \frac{Z'''}{Z'} = \kappa.$$

Thus, there are two possibilities: $\kappa \neq 0$ and $\kappa = 0$.

(1) $\kappa \neq 0$.

Solutions of X, Y and Z are

$$\begin{aligned} X(u) &= a_1 + b_1 e^{\sqrt{\kappa}u} + c_1 e^{-\sqrt{\kappa}u}, \\ Y(v) &= a_2 + b_2 e^{\sqrt{\kappa}v} + c_2 e^{-\sqrt{\kappa}v}, \\ Z(w) &= a_3 + b_3 e^{\sqrt{\kappa}w} + c_3 e^{-\sqrt{\kappa}w}, \end{aligned} \tag{2-4}$$

where a_i is a real number and b_i and c_i are non-zero real numbers for each $i = 1, 2, 3$ if $\kappa > 0$. Otherwise, b_i and c_i are non-zero complex numbers with $b_i = \overline{c_i}$ for each $i = 1, 2, 3$. Inserting the solutions into the equation (2-3) and comparing the coefficients of exponential-terms as independent terms, we obtain

$$(a_2+a_3)b_1 = 2c_2c_3, \quad (a_3+a_1)b_2 = 2c_3c_1, \quad (a_1+a_2)b_3 = 2c_1c_2,$$
$$(a_2+a_3)c_1 = 2b_2b_3, \quad (a_3+a_1)c_2 = 2b_3b_1, \quad (a_1+a_2)c_3 = 2b_1b_2. \tag{2-5}$$

(2) $\kappa = 0$.
Solutions of X, Y and Z are as follows:

$$\begin{aligned} X(u) &= a_1 + b_1u + c_1u^2, \\ Y(v) &= a_2 + b_2v + c_2v^2, \\ Z(w) &= a_3 + b_3w + c_3w^2, \end{aligned} \tag{2-6}$$

where a_i is a real number and b_i and c_i are non-zero real numbers. Substituting the solutions into the equation (2-3), the coefficients of the independent terms of polynomials of u and v yield

$$\begin{aligned} (a_1+a_2)b_3 + (a_2+a_3)b_1 + (a_3+a_1)b_2 &= 0, \\ c_1c_2 + c_2c_3 + c_3c_1 &= 0, \\ 2(a_1+a_2)c_3 - b_1b_2 = 2(a_2+a_3)c_1 - b_2b_3 &= 2(a_3+a_1)c_2 - b_3b_1, \\ (b_1-b_2)c_3 = (b_2-b_3)c_1 &= (b_3-b_1)c_2. \end{aligned} \tag{2-7}$$

Case 2. $X' = 0$ and $Y'Z' \neq 0$.
Without loss of generality, we can assume $X' = 0$ and $Y'Z' \neq 0$. The second item of the equation (2-3) yields

$$\begin{aligned} X &= a_1, \\ Y'' &= \lambda Y', \\ Z'' &= -\lambda Z', \end{aligned} \tag{2-8}$$

where a_1 is a real number and λ is a non-zero real number. These equations can appear in (1) of Case 1 with $\kappa = \lambda^2 > 0$, $b_1 = c_1 = 0$ and $b_3 = c_2 = 0$. Thus, we have

$$(a_3+a_1)b_2 = 0, \quad (a_1+a_2)c_3 = 0. \tag{2-9}$$

Case 3. $X' = Y' = 0$ and $Z' \neq 0$.
Without loss of generality, changing roles of variables x, y and z, we can assume $X' = Y' = 0$ and $Z' \neq 0$. The equation (2-3) yields $(X+Y)Z' = 0$ and thus there is a constant a_0 such that $X = -Y = a_0$, namely, $(f')^2 = -(g')^2 = a_0$. Then, we have $a_0 = 0$, $f = \alpha_1$ and $g = \alpha_2$ where α_1 and α_2 are constants. Therefore, $\Sigma = F^{-1}(\{\alpha_1 + \alpha_2 + h(z)\})$ is a plane.

We reorganize the cases as the following three cases: The first case contains $\kappa > 0$ in Case 1 and Case 2, the second case is $\kappa < 0$ in Case 1 and the third case is $\kappa = 0$ in Case 1. In each case, we first figure out the relation of all parameters: the equations (2-5), (2-7) and (2-9) and then secondly, explicit equations to be minimal surfaces are provided.

3. $\kappa > 0$ in Case 1 and Case 2

By scaling invariant of a minimal surface, we may assume $\kappa = 4$.

3.1. $\kappa > 0$ in Case 1

Let A, B and C be constants related to a_i for $i = 1, 2, 3$ as follows:

$$A = \frac{a_2 + a_3}{2}, \qquad B = \frac{a_1 + a_3}{2}, \qquad C = \frac{a_1 + a_2}{2}. \tag{3-1}$$

Then, the equations (2-5) can be rewritten as

$$\begin{aligned} Ab_1 &= c_2 c_3, & Bb_2 &= c_3 c_1, & Cb_3 &= c_1 c_2, \\ Ac_1 &= b_2 b_3, & Bc_2 &= b_3 b_1, & Cc_3 &= b_1 b_2. \end{aligned} \tag{3-2}$$

Lemma 3.1. *Suppose that b_i and c_i are real numbers so that $(b_i, c_i) \neq (0, 0)$ for $i = 1, 2, 3$ in the equations (3-2) with $\kappa = 4$ in Case 1. Then, there are the following four cases. In particular,* (A2)-(A4) *are limits of* (A1):

