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MULTIGRID METHODS FOR 3D H(curl) PROBLEMS WITH NONOVERLAPPING DOMAIN DECOMPOSITION SMOOTHERS

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Abstract. We propose V–cycle multigrid methods for vector field problems arising from the lowest order hexahedral Nédélec finite element. Since the conventional scalar smoothing techniques do not work well for the problems, a new type of smoothing method is necessary. We introduce new smoothers based on substructuring with nonoverlapping domain decomposition methods. We provide the convergence analysis and numerical experiments that support our theory.

1. Introduction

In this paper, the following boundary value problem in three dimensions will be considered:

$$\begin{aligned} Lu &:= \mathbf{curl}\ (\alpha\, \mathbf{curl}\ \boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \\ \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here, Ω is a bounded convex hexahedral domain in three dimensions whose edges are parallel to the coordinate axes and $\boldsymbol{n}$ is the outward unit normal vector of its boundary. We assume that the coefficient α is a strictly positive constant and $\boldsymbol{f}$ is in $\left(L^2(\Omega)\right)^3$.

Our model problem (1) is posed in the Hilbert space $H_0(\mathbf{curl};\Omega)$, the subspace of $H(\mathbf{curl};\Omega)$ with zero tangential components on the boundary $\partial\Omega$. Here, the space $H(\mathbf{curl};\Omega)$ is defined by

$$H(\mathbf{curl};\Omega) = \left\{ \boldsymbol{u} \in \left(L^2(\Omega)\right)^3 : \mathbf{curl}\, \boldsymbol{u} \in \left(L^2(\Omega)\right)^3 \right\}. \tag{2}$$

Applying integration by parts, the corresponding variational problem for (1) can be obtained as follows: Find $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in H_0(\mathbf{curl};\Omega), \tag{3}$$

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where

$$a(\boldsymbol{w}, \boldsymbol{v}) := \alpha \int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \boldsymbol{w} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \tag{4}$$
$$(\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}.$$

We will also define the following bilinear forms for any subdomain $D \subset \Omega$ by:

$$a_D(\boldsymbol{w}, \boldsymbol{v}) := \alpha \int_D \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_D \boldsymbol{w} \cdot \boldsymbol{v}\, d\boldsymbol{x} \tag{5}$$

and

$$(\boldsymbol{w}, \boldsymbol{v})_D = \int_D \boldsymbol{w} \cdot \boldsymbol{v}\, d\boldsymbol{x}. \tag{6}$$

The problem (1) is motivated by the eddy-current problem of Maxwell's equation; see [5, 26]. Specifically, time-dependent Maxwell's equations satisfy the following system:

$$\epsilon \frac{\partial}{\partial t} \boldsymbol{E} + \sigma \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \text{ in } \Omega \times [0, T] \tag{7}$$

$$\mu \frac{\partial}{\partial t} \boldsymbol{H} \qquad + \mathbf{curl}\, \boldsymbol{E} = 0 \text{ in } \Omega \times [0, T], \tag{8}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{J}$ is the intrinsic current. Eliminating $\boldsymbol{H}$ and employing an implicit method yield the following equation in each time step:

$$\frac{1}{4} \Delta t^2 \mathbf{curl} \left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{E}_n \right) + \left(\epsilon + \frac{1}{2} \sigma \Delta t \right) \boldsymbol{E}_n = \text{R.H.S. in } \Omega. \tag{9}$$

The problem (9) is equivalent to our model problem (1). Hence, efficient numerical methods for (1) are essential for solving time-dependent Maxwell's equations. There have been a number of attempts for designing fast solvers related to multigrid methods or domain decomposition methods for the problem (1). For more details, see [4, 11–13, 16, 18, 19, 21–23, 30–33].

Not like the elliptic problems posed in the H^1 Hilbert space, multigrid methods for vector field problems posed in $H(\text{div})$ or $H(\mathbf{curl})$ are challenging. This is because conventional smoothers designed for H^1 related problems, e.g., point-wise smoothers, are not performing well for vector field problems; see [34]. The structures of the null spaces of the differential operators make the hurdle. For the gradient operator, the kernel consists of all constants. However, all gradient fields and all curl fields are the null spaces of the curl and the divergence operators, respectively. Thus, a special treatment for handling the kernels is essential when building multigrid solvers for vector field problems. There have been several approaches in order to overcome the difficulties. In [15, 16], Hiptmair suggested function space splitting methods based on Helmholtz type decompositions. In the algorithms in [15, 16], the smoothing steps have been applied to the decomposed spaces separately, i.e, the range space and the null space. Later, Hiptmair and Xu developed nodal auxiliary space preconditioning techniques based on a new type of regular decomposition in [19]. In [2–4], smoothing methods based on geometric substructures have been proposed. Overlapping types of domain decomposition preconditioners have been applied to vector fields successfully. Another class of methods related to nonoverlapping substructure has been considered for $H(\mathrm{div})$ problems by the author and Brenner in [7, 8]. In [10], the authors suggested multigrid methods based on both overlapping and nonoverlapping methods for higher order finite elements.

In this paper, we suggest V–cycle multigrid methods for $H(\mathbf{curl})$ vector field problems (1) with smoothers based on nonoverlapping domain decomposition preconditioners. We note that our approaches are $H(\mathbf{curl})$ counterparts of the methods in [7, 8] and nonoverlapping alternatives of the method in [4], which reduce the computational complexity when applying the smoothers.

The rest of this paper is organized as follows. We introduce the edge finite elements for our model problem and the discretized problem in Section 2. The V–cycle multigrid algorithms are presented in Section 3. In Section 4, we provide the convergence analysis for the suggested methods. The numerical experiments which support our theory are presented in Section 5, followed by concluding remarks in Section 6.

2. Finite element discretization

We introduce a hexahedral triangulation $\mathcal{T}_h$ of the domain Ω. The edge finite element space, also known as Nédélec finite element space of the lowest order, is defined by

$$N_h := \{\boldsymbol{u} : \boldsymbol{u}|_T \in \mathcal{ND}(T), T \in \mathcal{T}_h \text{ and } \boldsymbol{u} \in H(\mathbf{curl};\Omega)\}, \tag{10}$$

where

$$\mathcal{ND}(T) := \begin{bmatrix} a_1 + a_2x_2 + a_3x_3 + a_4x_2x_3 \\ b_1 + b_2x_3 + b_3x_1 + b_4x_3x_1 \\ c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2 \end{bmatrix} \tag{11}$$

on each element with twelve constants $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$, $i = 1, 2, 3, 4$; see [27, 28] for more details. We note that on each hexahedral element T, the tangential components of vector fields of the form (11) are constants on the twelve edges of T. The twelve coefficients are completely determined by the average tangential components, which is obtained by

$$\lambda_e(\boldsymbol{v}) := \frac{1}{|e|}\int_e \boldsymbol{v} \cdot \boldsymbol{t}_e \, ds, \tag{12}$$

on the twelve edges. Here, e is one of the twelve edges of T, $|e|$ is the length of e, and $\boldsymbol{t}_e$ is the unit tangential vector along the edge e. The standard basis function for N_h associated with e is denoted by ϕ_e. We note that $\lambda_e(\phi_e) = 1$ and $\lambda_{e'}(\phi_e) = 0$ for $e' \neq e$.