(A1) *For $ABC \neq 0$, the following equations hold*:

$$a_1 = B + C - A, \quad a_2 = C + A - B, \quad a_3 = A + B - C,$$
$$b_1 = \frac{Cc_3}{b_2}, \quad b_3 = \frac{AB}{c_3}, \quad c_1 = \frac{Bb_2}{c_3}, \quad c_2 = \frac{AC}{b_2}, \quad b_2, c_3 \in \mathbb{R} \setminus \{0\}.$$

(A2) *For $A = 0$ and $BC \neq 0$, the following equations hold*:

$$a_1 = B + C, \quad a_2 = -a_3 = C - B,$$
$$b_1 = \frac{Cc_3}{b_2}, \quad c_1 = \frac{Bb_2}{c_3}, \quad b_3 = c_2 = 0, \quad b_2, c_3 \in \mathbb{R} \setminus \{0\}.$$

(A3) *For $A = C = 0$ and $B \neq 0$, the following equations hold*:

$$a_1 = -a_2 = a_3 = B, \quad b_1 = c_2 = b_3 = 0, \quad c_1 = \frac{Bb_2}{c_3}, \quad b_2, c_3 \in \mathbb{R} \setminus \{0\}.$$

(A4) *For $A = B = 0$ and $C \neq 0$, the following equations hold*:

$$a_1 = a_2 = -a_3 = C, \quad c_1 = c_2 = b_3 = 0, \quad b_1 = \frac{Cc_3}{b_2}, \quad b_2, c_3 \in \mathbb{R} \setminus \{0\}.$$

Proof. In the case of $A = B = C = 0$, it is easy to check that $b_i = c_i = 0$ for some $i = 1, 2, 3$, which is a contradiction. Thus, we have three possibilities: All of A, B and C are not zero, one of them is zero and two of them are zero. Without loss of generality, we consider four cases: (A1) $ABC \neq 0$, (A2) $A = 0$ and $BC \neq 0$, (A3) $A = C = 0$ and $B \neq 0$ and (A4) $A = B = 0$ and $C \neq 0$.

Assume that (A1), namely, $ABC \neq 0$. If we assume $c_1 = 0$, then we obtain that $b_2 = b_3 = 0$ yields $c_2 = c_3 = 0$ by the equations (2-5), which is a contradiction, namely, $c_1c_2c_3 \neq 0$. By the equations (3-2), we have

$$b_1 = \frac{c_2c_3}{A}, \quad b_2 = \frac{c_3c_1}{B}, \quad b_3 = \frac{c_1c_2}{C}, \quad c_1 = \frac{ABC}{c_2c_3},$$

and then

$$b_1 = \frac{Cc_3}{b_2}, \quad c_1 = \frac{Bb_2}{c_3}, \quad c_2 = \frac{AC}{b_2}, \quad b_3 = \frac{AB}{c_3}.$$

For the cases (A2), (A3) and (A4), by the similar argument, we get the assertion. □

We define $F_1 = e^f$, $F_2 = e^g$ and $F_3 = e^h$. Then, the equation (2-4) is equivalent to the following equation for any $i = 1, 2, 3$:

$$(F_i')^2 = a_iF_i^2 + b_iF_i^4 + c_i. \tag{3-3}$$

We first consider complex-valued solutions of the equation (3-3) up to the coordinate shift $t \mapsto t - t_0$ and coordinate change $t \mapsto -t$. We denote α and β as constant solutions of $b_ix^2 + a_ix + c_i = 0$:

$$\alpha = \frac{-a_i + \sqrt{a_i^2 - 4b_ic_i}}{2b_i}, \quad \beta = \frac{-a_i - \sqrt{a_i^2 - 4b_ic_i}}{2b_i}.$$

We can distinguish cases according to the value of the discriminant D of $b_ix^2 + a_ix + c_i = 0$:

$$D = a_1^2 - 4b_1c_1 = a_2^2 - 4b_2c_2 = a_3^2 - 4b_3c_3 = (A - B - C)^2 - 4BC.$$

We first consider $D \neq 0$ and then observe one-parameter family of separable minimal surfaces from $D \neq 0$ to $D = 0$. The following lemma and corollary provide solutions of the equations (3-3) up to coordinate changes.

Lemma 3.2. *Suppose that α and β are non-zero distinct solutions of $b_ix^2 + a_ix + c_i = 0$. Then, complex-valued solutions $F_i(t)$ of the equation* (3-3) *with $F_i' \neq 0$ have the following form*:

$$F_i = \pm\sqrt{\beta}\,\mathrm{sn}\left(\sqrt{\alpha b_i}\,(t + \mathcal{C})\,;\frac{\beta}{\alpha}\right),$$

where $\mathcal{C}$ is a complex number and the squared-modulus is $\frac{\beta}{\alpha}$.