Applying the finite element method with N_h, the discrete problem for (3) is given by the following form: Find $\boldsymbol{u}_h \in N_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in N_h. \tag{13}$$

The operator $A_h : N_h \longrightarrow N_h'$ is defined by

$$\langle A_h \boldsymbol{w}_h, \boldsymbol{v}_h \rangle = a(\boldsymbol{w}_h, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h, \boldsymbol{w}_h \in N_h, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $N_h' \times N_h$. We also define $f_h \in N_h'$ in the following way:

$$\langle f_h, \boldsymbol{v}_h \rangle = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in N_h. \tag{15}$$

Then, the discrete problem (13) can be written as

$$A_h \boldsymbol{u}_h = f_h. \tag{16}$$

3. Multigrid algorithms

3.1. Triangulations and grid transfer operators

We introduce $\mathcal{T}_0$, an initial triangulation of the domain Ω. The triangulations $\mathcal{T}_1, \mathcal{T}_2, \ldots$ are obtained from the initial triangulation $\mathcal{T}_0$ by uniform refinement with the relation $h_k = h_{k-1}/2$, where h_k is the mesh size of $\mathcal{T}_k$. The lowest order Nédélec space associated with $\mathcal{T}_k$ is denoted by N_k. Then, we can rewrite the corresponding $k-$th level discrete problem as

$$A_k \boldsymbol{u}_k = f_k. \tag{17}$$

In order to design V–cycle multigrid methods for solving (17), two essential ingredients, i.e., intergrid transfer operators and smoothers, have to be defined properly. We first focus on the grid transfer operators. Due to the fact that the finite element spaces are nested, we can use the natural injection to define the coarse-to-fine operator $I_{k-1}^k : N_{k-1} \longrightarrow N_k$. The associated fine-to-coarse operator $I_k^{k-1} : N_k' \longrightarrow N_{k-1}'$ can be defined by

$$\langle I_k^{k-1} \ell, \boldsymbol{v} \rangle = \langle \ell, I_{k-1}^k \boldsymbol{v} \rangle \qquad \forall\, \ell \in N_k',\ \boldsymbol{v} \in N_{k-1}. \tag{18}$$

3.2. Smoothers

We now concentrate on the other ingredient, smoothers. Nonoverlapping type domain decomposition methods will be used to construct the smoothers. In order to keep consistency with the notations for the standard two-level domain decomposition methods, we will denote $\mathcal{T}_{k-1}$ by $\mathcal{T}_H$ and $\mathcal{T}_k$ by $\mathcal{T}_h$. It means that all the coarse level and fine level settings are associated with $\mathcal{T}_{k-1}(= \mathcal{T}_H)$ and $\mathcal{T}_k(= \mathcal{T}_h)$, respectively. We also define geometric substructures. We will use $\mathcal{F}_H$, $\mathcal{E}_H$, and $\mathcal{V}_H$ to denote the sets of interior faces, edges, and vertices of $\mathcal{T}_H$, respectively. We also define $\mathcal{E}_h^D$ for any subdomain $D \subset \Omega$ by the set of interior edges associated with $\mathcal{T}_h$ that are parts of D. Similarly, we define $\mathcal{V}_h^D$ by the set of interior vertices related to $\mathcal{T}_h$ that are contained in D.

We first introduce the interior space. For each element $T \in \mathcal{T}_H$, we define the following subspace N_h^T of N_h:

$$N_h^T = \{\boldsymbol{v} \in N_h : \boldsymbol{v} = \mathbf{0} \text{ on } \Omega \setminus T\}. \tag{19}$$

We next denote by J_T the natural injection from N_h^T into N_h and we define the operator $A_T : N_h^T \longrightarrow (N_h^T)'$ by

$$\langle A_T \boldsymbol{w}, \boldsymbol{v} \rangle = a(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in N_h^T. \tag{20}$$

In the rest of this subsection, we will introduce two types of smoothing techniques, edge-based and vertex-based smoothers.

3.2.1. *Edge-based smoothers.* We first consider an edge-based smoother. For a given edge $E \in \mathcal{E}_H$, we can find four elements, $\left\{T_E^i\right\}_{i=1,2,3,4}$ in $\mathcal{T}_H$, and four faces, $\left\{F_E^i\right\}_{i=1,2,3,4}$ in $\mathcal{F}_H$, that are sharing the edge E. We define the edge space N_h^E of N_h by

$$\begin{aligned} N_h^E = \Bigg\{ & \boldsymbol{v} \in N_h : \boldsymbol{v} \cdot \boldsymbol{t}_e = 0 \\ & \text{for } e \in \mathcal{E}_h^\Omega \Bigg\backslash \left(\left(\cup_{i=1}^4 \mathcal{E}_h^{T_E^i} \right) \bigcup \left(\cup_{j=1}^4 \mathcal{E}_h^{F_E^j} \right) \bigcup \mathcal{E}_h^E \right), \\ & \text{and } a(\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall\, \boldsymbol{w} \in \left(N_h^{T_E^1} + N_h^{T_E^2} + N_h^{T_E^3} + N_h^{T_E^4} \right) \Bigg\}. \end{aligned} \tag{21}$$

We remark that due to (21), if $\boldsymbol{v} \in N_h^E$ and $\boldsymbol{w}$ have the same tangential components as $\boldsymbol{v}$ on the edges associated with $\partial T_E^i, i = 1, 2, 3, 4$, we have the following property:

$$a_{T_E^i}(\boldsymbol{v}, \boldsymbol{v}) \le a_{T_E^i}(\boldsymbol{w}, \boldsymbol{w}), \qquad i = 1, 2, 3, 4. \tag{22}$$

The operator $J_E : N_h^E \longrightarrow N_h$ is defined as the natural injection. We next define the operator $A_E : N_h^E \longrightarrow \left(N_h^E\right)'$ by

$$\langle A_E \boldsymbol{w}, \boldsymbol{v} \rangle = a(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in N_h^E. \tag{23}$$

The edge-based smoothing operator $M_{E,h}^{-1}$ is constructed as follows:

$$M_{E,h}^{-1} = \eta_E \left(\sum_{T \in \mathcal{T}_H} J_T A_T^{-1} J_T^t + \sum_{E \in \mathcal{E}_H} J_E A_E^{-1} J_E^t \right), \tag{24}$$

where η_E is a damping factor and $J_T^t : N_h' \longrightarrow \left(N_h^T\right)'$ and $J_E^t : N_h' \longrightarrow \left(N_h^E\right)'$ are the transposes of J_T and J_E, respectively. We can choose the damping factor η_E such that the spectral radius of $M_{E,h}^{-1} A_h \le 1$. We note that by using the fact that each fine edge is shared by at most 12 substructures and a standard coloring argument, the condition is satisfied if $\eta_E \le 1/12$, which is assumed to be the case from now on.

3.2.2. *Vertex-based smoothers.* We now consider a vertex-based method. In order to define the vertex space N_h^V, we need geometric substructures associated with the given coarse vertex $V \in \mathcal{V}_H$. For each $V \in \mathcal{V}_H$, there are eight elements, $\left\{T_V^i\right\}_{i=1,\dots,8}$ in $\mathcal{T}_H$, twelve faces, $\left\{F_V^i\right\}_{i=1,\dots,12}$ in $\mathcal{F}_H$, and six edges, $\left\{E_V^i\right\}_{i=1,\dots,6}$ in $\mathcal{E}_H$, that have the vertex V in common. The vertex space N_h^V is defined by

$$N_h^V = \Bigg\{ \boldsymbol{v} \in N_h : \boldsymbol{v} \cdot \boldsymbol{t}_e = 0 \qquad (25)$$
$$\text{for } e \in \mathcal{E}_h^\Omega \setminus \left(\left(\cup_{i=1}^{8} \mathcal{E}_h^{T_V^i} \right) \bigcup \left(\cup_{j=1}^{12} \mathcal{E}_h^{F_V^j} \right) \bigcup \left(\cup_{l=1}^{6} \mathcal{E}_h^{E_V^l} \right) \right),$$
$$\text{and } a(\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall\, \boldsymbol{w} \in \left(\sum_{i=1}^{8} N_h^{T_V^i} \right) \Bigg\}.$$

Note that (25) implies the following minimum energy property:

$$a_{T_V^i}\left(\boldsymbol{v}, \boldsymbol{v}\right) \le a_{T_V^i}\left(\boldsymbol{w}, \boldsymbol{w}\right), \qquad i = 1, \dots, 8, \qquad (26)$$

for $\boldsymbol{v} \in N_h^V$ and $\boldsymbol{w} \in N_h$ with the same degrees of freedom as $\boldsymbol{v}$ on $\partial T_V^i, i = 1, \dots, 8$.

The vertex-based preconditioner is given by

$$M_{V,h}^{-1} = \eta_V \left(\sum_{T \in \mathcal{T}_H} J_T A_T^{-1} J_T^t + \sum_{V \in \mathcal{V}_H} J_V A_V^{-1} J_V^t \right). \qquad (27)$$

Here, η_V is a damping factor and J_V, J_V^t, and A_V are defined in a similar way to those in the edge-based method. The operator $J_V : N_h^V \longrightarrow N_h$ is the natural injection and $J_V^t : N_h' \longrightarrow \left(N_h^V\right)'$ is the transpose of J_V. We define $A_V : N_h^V \to \left(N_h^V\right)'$ as follows:

$$\langle A_V \boldsymbol{w}, \boldsymbol{v} \rangle = a(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in N_h^V. \qquad (28)$$

We note that if $\eta_V \le 1/8$, the spectral radius of $M_{V,h}^{-1} A_h \le 1$ by using a similar argument to that of the edge-based method and we will use the condition for the rest of this paper.

3.3. V–cycle multigrid algorithm

Combining all together, we now construct the symmetric V–cycle multigrid algorithm. Let $MG(k, g, \boldsymbol{z}_0, m)$ be the output of the $k-$th level symmetric multigrid algorithm for solving $A_k \boldsymbol{z} = g$ with initial guess $\boldsymbol{z}_0 \in N_k$ and m smoothing steps. The algorithm is defined in Figure 1.

The smoothing operator M_k^{-1} will be either $M_{E,k}^{-1}$ or $M_{V,k}^{-1}$. We note that given $\ell \in N_k'$, the cost of computing $M_k^{-1}\ell$ is $O(n_k)$ for both edge-based and vertex-based smoothers, where n_k is the number of degrees of freedom of N_k. Hence, the overall computational complexity for $MG(k, g, \boldsymbol{z}_0, m)$ is also $O(n_k)$.

For $k = 0$,

$$MG\left(0, g, \boldsymbol{z}_0, m\right) = A_0^{-1} g.$$

For $k \geq 1$, we set

$$\begin{aligned}
\boldsymbol{z}_l &= \boldsymbol{z}_{l-1} + M_k^{-1}\left(g - A_k \boldsymbol{z}_{l-1}\right) && \text{for } 1 \leq l \leq m, \\
\overline{g} &= I_k^{k-1}\left(g - A_k \boldsymbol{z}_m\right), \\
\boldsymbol{z}_{m+1} &= \boldsymbol{z}_m + I_{k-1}^{k} MG\left(k-1, \overline{g}, 0, m\right), \\
\boldsymbol{z}_l &= \boldsymbol{z}_{l-1} + M_k^{-1}\left(g - A_k \boldsymbol{z}_{l-1}\right) && \text{for } m+2 \leq l \leq 2m+1.
\end{aligned}$$

The output of $MG\left(k, g, \boldsymbol{z}_0, m\right)$ is $\boldsymbol{z}_{2m+1}$.

FIGURE 1. V–cycle Multigrid Method

4. Convergence analysis

Firstly, we remark that the authors in [23] suggested a sufficient condition, assumption **(A1)**, for the convergence of the multigrid methods for problems have large null space. However, the kernel splitting condition does not hold for the smoothers constructed in Section 3.2. This is because the restriction of a gradient field to the edge space N_h^E or the vertex space N_h^V is no longer curl-free. This fact makes the convergence analysis for our suggested multigrid methods challenging. We rephrase the assumption for our methods in Assumption 4.1.

Assumption 4.1 (Assumption **(A1)** of [23]). *Let K_h be the kernel of the curl operator and $\mathcal{G}_H$ correspond $\mathcal{E}_H$ or $\mathcal{V}_H$. The decomposition*

$$N_h = \sum_{T \in \mathcal{T}_H} N_h^T + \sum_{G \in \mathcal{G}_H} N_h^G$$

satisfies

$$K_h = \sum_{T \in \mathcal{T}_H} \left(N_h^T \cap K_h\right) + \sum_{G \in \mathcal{G}_H} \left(N_h^G \cap K_h\right).$$

In another word, Assumption 4.1 implies that any element in K_h can be decomposed into a sum of elements in $\left(N_h^T \cap K_h\right)$ and $\left(N_h^G \cap K_h\right)$.

We define operators that are useful for our analysis. The projection operator P_H is defined by the Ritz projection from the fine level space N_h to the coarse level space N_H with respect to the bilinear form $a(\cdot, \cdot)$ and the identity operator on N_h is denoted by I.

We will also need the Lagrange finite element space of order one, W_h for our analysis. The degree of freedoms are chosen as the function evaluations at vertex points $v \in \mathcal{V}_h$ and are denoted by $\nu_v(p) := p(v)$. The standard basis function associated with the vertex v is denoted by ψ_v, i.e., $\nu_v(\psi_v) = 1$ and $\nu_{v'}(\psi_v) = 0$ for $v' \neq v$.

4.1. Stability estimates

The next lemmas, which are useful for the stability in the edge space, can be obtained by direct calculations.

Lemma 4.2. *For a given coarse edge $E \in \mathcal{E}_H$, which is parallel to the x_1 axis, there are four elements $T_E^i \in \mathcal{T}_H, i = 1,2,3,4$ sharing E. Let v be the midpoint of E. Then, there are six fine edges $e_i \in \mathcal{E}_h, i = 1,\dots,6$, having v in common as an endpoint, such that e_{2i-1} and e_{2i} are parallel to the x_i axis for $i = 1,2,3$, and let us fix the directions for the tangential vectors, $\boldsymbol{t}_{x_i}, i = 1,2,3$, for all corresponding fine edges. Without loss of generality, let v be the endpoint of e_1, e_3, and e_5 with respect to the given tangential directions. We construct $\boldsymbol{u} \in N_h$ supported in $\bigcup_{i=1}^4 T_E^i$ by the properties that*

- *$\boldsymbol{u} \cdot \boldsymbol{t}_{x_i} = -1$ on e_{2i-1} and $\boldsymbol{u} \cdot \boldsymbol{t}_{x_i} = 1$ on e_{2i} for $i = 1,2,3$.*
- *On the other edges in $\bigcup_{i=1}^4 \mathcal{E}_h^{\partial T_E^i}$, the tangential component of $\boldsymbol{u}$ vanishes.*
- *$\boldsymbol{u}$ is orthogonal to $N_h^{T_E^i}$ with respect to the innerproduct $(\cdot,\cdot)_{T_E^i}$ for $i = 1,2,3,4$.*