Proof. It is easy to verify that all cases satisfying $b_ic_i = 0$ contradict to the equations (2-4) and (2-5). We consider only the case $b_ic_i \neq 0$. By direct computation, we obtain

$$1 = \pm\frac{F_i'}{\sqrt{a_iF_i^2 + b_iF_i^4 + c_i}} = \pm\frac{F_i'}{\sqrt{b_i(F_i^2 - \alpha)(F_i^2 - \beta)}} = \pm\frac{F_i'}{\sqrt{(b_i\beta F_i^2 - c_i)(\frac{1}{\beta}F_i^2 - 1)}}.$$

We consider a function θ of t such that $\sin(\theta) = \sqrt{\frac{1}{\beta}} F_i$. Then, we have

$$\begin{aligned} 1 &= \pm \frac{\theta' \cos(\theta)\sqrt{\beta}}{\sqrt{(b_i \beta^2 \sin^2(\theta) - c_i)(\sin^2(\theta) - 1)}} \\ &= \pm \frac{\theta' \sqrt{\beta}}{\sqrt{c_i - b_i \beta^2 \sin^2(\theta)}} = \pm \sqrt{\frac{\beta}{c_i}} \frac{\theta'}{\sqrt{1 - \frac{\beta}{\alpha} \sin^2(\theta)}}. \end{aligned}$$

After integrating both sides of the above equation, we obtain

$$\pm \sqrt{\frac{\beta}{c_i}} \mathrm{F}\left(\varphi \,\middle|\, \frac{\beta}{\alpha}\right) = t + \mathcal{C},$$

where $\mathcal{C}$ is a complex number. Therefore, the solution is

$$F_i = \pm \sqrt{\beta} \mathrm{sn}\left(\sqrt{\alpha b_i}\,(t + \mathcal{C})\,, \frac{\beta}{\alpha}\right).$$

□

From now on, we use the notation $c = +$ (resp. $c = -$) as $c > 0$ (resp. $c < 0$) for convenience.

Corollary 3.1. *Let α and β be non-zero distinct solutions of $b_i x^2 + a_i x + c_i = 0$. Suppose that b_i, c_i and $D = a_i^2 - 4b_i c_i$ do not vanish. Then, up to the coordinate shift, real-valued solutions $F_i(t)$ of the equation* (3-3) *with $F_i' \neq 0$ are obtained as follows*:

(1) *For $(a_i, b_i, c_i) = (\pm, +, +)$ and $D < 0$, the solution is*

$$F_i = \sqrt{\beta} \mathrm{sn}\left(\sqrt{\alpha b_i} t, \frac{\beta}{\alpha}\right).$$

(2) *For $(a_i, b_i, c_i) = (+, +, +)$ and $D > 0$, the solution is*

$$F_i = \sqrt{-\beta} \mathrm{sc}\left(\sqrt{-\alpha b_i} t, 1 - \frac{\beta}{\alpha}\right).$$

(3) *For $(a_i, b_i, c_i) = (\pm, +, -)$ and $D > 0$, the solution is*

$$F_i = \sqrt{\alpha} \mathrm{dc}\left(\sqrt{\alpha b_i} t, \frac{\beta}{\alpha}\right).$$

(4) *For $(a_i, b_i, c_i) = (-, +, +)$ and $D > 0$, the solution is*

$$F_i(t) = \begin{cases} \sqrt{\beta} \mathrm{sn}\left(\sqrt{\alpha b_i} t, \frac{\beta}{\alpha}\right), \\ \sqrt{\alpha} \mathrm{ns}\left(\sqrt{\alpha b_i} t, \frac{\beta}{\alpha}\right). \end{cases}$$

(5) *For $(a_i, b_i, c_i) = (\pm, -, +)$ and $D > 0$, the solution is*

$$F_i = \sqrt{\beta} \mathrm{sn}\left(\sqrt{\alpha b_i} t, \frac{\beta}{\alpha}\right).$$

(6) *For* $(a_i, b_i, c_i) = (+, -, -)$ *and* $D > 0$*, the solution is*

$$F_i = \sqrt{\beta}\mathrm{nd}\left(\sqrt{-\alpha b_i}t, 1 - \frac{\beta}{\alpha}\right).$$

Proof. In Lemma 3.2, in order to get a real-valued solution $F_i(t)$, we can distinguish several cases using b_i, D, α and β. For each case, an appropriate integral constant $\mathcal{C}$ is obtained by taking a suitable initial value and using a change of variables.

1. $b_i > 0$
   (1) $D < 0$
   This is the case of $(a_i, b_i, c_i) = (\pm, +, +)$. If we assume $\alpha\beta \neq 0$, then let $\mathcal{C} = 0$ and we get the assertion.
   (2) $D > 0$
      i. $\alpha < 0$ and $\beta < 0$
      This is the case of $(a_i, b_i, c_i) = (+, +, +)$. Taking $\mathcal{C} = 0$, we obtain the similar form of F_i with (1).
      ii. $\alpha > 0$ and $\beta < 0$
      This is the case of $(a_i, b_i, c_i) = (\pm, +, -)$. Taking $\mathcal{C} = \sqrt{\frac{1}{\alpha b_i}}(K + iK')$ yields

$$\begin{aligned} F_i(t) &= \pm\sqrt{\beta}\mathrm{sn}\left(\sqrt{\alpha b_i}t + K + iK', m\right) \\ &= \pm\sqrt{\beta}\mathrm{cd}\left(\sqrt{\alpha b_i}t + iK', m\right) \\ &= \pm\sqrt{\alpha}\mathrm{dc}\left(\sqrt{\alpha b_i}t, m\right). \end{aligned}$$

      iii. $\alpha > 0$ and $\beta > 0$
      This is the case of $(a_i, b_i, c_i) = (-, +, +)$. There are two types of solutions. First, we let $\mathcal{C} = 0$ and we obtain the same form of F_i with (1). Secondly, we consider $\mathcal{C} = \sqrt{\frac{1}{\alpha b_i}}K'i$ and obtain

$$F_i(t) = \pm\sqrt{\beta}\mathrm{sn}\left(\sqrt{\alpha b_i}t + iK', m\right) = \pm\sqrt{\alpha}\mathrm{ns}\left(\sqrt{\alpha b_i}t, m\right).$$