Then, $\mathbf{curl}\,\boldsymbol{u}$ *does not vanish.*

Proof. It suffices to prove the argument on the edge $E = (0,1)$ and the reference cube $\widehat{T}(:= T_E^1) = (0,1)^3$. The general case can be done with a suitable scaling and symmetry. The fine edges $e_i, i = 1,\dots,6$ are defined in the lemma statement. The interior fine edges $e_{I,j}, j = 1,\dots,6$ associated with the midpoint of $\widehat{T}$, $[1/2,1/2,1/2]$, can be defined in a similar way. We note that only e_1, e_2, e_4, and e_6 are associated with $\partial\widehat{T}$ and the degrees of freedom on the other fine edges on $\partial\widehat{T}$ vanish. By a direct calculation, we obtain the following linear system:

$$\frac{2}{9} I_{6\times 6} \begin{bmatrix} u_{I,1} \\ \vdots \\ u_{I,6} \end{bmatrix} + \begin{bmatrix} \frac{1}{72} & 0 & 0 & 0 \\ 0 & \frac{1}{72} & 0 & 0 \\ 0 & 0 & \frac{1}{18} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{18} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{29}$$

where $I_{6\times 6}$ is the six by six identity matrix, $u_{I,j}, j = 1,\dots,6$ are the degrees of freedom related to the interior fine edges $e_{I,j}, j = 1,\dots,6$, respectively. Here, from the construction, $[u_1, u_2, u_4, u_6]^t = [-1,1,1,1]^t$. The solution of (29) is given by $[1/16, -1/16, -1/4, 0, -1/4, 0]^t$. Hence, by a direct calculation, we can find

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{L^2(\widehat{T})}^2 = \frac{17}{24} > 0. \tag{30}$$

□

Lemma 4.3. *For a given vertex $V \in \mathcal{V}_H$, there are eight elements $T_V^i \in \mathcal{T}_H, i = 1, \ldots, 8$, sharing V. Also, there are six fine edges $e_i \in \mathcal{E}_h, i = 1, \ldots, 6$, having V in common as endpoint, such that e_{2i-1} and e_{2i} are parallel to the x_i axis for $i = 1, 2, 3$, and let us fix the directions for the tangential vectors, $\boldsymbol{t}_{x_i}, i = 1, 2, 3$, for all corresponding fine edges. Without loss of generality, let V be the endpoint of e_1, e_3, and e_5 with respect to the given tangential directions. We construct $\boldsymbol{u} \in N_h$ supported in $\bigcup_{i=1}^8 T_V^i$ by the properties that*

- *$\boldsymbol{u} \cdot \boldsymbol{t}_{x_i} = -1$ on e_{2i-1} and $\boldsymbol{u} \cdot \boldsymbol{t}_{x_i} = 1$ on e_{2i} for $i = 1, 2, 3$.*
- *On the other edges in $\bigcup_{i=1}^8 \mathcal{E}_h^{\partial T_V^i}$, the tangential component of $\boldsymbol{u}$ vanishes.*
- *$\boldsymbol{u}$ is orthogonal to $N_h^{T_V^i}$ with respect to the innerproduct $(\cdot,\cdot)_{T_V^i}$ for $i = 1, \ldots, 8$.*

Then, $\mathbf{curl}\,\boldsymbol{u}$ *does not vanish.*

Proof. In a similar way to the proof of Lemma 4.2, we consider the argument for $V = (0,0,0)$ and $\widehat{T}(:= T_V^1)$. The same approach with that of Lemma 4.2 gives

$$\frac{2}{9} I_{6\times 6} \begin{bmatrix} u_{I,1} \\ \vdots \\ u_{I,6} \end{bmatrix} + \begin{bmatrix} \frac{1}{72} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{72} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{72} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{31}$$

where $[u_2, u_4, u_6]^t = [1,1,1]^t$. From the result, $[-1/16, 0, -1/16, 0, -1/16, 0]^t$, of (31) and a direct calculation, we have

$$\|\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2(\widehat{T})} = \frac{1}{32} > 0. \tag{32}$$

□

In [4, Proposition 4.4], Arnold, Falk, and Winther suggested the following discrete orthogonal Helmholtz decomposition that plays an essential role in the analysis.

Lemma 4.4. [Discrete Helmholtz decomposition] *For any $\boldsymbol{w} \in (I - P_H)\, N_h$, there exist $\boldsymbol{r} \in N_h$ and $q \in W_h$ such that*

$$\boldsymbol{w} = \boldsymbol{r} + \nabla q, \tag{33}$$

and

$$\|\boldsymbol{r}\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{L^2(\Omega)} = \|\boldsymbol{w}\|^2_{L^2(\Omega)}\,, \tag{34}$$

$$\alpha\, \|\boldsymbol{r}\|^2_{L^2(\Omega)} \le C H^2 a\,(\boldsymbol{w}, \boldsymbol{w})\,, \tag{35}$$

$$\|q\|^2_{L^2(\Omega)} \le C H^2\, \|\boldsymbol{w}\|^2_{L^2(\Omega)}\,, \tag{36}$$

where the positive constant C does not depend on the mesh size h.

The edge-based smoother has the following stable decomposition result:

Lemma 4.5. *For any $\boldsymbol{w} \in (I - P_H)\, N_h$, there exist a constant $C_{E,\dagger}$ that does not depend on α, h and the number of elements in $\mathcal{T}_H$ and a decomposition*

$$\boldsymbol{w} = \sum_{T \in \mathcal{T}_H} \boldsymbol{w}_T + \sum_{E \in \mathcal{E}_H} \boldsymbol{w}_E,$$

such that

$$\sum_{T \in \mathcal{T}_H} a\left(\boldsymbol{w}_T, \boldsymbol{w}_T\right) + \sum_{E \in \mathcal{E}_H} a\left(\boldsymbol{w}_E, \boldsymbol{w}_E\right) \le C_{E,\dagger} a\left(\boldsymbol{w}, \boldsymbol{w}\right). \tag{37}$$

Proof. For given $\boldsymbol{w} \in (I - P_h) N_h$, we consider the decomposition (33) in Lemma 4.4, i.e., $\boldsymbol{w} = \boldsymbol{r} + \nabla\, q$.

For each coarse edge $E \in \mathcal{E}_H$, we have four coarse faces $F_E^i \in \mathcal{F}_H, i = 1, 2, 3, 4$ and four elements $T_E^i \in \mathcal{T}_H, i = 1, 2, 3, 4$, that are sharing E. We denote by $\mathcal{N}_{F_E^i}$ the number of edges in $\mathcal{E}_H$ that are parts of ∂F_E^i. We now construct $\boldsymbol{r}_E \in N_h^E$ in the following way:

$$\boldsymbol{r}_E \cdot \boldsymbol{t}_e = \begin{cases} \dfrac{1}{\mathcal{N}_{F_E^i}} \boldsymbol{r} \cdot \boldsymbol{t}_e \text{ for } e \in \mathcal{E}_h^{F_E^i}, i = 1, 2, 3, 4, \\ \boldsymbol{r} \cdot \boldsymbol{t}_e \qquad \text{for } e \in \mathcal{E}_h^E, \end{cases} \tag{38}$$

and (21). Then, $\boldsymbol{r}$ and $\sum_{E \subset \mathcal{E}_H} \boldsymbol{r}_E$ have identical degrees of freedom on the edges contained in the boundaries of elements in $\mathcal{T}_H$. Thus, we can find $\boldsymbol{r}_T \in N_h^T$ such that

$$\boldsymbol{r} = \sum_{T \in \mathcal{T}_H} \boldsymbol{r}_T + \sum_{E \in \mathcal{E}_H} \boldsymbol{r}_E. \tag{39}$$

Let $\boldsymbol{g} = \nabla\, q$. We construct $\boldsymbol{g}_E$ in exactly the same way with $\boldsymbol{r}_E$. Now that $\boldsymbol{g}$ and $\sum_{E \in \mathcal{E}_H} \boldsymbol{g}_E$ have the same degrees of freedom on the edges of N_H, we have

$$\boldsymbol{g} = \sum_{T \in \mathcal{T}_H} \boldsymbol{g}_T + \sum_{E \in \mathcal{E}_H} \boldsymbol{g}_E \tag{40}$$

for unique vector fields $\boldsymbol{g}_T \in N_h^T$.