Similarly, we can check the assertion for the case of $b_i < 0$. □

3.1.1. Case (A1).

In the rest of this section, we denote

$$\begin{aligned} n_1 &= \frac{-(B + C - A) + \sqrt{D}}{2}, \quad m_1 = \frac{B + C - A + \sqrt{D}}{B + C - A - \sqrt{D}}, \\ n_2 &= \frac{-(C + A - B) + \sqrt{D}}{2}, \quad m_2 = \frac{C + A - B + \sqrt{D}}{C + A - B - \sqrt{D}}, \\ n_3 &= \frac{-(A + B - C) + \sqrt{D}}{2}, \quad m_3 = \frac{A + B - C + \sqrt{D}}{A + B - C - \sqrt{D}}. \end{aligned}$$

Proposition 3.2. *Let Σ be a non-planar separable minimal surface in $\mathbb{R}^3$ corresponding to Case* (A1) *in Lemma 3.1. Then, up to the coordinate change, Σ is one of the following families of triply periodic minimal surfaces* (*see Figure 1*):

(1) *For $D < 0$, $A > 0$, $B > 0$ and $C > 0$,*

$$\mathrm{sn}\left(\sqrt{n_1}x, m_1\right)\mathrm{sn}\left(\sqrt{n_2}y, m_2\right)\mathrm{sn}\left(\sqrt{n_3}z, m_3\right) = 1.$$

(2) *For $D > 0$, $B > 0$, $C > 0$ and $A > B + C + 2\sqrt{BC}$,*

$$\mathrm{sn}\left(\sqrt{n_1}x, m_1\right)\mathrm{sc}\left(\sqrt{-n_2}y, 1 - m_2\right)\mathrm{sc}\left(\sqrt{-n_3}z, 1 - m_3\right) = 1.$$

(3) *For $D > 0$, $A < 0$, $C < 0$, $B > 0$ and $B \neq |A - C|$,*

$$\mathrm{ns}\left(\sqrt{n_1}x, m_1\right)\mathrm{sn}\left(\sqrt{n_2}y, m_2\right)\mathrm{ns}\left(\sqrt{n_3}z, m_3\right) = \sqrt{m_2 m_3}.$$

(4) *For $B > 0$, $C > 0$, $A < 0$ and $A \neq -|B - C|$,*

$$\mathrm{sc}\left(\sqrt{-n_1}x, 1 - m_1\right)\mathrm{ns}\left(\sqrt{n_2}y, m_2\right)\mathrm{sn}\left(\sqrt{n_3}z, m_3\right) = \sqrt{-m_2}.$$

(5) *For $D > 0$, $C < B < 0$, $B + C + 2\sqrt{BC} < A < 0$ and $A > C - B$,*

$$\mathrm{ns}\left(\sqrt{n_1}x, m_1\right)\mathrm{sn}\left(\sqrt{n_2}y, m_2\right)\mathrm{nd}\left(\sqrt{-n_3}z, 1 - m_3\right) = \sqrt{m_1}.$$

Proof. For $D < 0$, we only have (1) because $F_i = \sqrt{\beta}\left(\sqrt{\alpha b_i} t \frac{\beta}{\alpha}\right)$ by Corollary 3.1. By Corollary 3.1, F_i $(i = 1, 2, 3)$ are one of 6-types of solutions. Moreover, considering the change of roles of x, y and z, we only need to check their duplicate combinations, namely, $\frac{(5+3)!}{5!\,3!} = 56$ cases. By direct computation, most cases do not occur because parameters a_i, b_i and c_i should satisfy the conditions of Case (A1) in Lemma 3.1 and the range of elliptic functions should satisfy $f + g + h = 0$. Thus, we get the assertion. □

Remark 3.1. In Proposition 3.2, the cases of (1), (2), (3), (4) and (5) are included in the Meeks family [8] because they have genus 3 and are represented by elliptic functions. They also have two parameters which change the ratios of edges in the period cuboid.

We call *snsnsn-type*, *snscsc-type*, *nssnns-type*, *scnssn-type* and *nssnnd-type* as the case of (1), (2), (3), (4) and (5) in Proposition 3.2, respectively. We second consider that the discriminant D tending to 0.