Term $\boldsymbol{r}_T$: We first consider the vector fields associated with the interior spaces N_h^T. We note that the interior spaces are orthogonal to all the edge spaces N_h^E with respect to the bilinear form $a(\cdot, \cdot)$. Also, the interior spaces are mutually orthogonal. Thus, we have the following estimate putting together with (34), (35), and a standard inverse inequality:

$$\begin{aligned} \sum_{T \in \mathcal{T}_H} a\left(\boldsymbol{r}_T, \boldsymbol{r}_T\right) &= a\left(\sum_{T \in \mathcal{T}_H} \boldsymbol{r}_T, \sum_{T \in \mathcal{T}_H} \boldsymbol{r}_T \right) \\ &\le a(\boldsymbol{r}, \boldsymbol{r}) \end{aligned}$$

$$= \sum_{T\in\mathcal{T}_H} \left[\alpha \|\mathbf{curl}\,\boldsymbol{r}\|^2_{L^2(T)} + \|\boldsymbol{r}\|^2_{L^2(T)}\right] \tag{41}$$
$$\le \sum_{T\in\mathcal{T}_H} \left[C\frac{\alpha}{h^2} \|\boldsymbol{r}\|^2_{L^2(T)} + \|\boldsymbol{r}\|^2_{L^2(T)}\right] \le Ca(\boldsymbol{w},\boldsymbol{w}).$$

Term $\boldsymbol{r}_E$: We next consider the vector fields associated with edges. For any $E\in\mathcal{E}_H$, we construct $\widetilde{\boldsymbol{r}}_{E,F}$ in the following way:

$$\widetilde{\boldsymbol{r}}_E = \sum_{e\in\mathcal{M}} \lambda_e(\boldsymbol{r}_E)\phi_e, \tag{42}$$

where $\mathcal{M} = \left(\cup_{i=1}^4 \mathcal{E}_h^{F_E^i}\right) \cup \mathcal{E}_h^E$. From (22) and a scaling argument, we obtain

$$a(\boldsymbol{r}_E,\boldsymbol{r}_E) \le a(\widetilde{\boldsymbol{r}}_E,\widetilde{\boldsymbol{r}}_E) \tag{43}$$

and

$$\|\widetilde{\boldsymbol{r}}_E\|_{L^2(T_E^i)} \le C\,\|\boldsymbol{r}\|_{L^2(T_E^i)}, \qquad i=1,2,3,4. \tag{44}$$

Using (34), (35), (43), (44), and an inverse inequality, we obtain

$$\sum_{E\in\mathcal{E}_H} a(\boldsymbol{r}_E,\boldsymbol{r}_E) \le \sum_{E\in\mathcal{E}_H} a(\widetilde{\boldsymbol{r}}_E,\widetilde{\boldsymbol{r}}_E)$$
$$= \sum_{E\in\mathcal{E}_H}\sum_{i=1}^4 \left[\alpha\|\mathbf{curl}\,\widetilde{\boldsymbol{r}}_E\|^2_{L^2(T_E^i)} + \|\widetilde{\boldsymbol{r}}_E\|^2_{L^2(T_E^i)}\right] \tag{45}$$
$$\le C\sum_{E\in\mathcal{E}_H}\sum_{i=1}^4 \left[\frac{\alpha}{h^2}\|\boldsymbol{r}\|^2_{L^2(T_E^i)} + \|\boldsymbol{r}\|^2_{L^2(T_E^i)}\right] \le Ca(\boldsymbol{w},\boldsymbol{w}).$$

We therefore have by (41) and (45)

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{r}_T,\boldsymbol{r}_T) + \sum_{E\in\mathcal{E}_H} a(\boldsymbol{r}_E,\boldsymbol{r}_E) \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{46}$$

Term $\boldsymbol{g}_T$: The orthogonal properties and (34) imply the estimate

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{g}_T,\boldsymbol{g}_T) = a\left(\sum_{T\in\mathcal{T}_H}\boldsymbol{g}_T, \sum_{T\in\mathcal{T}_H}\boldsymbol{g}_T\right)$$
$$\le a(\boldsymbol{g},\boldsymbol{g}) = \|\nabla q\|^2_{L^2(\Omega)} \le \|\boldsymbol{w}\|^2_{L^2(\Omega)} \le a(\boldsymbol{w},\boldsymbol{w}). \tag{47}$$

Term $\boldsymbol{g}_E$: For a scalar function z, we define z_v for $v\in\mathcal{V}_h$ by

$$z_v := \nu_v(z)\psi_v. \tag{48}$$

For a given $E\in\mathcal{E}_H$, let $v_E\in\mathcal{V}_h$ be the midpoint of E. Similarly, we denote by $v_F\in\mathcal{V}_h$ the midpoint of $F\in\mathcal{F}_H$. We also consider $\mathcal{R}_E$ defined by the set of all fine edges that share v_E, the midpoint of E.

For each $E \in \mathcal{E}_H$, we construct $\widetilde{\boldsymbol{g}}_E^{(1)} \in N_h$ and $\widetilde{\boldsymbol{g}}_E^{(2)} \in N_h^E$. The vector field $\widetilde{\boldsymbol{g}}_E^{(1)}$ is defined by

$$\widetilde{\boldsymbol{g}}_E^{(1)} = \nabla \left(q_{v_E} + \sum_{i=1}^{4} \frac{1}{\mathcal{N}_{F_E^i}} q_{v_{F_E^i}} \right). \tag{49}$$

For $e \in \mathcal{E}_h^{F_E^i} \setminus \mathcal{R}_E$, let $E_e^i \in \mathcal{E}_H^{\partial F_E^i \setminus E}$ be the coarse edge that shares one vertex point with e. Then, $\widetilde{\boldsymbol{g}}_E^{(2),e} \in N_h^E$ is defined by

$$\widetilde{\boldsymbol{g}}_E^{(2),e} \cdot \boldsymbol{t}_e = \nabla \left(\frac{1}{\mathcal{N}_{F_E^i}} q_{v_{E_e^i}} \right) \cdot \boldsymbol{t}_e \qquad \text{for } e, \tag{50}$$

$$\widetilde{\boldsymbol{g}}_E^{(2),e} \cdot \boldsymbol{t}_{e'} = 0 \text{ for } e' \neq e \text{ and } e' \in \mathcal{E}_h^{F_E^i} \backslash \mathcal{R}_E, i = 1, 2, 3, 4, \tag{51}$$

and (21). For $V \in \mathcal{V}_H^{\partial E}$, we construct $\widetilde{\boldsymbol{g}}_E^{(2),V} \in N_h^E$ as follows:

$$\widetilde{\boldsymbol{g}}_E^{(2),V} \cdot \boldsymbol{t}_e = \begin{cases} \nabla q_V \cdot \boldsymbol{t}_e & \text{for } e, \\ 0 & \text{for } e' \in \mathcal{E}_h^E \text{ and } e' \neq e, \end{cases} \tag{52}$$

and (21), where $e \in \mathcal{E}_h^E$ and V is one of the endpoint of e.

We then construct $\widetilde{\boldsymbol{g}}_E^{(2)}$ as follows:

$$\widetilde{\boldsymbol{g}}_E^{(2)} = \left(\sum_{i=1}^{4} \sum_{e \in \mathcal{E}_h^{F_E^i} \backslash \mathcal{R}_E} \widetilde{\boldsymbol{g}}_E^{(2),e} \right) + \left(\sum_{V \in \mathcal{V}_H^{\partial E}} \widetilde{\boldsymbol{g}}_E^{(2),V} \right). \tag{53}$$

We note that $\widetilde{\boldsymbol{g}}_E^{(1)} + \widetilde{\boldsymbol{g}}_E^{(2)}$ and $\boldsymbol{g}_E$ have the same degrees of freedom on the edges in $(\cup_{j=1}^4 \mathcal{E}_h^{F_E^j}) \bigcup \mathcal{E}_h^E$.