Proposition 3.3. *Let Σ_D be a surface in a family of snsnsn, snscsc or nssnns-type minimal surfaces in Proposition 3.2. Then, up to the coordinate change, the following generalized Scherk's surface is a limit of Σ_D as $D \to 0$* (*see Figure 1*): *For $|BC| \neq 0$,*

$$\tanh\left(\sqrt[4]{|BC|}x\right)\tan\left(\sqrt{|C| + \sqrt{|BC|}}y\right)\tan\left(\sqrt{|B| + \sqrt{|BC|}}z\right) = 1. \tag{3-4}$$

Proof. Letting $D \to 0$, we have $(F_i')^2 = b_i(F_i + \frac{a_i}{2b_i})^2$, $\alpha = \beta$ and $m_i = 1$. This implies $b_i > 0$ and $c_i > 0$. Thus, the limit of A as D approaches 0 equals $B + C \pm 2\sqrt{BC}$. Then, snsnsn-type, snscsc-type and nssnns-type converge to the following equations, respectively:

$$\tanh\left(\sqrt[4]{BC}x\right)\tan\left(\sqrt{C+\sqrt{BC}}y\right)\tan\left(\sqrt{B+\sqrt{BC}}z\right) = 1,$$

$$\tanh\left(\sqrt[4]{BC}x\right)\tan\left(\sqrt{C+\sqrt{BC}}y\right)\tan\left(\sqrt{B+\sqrt{BC}}z\right) = 1,$$

$$\coth\left(\sqrt[4]{BC}x\right)\tan\left(\sqrt{-C+\sqrt{BC}}y\right)\cot\left(\sqrt{-B+\sqrt{BC}}z\right) = 1.$$

For the last equation, we obtain the required equation from $\cot(\pi/2 - t) = \tan(t)$. □

3.1.2. Case (A2).

Proposition 3.4. *Let Σ_A be a surface in a family of nssnns, scnssn or nssnnd-type minimal surfaces in Proposition 3.2. Then, up to the coordinate change, there is a one-parameter family $\{\Sigma_A\}$ of surfaces as follows (see Figure 1):*

(1) *From a nssnns-type minimal surface Σ_A to the following minimal surface Σ_0: For $B > 0,\ C < 0$ and $B \neq -C$,*

$$\sqrt{-\frac{B}{C}}\mathrm{sn}\left(\sqrt{-C}x, \frac{B}{C}\right)\sinh\left(\sqrt{B-C}z\right) = \sin\left(\sqrt{B-C}y\right). \tag{3-5}$$

(2) *From a scnssn-type minimal surface Σ_A to the following minimal surface Σ_0: For $B > 0,\ C > 0$ and $B \geq C$,*

$$\sqrt{\frac{B}{C}}\mathrm{sc}\left(\sqrt{C}x, 1-\frac{B}{C}\right)\sinh\left(\sqrt{B-C}z\right) = \sin\left(\sqrt{B-C}y\right). \tag{3-6}$$

In particular, the limit of the equation (3-6) *as $B \to C$ is a helicoid*

$$\tan\left(\sqrt{B}x\right) = \frac{y}{z}.$$

(3) *From a nssnnd-type minimal surface Σ_A to the following minimal surface Σ_0: For $C < B < 0$,*

$$\sqrt{\frac{B}{C}}\mathrm{sn}\left(\sqrt{-C}x, \frac{B}{C}\right)\cosh\left(\sqrt{B-C}z\right) = \sin\left(\sqrt{B-C}y\right). \tag{3-7}$$

Proof. Let us consider (A2) as the limit of (A1) as $A \to 0$. It is easy verified that $D = (B-C)^2 \geq 0$ and snsnsn-type is not considered. Thus, we distinguish two cases: $B \neq C$ and $B = C$. We have $n_1 = -C$, $n_2 = B - C$, $n_3 = m_2 = 0$, $m_1 = B/C$ and $m_3 = \pm\infty$. For snscsc-type with $B > C$, it is a contradiction from $B > 0$, $C > 0$ and $A > B + C + 2\sqrt{BC}$. Even if we change the roles of coordinates x, y and z, we obtain a contradiction. Similarly, we can check other cases by same calculation, and we have the equation (3-5) from nssnns-type, the equation (3-6) from scnssn-type and the equation (3-7) from nssnnd-type.

In a particular case: $B = C$, we can consider the limit of

$$\mathrm{sc}\left(\sqrt{C}x, 1 - \frac{B}{C}\right)\sinh\left(\sqrt{B-C}z\right) = \sin\left(\sqrt{B-C}y\right).$$

Then, we have

$$\mathrm{sc}\left(\sqrt{C}x, 1 - \frac{B}{C}\right) = \frac{\sin\left(\sqrt{B-C}y\right)}{\sinh\left(\sqrt{B-C}z\right)}.$$

Letting B tend to C, we have

$$\tan\left(\sqrt{B}x\right) = \frac{y}{z},$$

which is a helicoid. □

Remark 3.2. In Proposition 3.4, the cases of (1), (2) with $B \neq C$ and (3) are included in the Rodríguez family [10] because they have genus 1 and parallel planar ends. Up to scaling, we can assume $C = \pm 1$, and they have one parameter which changes the ratio of the edges in the period rectangle lattice. In particular, a generalized Scherk's surface and a helicoid can converge to a plane.