We first estimate $\widetilde{\boldsymbol{g}}_E^{(1)}$. By a standard inverse inequality and a scaling argument, we obtain

$$\left\| \widetilde{\boldsymbol{g}}_E^{(1)} \right\|_{L^2(T_E^i)}^2 \leq \frac{C}{h^2} \left\| q \right\|_{L^2(T_E^i)}^2, \qquad i = 1, 2, 3, 4. \tag{54}$$

Hence, from (36), and (54) we have

$$\sum_{E \in \mathcal{E}_H} a(\widetilde{\boldsymbol{g}}_E^{(1)}, \widetilde{\boldsymbol{g}}_E^{(1)}) \leq \frac{C}{h^2} \left\| q \right\|_{L^2(\Omega)}^2 \leq C \left\| \boldsymbol{w} \right\|_{L^2(\Omega)}^2 \leq C a(\boldsymbol{w}, \boldsymbol{w}). \tag{55}$$

We next consider $\widetilde{\boldsymbol{g}}_E^{(2)}$. For each $v_{E_e^i}$, there exist six fine edges $\{e_j\}, j = 1, \ldots, 6$ in $\mathcal{E}_h$ that have $v_{E_e^i}$ in common. We define $\widehat{\boldsymbol{g}}_E^{(2),e} \in N_h^E$ as follows:

$$\widehat{\boldsymbol{g}}_E^{(2),e} \cdot \boldsymbol{t}_{e'} = \nabla \left(\frac{1}{\mathcal{N}_{F_E^i}} q_{v_{E_e^i}} \right) \cdot \boldsymbol{t}_{e'}, \qquad \text{for } e' = e_j, j = 1, \ldots, 6 \tag{56}$$

and (21). We compare $\widetilde{\boldsymbol{g}}_E^{(2),e}$ and $\widehat{\boldsymbol{g}}_E^{(2),e}$. Let $\{T^j_{E^i_e}\}, j = 1, 2, 3, 4$ be four elements in $\mathcal{T}_H$, that are sharing E^i_e. Because

$$\left\|\widehat{\boldsymbol{g}}_E^{(2),e}\right\|_{L^2(T^j_{E^i_e})} = 0$$

if and only if $\widetilde{\boldsymbol{g}}_E^{(2),e} = 0$, we obtain

$$\left\|\widetilde{\boldsymbol{g}}_E^{(2),e}\right\|_{L^2(T^j_{E^i_e})} \le C \left\|\widehat{\boldsymbol{g}}_E^{(2),e}\right\|_{L^2(T^j_{E^i_e})}, \qquad j = 1, 2, 3, 4. \tag{57}$$

Furthermore, it follows from Lemma 4.2 that $\mathbf{curl}\, \widehat{\boldsymbol{g}}_E^{(2),e} = 0$ if and only if $\widetilde{\boldsymbol{g}}_E^{(2),e} = 0$. Thus, we have, by a scaling argument again,

$$\left\|\mathbf{curl}\, \widetilde{\boldsymbol{g}}_E^{(2),e}\right\|_{L^2(T^j_{E^i_e})} \le C \left\|\mathbf{curl}\, \widehat{\boldsymbol{g}}_E^{(2),e}\right\|_{L^2(T^j_{E^i_e})}, \qquad j = 1, 2, 3, 4. \tag{58}$$

Additionally, the construction of $\widehat{\boldsymbol{g}}_E^{(2),e}$, (22), a scaling argument, and an inverse estimate give the estimate

$$a_{T^j_{E^i_e}}(\widehat{\boldsymbol{g}}_E^{(2),e}, \widehat{\boldsymbol{g}}_E^{(2),e}) \le \frac{C}{h^2} \|q\|^2_{L^2(T^j_{E^i_e})}, \qquad j = 1, 2, 3, 4. \tag{59}$$

For each $V \in \mathcal{V}_H^{\partial E}$, there are six edges in $\mathcal{E}_h$ sharing V in common. We can then construct $\widehat{\boldsymbol{g}}_E^{(2),V} \in N_h^E$ in an exactly same way with $\widehat{\boldsymbol{g}}_E^{(2),e}$. Combining Lemma 4.3 and a similar scaling argument to that of $\widetilde{\boldsymbol{g}}_E^{(2),e}$ and $\widehat{\boldsymbol{g}}_E^{(2),e}$, we obtain

$$\left\|\widetilde{\boldsymbol{g}}_E^{(2),V}\right\|_{L^2(T^j_V)} \le C \left\|\widehat{\boldsymbol{g}}_E^{(2),V}\right\|_{L^2(T^j_V)}, \qquad j = 1, \dots, 8 \tag{60}$$

and

$$\left\|\mathbf{curl}\, \widetilde{\boldsymbol{g}}_E^{(2),V}\right\|_{L^2(T^j_V)} \le C \left\|\mathbf{curl}\, \widehat{\boldsymbol{g}}_E^{(2),V}\right\|_{L^2(T^j_V)}, \qquad j = 1, \dots, 8, \tag{61}$$

where $\{T^j_V\}, j = 1, \dots, 8$, are the eight elements in $\mathcal{T}_H$ sharing V in common. We then have

$$a_{T^j_V}(\widehat{\boldsymbol{g}}_E^{(2),V}, \widehat{\boldsymbol{g}}_E^{(2),V}) \le \frac{C}{h^2} \|q\|^2_{L^2(T^j_V)}, \qquad j = 1, \dots, 8. \tag{62}$$

By summing over all $E \in \mathcal{E}_H$, i, and $e \in \mathcal{E}_h^{F^i_E}$ and by (53), (57), (58), (59), (60), (61), (62), and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{E\in\mathcal{E}_H} a\left(\widetilde{\boldsymbol{g}}_E^{(2)}, \widetilde{\boldsymbol{g}}_E^{(2)}\right) &\le C \sum_{E\in\mathcal{E}_H} \sum_{i=1}^{4} \sum_{e\in\mathcal{E}_h^{F^i_E}\setminus\mathcal{R}_E} a\left(\widetilde{\boldsymbol{g}}_E^{(2),e}, \widetilde{\boldsymbol{g}}_E^{(2),e}\right) \\ &\quad + C \sum_{E\in\mathcal{E}_H} \sum_{V\in\mathcal{V}_H^{\partial E}} a\left(\widetilde{\boldsymbol{g}}_E^{(2),V}, \widetilde{\boldsymbol{g}}_E^{(2),V}\right) \\ &\le \frac{C}{h^2} \|q\|^2_{L^2(\Omega)} \le C \|\boldsymbol{w}\|^2_{L^2(\Omega)} \le C a(\boldsymbol{w}, \boldsymbol{w}).\end{aligned} \tag{63}$$

Putting all together with (22), (55), (63), and Cauchy-Schwarz inequality, we obtain

$$\sum_{E\in\mathcal{E}_H} a(\boldsymbol{g}_E,\boldsymbol{g}_E) \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{64}$$

With $\boldsymbol{w}_T=\boldsymbol{r}_T+\boldsymbol{g}_T$ and $\boldsymbol{w}_E=\boldsymbol{r}_E+\boldsymbol{g}_E$, we have the estimate (37) by (46), (47), and (64). □

The following lemma shows a stability estimate for the vertex-based method:

Lemma 4.6. *For any $\boldsymbol{w}\in(I-P_H)\,N_h$, we can find a decomposition*

$$\boldsymbol{w}=\sum_{T\in\mathcal{T}_H}\boldsymbol{w}_T+\sum_{V\in\mathcal{V}_H}\boldsymbol{w}_V$$

and a constant $C_{V,\dagger}$ that does not depend on α, h and the number of elements in $\mathcal{T}_H$, such that

$$\sum_{T\in\mathcal{T}_H} a\left(\boldsymbol{w}_T,\boldsymbol{w}_T\right)+\sum_{V\in\mathcal{V}_H} a\left(\boldsymbol{w}_V,\boldsymbol{w}_V\right)\le C_{V,\dagger}\,a\left(\boldsymbol{w},\boldsymbol{w}\right). \tag{65}$$