3.1.3. Cases (A3) and (A4).

Proposition 3.5. *Let Σ_C be a surface in a family of minimal surfaces with the form* (1) *in Proposition 3.4. Then, up to the coordinate change, the following surface is a limit of Σ_C as $C \to 0$* (*see Figure 1*): *For $B > 0$,*

$$\sinh\left(\sqrt{B}x\right)\sinh\left(\sqrt{B}z\right) = \sin\left(\sqrt{B}y\right). \tag{3-8}$$

Proof. By Proposition 3.4, we have the following possible case for a non-planar separable minimal surface: for $B > 0,\ C < 0,\ B \neq -C$,

$$\sqrt{-\frac{B}{C}}\mathrm{sn}\left(\sqrt{-C}x, \frac{B}{C}\right)\sinh\left(\sqrt{B-C}z\right) = \sin\left(\sqrt{B-C}y\right).$$

Letting $C \to 0$, we obtain for $B > 0$,

$$\sinh\left(\sqrt{B}x\right)\sinh\left(\sqrt{B}z\right) = \sin\left(\sqrt{B}y\right).$$

□

Remark 3.3. The limits of (2) in Proposition 3.4 as $C \to 0$ is a plane. The limit of (3) in Proposition 3.4 and limit of Proposition 3.5 as $B \to 0$ are planes. As the same way in Section 3.1.2, we can consider (A4) as the limit of (A2) as $B \to 0$. By Proposition 3.4, we only have the trivial cases. Even if we apply the coordinate change as $(x, y, z) \to (x, z, y)$, we only obtain the same equation with (3-8).

3.2. Case 2

We define $2A' := a_2 + a_3$, $2B' := a_1 + a_3$ and $2C' := a_1 + a_2$.

Lemma 3.3. *Suppose that a_i is a real number and b_i and c_i are non-zero real numbers for $i = 1, 2, 3$ in the equations (2-9) with $\kappa = \lambda^2$. Then, there is only one case: $-a_1 = a_2 = a_3 = A'$.*

Define $F_1 = e^f$, $F_2 = e^g$ and $F_3 = e^h$. The following equation is equivalent to the equation (2-8):

$$
\begin{aligned}
(F_1')^2 &= a_1 F_1^2, \\
(F_2')^2 &= a_2 F_2^2 + b_2 F_2^4, \\
(F_3')^2 &= a_3 F_3^2 + c_3.
\end{aligned}
\tag{3-9}
$$

By the same argument in Case (A), we have the result.

Proposition 3.6. *Let Σ be a non-planar separable minimal surface corresponding to Case 2. Then, up to the coordinate change, Σ is a doubly periodic Scherk's surface (see Figure 1):*

$$
\frac{\sin(y)}{\sin(z)} = e^x. \tag{3-10}
$$

After a suitable rescaling $(x, y, z) \to \Lambda(x, y, z)$ for a constant Λ, its limit as $\Lambda \to 0$ is a plane.

4. $\kappa < 0$ in Case 1

We use the notations (3-1) and (3-2).

Lemma 4.1. *Suppose that b_i and c_i are real numbers so that $(b_i, c_i) \neq (0, 0)$ for $i = 1, 2, 3$ in the equations (2-5). Then, there is only one case:*

$$
\begin{aligned}
&a_1 = B + C - A, \quad a_2 = C + A - B, \quad a_3 = A + B - C, \\
&b_1 = \sqrt{|BC|}e^{i\theta_1}, \quad b_2 = \sqrt{|AC|}e^{i\theta_2}, \quad b_3 = \sqrt{|AB|}e^{-i(\theta_1+\theta_2)}, \quad c_i = \overline{b_i},
\end{aligned}
$$

where $ABC \neq 0$, θ_1 and θ_2 are real numbers. In particular, one of $a_i + 2|b_i|$ for $i = 1, 2, 3$ becomes zero if and only if $A = B + C + 2\sqrt{|BC|}$, $B = A + C + 2\sqrt{|AC|}$ or $C = A + B + 2\sqrt{|AB|}$.

We define $F_1 = f$, $F_2 = g$ and $F_3 = h$. Then, the equation (2-4) is equivalent to the following equations for $\kappa < 0$ and any $i = 1, 2, 3$:

$$
\begin{aligned}
(F_i')^2 &= a_i + b_i e^{i\sqrt{-\kappa}F_i} + \overline{b_i} e^{-i\sqrt{-\kappa}F_i} \\
&= a_i + 2\mathrm{Re}(b_i)\cos(\sqrt{-\kappa}F_i) - 2\mathrm{Im}(b_i)\sin(\sqrt{-\kappa}F_i).
\end{aligned}
\tag{4-1}
$$

The following lemmas provide solutions of the equations (4-1) up to the coordinate shift.

Lemma 4.2. *Up to the coordinate shift, a solution F_i of the equation $(F_i')^2 = a_i + b_i e^{i\sqrt{-\kappa}F_i} + \overline{b_i}e^{-i\sqrt{-\kappa}F_i}$ is as follows*:

$$F_i(t) = \frac{2}{\sqrt{-\kappa}}\mathrm{am}\left(\frac{\sqrt{-\kappa(a_i+r_i)}}{2}t, \frac{2r_i}{a_i+r_i}\right)$$

for $b_i^2 - \overline{b_i}^2 \neq 0$, $a_i + 2|b_i| > 0$ and $r_i = 2|b_i|$.