Proof. We will consider two terms $\boldsymbol{r}$ and ∇q in (33) separately as in the approach for Lemma 4.5. For each $V\in\mathcal{V}_H$, we consider the geometric structures $\left\{T_V^i\right\}_{i=1,\ldots,8}$ in $\mathcal{T}_H$, twelve faces, $\left\{F_V^i\right\}_{i=1,\ldots,12}$ in $\mathcal{F}_H$, and six edges, $\left\{E_V^i\right\}_{i=1,\ldots,6}$ in $\mathcal{E}_H$, considered in Section 3.2.2. The numbers $\mathcal{N}_{F_V^i}$ and $\mathcal{N}_{E_V^j}$ are denoted by the numbers of vertices in $\mathcal{V}_H$ that are parts of ∂F_V^i and ∂E_V^j, respectively. We now construct $\boldsymbol{r}_V\in N_h^V$ in the following way:

$$\boldsymbol{r}_V\cdot\boldsymbol{t}_e=\begin{cases}\dfrac{1}{\mathcal{N}_{F_V^i}}\boldsymbol{r}\cdot\boldsymbol{t}_e & \text{for } e\in\mathcal{E}_h^{F_V^i}, i=1,\ldots,12,\\ \dfrac{1}{\mathcal{N}_{E_V^j}}\boldsymbol{r}\cdot\boldsymbol{t}_e & \text{for } e\in\mathcal{E}_h^{E_V^j}, i=1,\ldots,6,\end{cases} \tag{66}$$

and (25). We note that $\boldsymbol{r}-\sum_{V\in\mathcal{V}_H}\boldsymbol{r}_V$ belongs to $\sum_{T\in\mathcal{T}_H}N_h^T$ since $\boldsymbol{r}$ and $\sum_{V\in\mathcal{V}_H}\boldsymbol{r}_V$ have the same degrees of freedom on the edges contained in $\partial T, T\in\mathcal{T}_H$. Hence, we have the following decomposition:

$$\boldsymbol{r}=\sum_{T\in\mathcal{T}_H}\boldsymbol{r}_T+\sum_{V\in\mathcal{V}_H}\boldsymbol{r}_V. \tag{67}$$

Using the same arguments in (41), we have

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{r}_T,\boldsymbol{r}_T)\le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{68}$$

Let $\widetilde{\boldsymbol{r}}_V$ be defined by

$$\widetilde{\boldsymbol{r}}_V:=\sum_{i=1}^{12}\sum_{e\in\mathcal{E}_h^{F_V^i}}\lambda_e(\boldsymbol{r}_V)\phi_e+\sum_{j=1}^{6}\sum_{e\in\mathcal{E}_h^{E_V^j}}\lambda_e(\boldsymbol{r}_V)\phi_e. \tag{69}$$

We then have

$$a(\boldsymbol{r}_V,\boldsymbol{r}_V) \le a(\widetilde{\boldsymbol{r}}_V,\widetilde{\boldsymbol{r}}_V) \tag{70}$$

and

$$\|\widetilde{\boldsymbol{r}}_V\|_{L^2(T_V^i)} \le C\,\|\boldsymbol{r}\|_{L^2(T_V^i)}\,, \qquad i=1,\dots,8, \tag{71}$$

by (26) and a standard scaling argument. Combining (34), (35), (70), (71), and an inverse estimate, we obtain

$$\begin{aligned}\sum_{V\in\mathcal{V}_H} a\left(\boldsymbol{r}_V,\boldsymbol{r}_V\right) &\le \sum_{V\in\mathcal{V}_H} a(\widetilde{\boldsymbol{r}}_V,\widetilde{\boldsymbol{r}}_V)\\ &= \sum_{V\in\mathcal{V}_H}\sum_{i=1}^{8}\left[\alpha\,\|\mathbf{curl}\,\widetilde{\boldsymbol{r}}_V\|^2_{L^2(T_V^i)} + \|\widetilde{\boldsymbol{r}}_V\|^2_{L^2(T_V^i)}\right]\\ &\le \sum_{V\in\mathcal{V}_H}\sum_{i=1}^{8} C\left[\frac{\alpha}{h^2}\,\|\widetilde{\boldsymbol{r}}_V\|^2_{L^2(T_V^i)} + \|\widetilde{\boldsymbol{r}}_V\|^2_{L^2(T_V^i)}\right]\\ &\le \sum_{V\in\mathcal{V}_H}\sum_{i=1}^{8} C\left[\frac{\alpha}{h^2}\,\|\boldsymbol{r}\|^2_{L^2(T_V^i)} + \|\boldsymbol{r}\|^2_{L^2(T_V^i)}\right] \le Ca(\boldsymbol{w},\boldsymbol{w}).\end{aligned} \tag{72}$$

Together with (68) and (72), we have

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{r}_T,\boldsymbol{r}_T) + \sum_{V\in\mathcal{V}_H} a\left(\boldsymbol{r}_V,\boldsymbol{r}_V\right) \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{73}$$

Next, we consider $\boldsymbol{g} = \nabla\, q$.

Let $\widetilde{\boldsymbol{g}}_V$ be defined by

(74)

$$\widetilde{\boldsymbol{g}}_V := \nabla\left(\nu_V(q)\psi_V + \sum_{i=1}^{12}\sum_{v\in\mathcal{V}_h^{F_V^i}}\frac{1}{\mathcal{N}_{F_V^i}}\nu_v(q)\psi_v + \sum_{j=1}^{6}\sum_{v\in\mathcal{V}_h^{E_V^j}}\frac{1}{\mathcal{N}_{E_V^j}}\nu_v(q)\psi_v\right).$$

Using a standard inverse estimate and a scaling argument, we obtain

$$\|\widetilde{\boldsymbol{g}}_V\|^2_{L^2(T)} \le \frac{C}{h^2}\,\|q\|^2_{L^2(T)} \qquad \forall T\in\mathcal{T}_H. \tag{75}$$

We then construct $\boldsymbol{g}_V \in N_h^V$ so that

$$\boldsymbol{g}_V\cdot\boldsymbol{t}_e = \widetilde{\boldsymbol{g}}_V\cdot\boldsymbol{t}_e \text{ for } e\in\left(\bigcup_{i=1}^{12}\mathcal{E}_h^{F_V^i}\right)\bigcup\left(\bigcup_{j=1}^{6}\mathcal{E}_h^{E_V^j}\right). \tag{76}$$

Now that $\boldsymbol{g}$ and $\sum_{V\in\mathcal{V}_H}\boldsymbol{g}_V$ have the identical degrees of freedom on the edges in $\bigcup_{T\in\mathcal{T}_H}\mathcal{E}_h^{\partial T}$, we have

$$\boldsymbol{g} = \sum_{T\in\mathcal{T}_H}\boldsymbol{g}_T + \sum_{V\in\mathcal{V}_H}\boldsymbol{g}_V \tag{77}$$

for unique vector fields $\boldsymbol{g}_T \in N_h^T$. For $\boldsymbol{g}_T$, approach with (47) to obtain

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{g}_T,\boldsymbol{g}_T) \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{78}$$

By the construction of $\boldsymbol{g}_V$ and (26), we obtain

$$a(\boldsymbol{g}_V,\boldsymbol{g}_V) \le a(\widetilde{\boldsymbol{g}}_V,\widetilde{\boldsymbol{g}}_V) = \sum_{i=1}^{8} \|\widetilde{\boldsymbol{g}}_V\|^2_{L^2(T_V^i)}\,. \tag{79}$$