Let us denote $E_1 = -A+B+C+2\sqrt{|BC|}$, $E_2 = -B+A+C+2\sqrt{|AC|}$ and $E_3 = -C+A+B+2\sqrt{|AB|}$. According to Lemma 4.1, we have the following proposition:

Proposition 4.1. *Let Σ be a non-planar separable minimal surface corresponding to $\kappa < 0$ in Case 1. Then, up to the coordinate change, Σ is a triply periodic minimal surface with the following form* (*see Figure 1*):

$$\begin{aligned}\mathrm{am}\left(\frac{\sqrt{-\kappa E_1}}{2}x\ , \frac{4\sqrt{|BC|}}{E_1}\right) + \ \mathrm{am}\left(\frac{\sqrt{-\kappa E_2}}{2}y\ , \frac{4\sqrt{|AC|}}{E_2}\right)\\ +\ \mathrm{am}\left(\frac{\sqrt{-\kappa E_3}}{2}z\ , \frac{4\sqrt{|AB|}}{E_3}\right) = 0.\end{aligned} \tag{4-2}$$

Proof. By Lemma 4.1, we can assume $ABC \neq 0$. If $E_1 > 0$, $E_2 > 0$ and $E_3 > 0$, then we can find solutions f, g and h as follows:

$$\begin{aligned} f &= \frac{2}{\sqrt{-\kappa}}\ \mathrm{am}\left(\frac{\sqrt{-\kappa E_1}}{2}x\ , \frac{4\sqrt{|BC|}}{E_1}\right) - \theta_1,\\ g &= \frac{2}{\sqrt{-\kappa}}\ \mathrm{am}\left(\frac{\sqrt{-\kappa E_2}}{2}y\ , \frac{4\sqrt{|AC|}}{E_2}\right) - \theta_2,\\ h &= \frac{2}{\sqrt{-\kappa}}\ \mathrm{am}\left(\frac{\sqrt{-\kappa E_3}}{2}z\ , \frac{4\sqrt{|AB|}}{E_3}\right) + (\theta_1+\theta_2).\end{aligned}$$

Therefore, this equation is represented by a triply periodic minimal surface in $\mathbb{R}^3$. □

Remark 4.1. The surfaces in Proposition 4.1 are included in the Meeks family [8], because they have genus 3 and are represented by elliptic functions. They also have two parameters which change the ratio of edges in the period cuboid. In particular, the limit as $\kappa \to 0$ in the equation (4-2) is a plane.

5. $\kappa = 0$ in Case 1

5.1. $\kappa = 0$ in Case 1

We use the notations in (3-1) and define $\tilde{A} = b_1 - b_2$, $\tilde{B} = b_2 - b_3$ and $\mu = \frac{-1}{4(A+B+C)}$. The following lemma is obtained similarly to Lemma 3.1:

Lemma 5.1. *Suppose that a_i is a real number and b_i and c_i are non-zero real numbers for $i = 1, 2, 3$ in the equations* (2-7) *with $\kappa = 0$ in Case 1. Then, there are three cases* (C1), (C2) *and* (C3). *In particular,* (C2) *and* (C3) *are limits of* (C1):

(C1) *For $\tilde{A}\tilde{B}(\tilde{A}+\tilde{B})(A+B+C) \neq 0$, the following equations hold*:

$$a_1 = B + C - A, \quad a_2 = C + A - B, \quad a_3 = A + B - C,$$
$$b_1 = -4(C(\tilde{A}+\tilde{B}) + B\tilde{A})\mu, \quad b_2 = b_1 - \tilde{A}, \quad b_3 = 4(A(\tilde{A}+\tilde{B}) + B\tilde{B})\mu,$$
$$c_1 = -\tilde{A}(\tilde{A}+\tilde{B})\mu, \quad c_2 = \tilde{A}\tilde{B}\mu, \quad c_3 = -\tilde{B}(\tilde{A}+\tilde{B})\mu,$$

(C2) *For $\tilde{A}(A+B+C) \neq 0$, the following equations hold*:

$$a_1 = B + C - A, \quad a_2 = C + A - B, \quad a_3 = A + B - C,$$
$$b_1 = b_3 = b_2 + \tilde{A} = -4B\tilde{A}\mu, \quad c_1 = c_3 = 0, \quad c_2 = -\tilde{A}^2\mu,$$

(C3) *For $A, B \in \mathbb{R}$ and $b_1 \neq 0$, the following equations hold*:

$$a_1 = -2A, \quad a_2 = -2B, \quad a_3 = 2A + 2B, \quad b_1 = b_2 = b_3,$$
$$c_1 = c_2 = c_3 = 0.$$

5.1.1. Case (C1). Let us define $F_1 = f$, $F_2 = g$ and $F_3 = h$. The equations (2-6), which are considered as the case of $\kappa = 0$, yield

$$(F_i')^2 = a_i + b_i F_i + c_i F_i^2, \tag{5-1}$$

and we define $D' = b_i^2 - 4a_i c_i$. The following lemma is obtained similarly to Lemma 4.2.

Lemma 5.2. *Up to the coordinate shift, solutions of $(F_i')^2 = a_i + b_i F_i + c_i F_i^2$ are as follows*:

$$F_i(t) = \begin{cases} \frac{b_i}{4}t^2 - \frac{a_i}{b_i}, & b_i \neq 0 \text{ and } c_i = 0, \\ e^{\sqrt{c_i}t} - \frac{b_i}{2c_i}, & c_i > 0 \text{ and } D' = 0, \\ -\frac{b_i \pm \sqrt{D'}\cosh(\sqrt{c_i}t)}{2c_i}, & c_i > 0 \text{ and } D' > 0, \\ -\frac{b_i - \sqrt{D'}\sin(\sqrt{-c_i}t)}{2c_i}, & c_i < 0 \text{ and } D' > 0, \\ -\frac{b_i - \sqrt{-D'}\sinh(\sqrt{c_i}t)}{2c_i}, & c_i > 0 \text{ and } D' < 0. \end{cases}$$

Proposition 5.1. *Let Σ be a non-planar separable minimal surface corresponding to* (C1) *in Lemma 5.1. Then, up to the coordinate change, Σ is a generalized Scherk's tower as follows* (*see Figure 1*):

$$\frac{\cosh\left(\sqrt{\tilde{A}(\tilde{A}+\tilde{B})}x\right)}{\tilde{A}(\tilde{A}+\tilde{B})} + \frac{\cosh\left(\sqrt{-\tilde{A}\tilde{B}}y\right)}{\tilde{A}\tilde{B}} + \frac{\sin\left(\sqrt{-\tilde{B}(\tilde{A}+\tilde{B})}z\right)}{\tilde{B}(\tilde{A}+\tilde{B})} = 0, \tag{5-2}$$

where $\tilde{A} > 0$, $\tilde{B} < 0$ and $\tilde{A}+\tilde{B} > 0$.

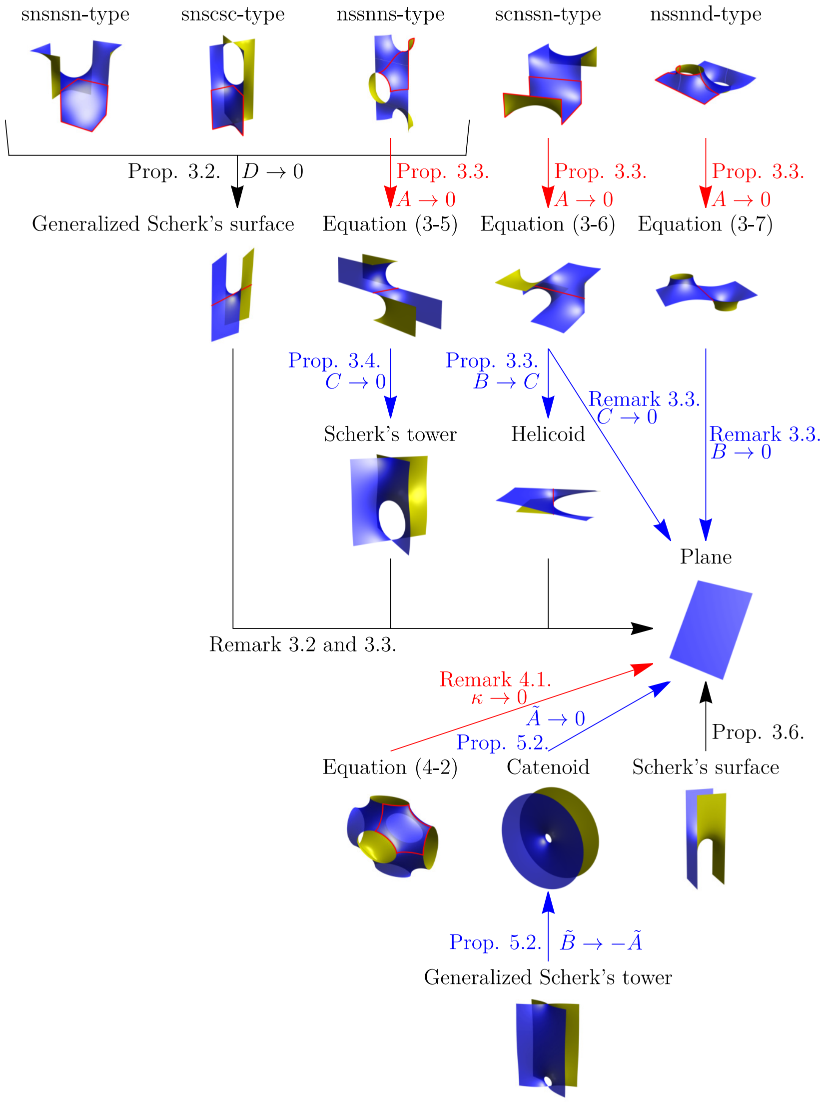

FIGURE 1. Continuous deformation of separable minimal surfaces.

5.1.2. Case (C2).

Proposition 5.2. *Let $\Sigma_{\tilde{B}}$ be a non-planar separable minimal surface in Proposition 5.1. Then, the following catenoid $\Sigma_{\tilde{A}}$ is a limit of $\Sigma_{\tilde{B}}$ as $\tilde{B} \to \tilde{A}$ (see*

Figure 1): *For* $|\tilde{A}| > 0$,

$$\frac{\tilde{A}^2}{4}(x^2+z^2) = \cosh^2\left(\frac{|\tilde{A}|}{2}y\right). \tag{5-3}$$

After suitable translation $x \to x + \frac{2}{\tilde{A}}$, the limit $\tilde{A} \to 0$ in the equation (5-3) is a plane. It is easy to verify that there are no examples of non-planar separable minimal surfaces in (C3).

We will apply our technique to get the complete classification of zero mean curvature surfaces in the Lorentzian 3-space in future work.

Acknowledgements. The authors would like to thank their advisors Juncheol Pyo and Wayne Rossman. We also warmly thank the referees for their careful reading of our paper and giving us valuable comments. The first author was supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MSIT) (No. NRF-2019R1C1C1004819 and No. NRF-2022R1H1A2091877). The second author was supported by a grants-in-aid from JSPS Research Fellowships for Young Scientist (No. 21K13799).

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