Moreover, we have the following estimate using (36), (75), and (79):

$$\sum_{V\in\mathcal{V}_H} a(\boldsymbol{g}_V,\boldsymbol{g}_V) \le \frac{C}{h^2}\|q\|^2_{L^2(\Omega)} \le C\|\boldsymbol{w}\|^2_{L^2(\Omega)} \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{80}$$

From (78) and (80), we therefore have

$$\sum_{T\in\mathcal{T}_H} a(\boldsymbol{g}_T,\boldsymbol{g}_T) + \sum_{V\in\mathcal{V}_H} a\left(\boldsymbol{g}_V,\boldsymbol{g}_V\right) \le Ca(\boldsymbol{w},\boldsymbol{w}). \tag{81}$$

With $\boldsymbol{w}_T = \boldsymbol{r}_T + \boldsymbol{g}_T$ and $\boldsymbol{w}_V = \boldsymbol{r}_V + \boldsymbol{g}_V$, we obtain the desired estimate (65) from (73) and (81). $\square$

4.2. Convergence analysis of the V–cycle multigrid algorithms

We now consider the convergence analysis for the V–cycle multigrid. The error propagation operator $E_k : N_k \longrightarrow N_k$ for the V–cycle multigrid methods with m smoothing steps is given by

(82)

$$E_k = \begin{cases} 0 & \text{if } k=0, \\ R_k^m\left(Id_k - I_{k-1}^k P_k^{k-1}\right)R_k^m + R_k^m\left(I_{k-1}^k E_{k-1} P_k^{k-1}\right)R_k^m & \text{if } k\ge 1; \end{cases}$$

see [14,24]. Here, I_{k-1}^k is defined in Section 3.1 and the operator $P_k^{k-1} : N_k \longrightarrow N_{k-1}$ is the Ritz projection operator defined by

$$a\left(P_k^{k-1}\boldsymbol{w},\boldsymbol{v}\right) = a\left(\boldsymbol{w}, I_{k-1}^k\boldsymbol{v}\right) \qquad \forall\, \boldsymbol{w}\in N_k,\ \boldsymbol{v}\in N_{k-1}. \tag{83}$$

Moreover, we define $R_k : N_k \longrightarrow N_k$ by

$$R_k = Id_k - M_k^{-1}A_k, \tag{84}$$

where Id_k is the identity operator on N_k.

Remark 4.7. The operator R_k in (84) is symmetric with respect to the inner product $a(\cdot,\cdot)$ and E_k is symmetric positive semidefinite with respect to $a(\cdot,\cdot)$. For more detail, see Chapter 6 of [9].

We will follow the framework in Bramble and Pasciak [6]. We can also refer to Chapter 6 of [9]. We note that the spectral conditions in Section 3.2.1 and Section 3.2.2 and stability estimates in Lemma 4.5 and Lemma 4.6 play main roles in the framework.

We first consider a smoothing property.

Lemma 4.8. *For $m \geq 1$, we have*

$$a\left(\left(Id_k - R_k\right) R_k^m \boldsymbol{v}, R_k^m \boldsymbol{v}\right) \leq \frac{1}{2m} a\left(\left(Id_k - R_k^{2m}\right) \boldsymbol{v}, \boldsymbol{v}\right) \qquad \forall \boldsymbol{v} \in N_k,\ k \geq 1.$$

Proof. Let $\boldsymbol{v} \in N_k$ be arbitrary. Since R_k is symmetric with respect to the inner product $a(\cdot,\cdot)$, it follows from the spectral conditions in Section 3.2.1 and Section 3.2.2 and the spectral theorem that

$$a\left(\left(Id_k - R_k\right) R_k^l \boldsymbol{v}, \boldsymbol{v}\right) \leq a\left(\left(Id_k - R_k\right) R_k^j \boldsymbol{v}, \boldsymbol{v}\right) \qquad \text{for } 0 \leq j \leq l,$$

and thus we have

$$\begin{aligned}(2m) a\left(\left(Id_k - R_k\right) R_k^m \boldsymbol{v}, R_k^m \boldsymbol{v}\right) &= (2m) a\left(\left(Id_k - R_k\right) R_k^{2m} \boldsymbol{v}, \boldsymbol{v}\right) \\ &\leq \sum_{j=0}^{2m-1} a\left(\left(Id_k - R_k\right) R_k^j \boldsymbol{v}, \boldsymbol{v}\right) \\ &= a\left(\left(Id_k - R_k^{2m}\right) \boldsymbol{v}, \boldsymbol{v}\right). \end{aligned}$$ □

We next derive two approximation properties.

Lemma 4.9. *For all $\boldsymbol{v} \in N_k$ and $k \geq 1$, let $\boldsymbol{w} = \left(Id_k - I_{k-1}^k P_k^{k-1}\right) \boldsymbol{v}$. We then have the following estimates*:

$$\langle M_{E,k} \boldsymbol{w}, \boldsymbol{w} \rangle \leq \frac{C_{E,\dagger}}{\eta_E} a\left(\boldsymbol{w}, \boldsymbol{w}\right)$$

and

$$\langle M_{V,k} \boldsymbol{w}, \boldsymbol{w} \rangle \leq \frac{C_{V,\dagger}}{\eta_V} a\left(\boldsymbol{w}, \boldsymbol{w}\right).$$

Proof. We will use a well-know additive Schwarz theory. For more details, see Chapter 7 of [9]. For any $\boldsymbol{w} \in N_h$, we have

$$\langle M_{E,k} \boldsymbol{w}, \boldsymbol{w} \rangle = \eta_E^{-1} \inf_{\substack{\boldsymbol{w} = \sum_{T \in \mathcal{T}_H} \boldsymbol{w}_T \\ + \sum_{E \in \mathcal{E}_H} \boldsymbol{w}_E, \\ \boldsymbol{w}_T \in N_h^T,\, \boldsymbol{w}_E \in N_h^E}} \left(\sum_{T \in \mathcal{T}_H} a\left(\boldsymbol{w}_T, \boldsymbol{w}_T\right) + \sum_{E \in \mathcal{E}_H} a\left(\boldsymbol{w}_E, \boldsymbol{w}_E\right) \right). \tag{85}$$

We therefore have the estimate for $M_{E,k}$ from Lemma 4.5 and (85) with $\boldsymbol{w} = \left(Id_k - I_{k-1}^k P_k^{k-1}\right) \boldsymbol{v}$.

Similarly, for any $\boldsymbol{w} \in N_h$, the following relation holds:

$$\langle M_{V,k} \boldsymbol{w}, \boldsymbol{w} \rangle = \eta_V^{-1} \inf_{\substack{\boldsymbol{w} = \sum_{T \in \mathcal{T}_H} \boldsymbol{w}_T \\ + \sum_{V \in \mathcal{V}_H} \boldsymbol{w}_V, \\ \boldsymbol{w}_T \in N_h^T,\, \boldsymbol{w}_V \in N_h^V}} \left(\sum_{T \in \mathcal{T}_H} a\left(\boldsymbol{w}_T, \boldsymbol{w}_T\right) + \sum_{V \in \mathcal{V}_H} a\left(\boldsymbol{w}_V, \boldsymbol{w}_V\right) \right). \tag{86}$$

In a similar way, we obtain the estimate for $M_{V,k}$ from Lemma 4.6 and (86) with $\boldsymbol{w} = \left(Id_k - I_{k-1}^k P_k^{k-1}\right) \boldsymbol{v}$. □

Lemma 4.10. *For all $\boldsymbol{v} \in N_k, k \geq 1$, we have*

$$a\left(\left(Id_k - I_{k-1}^k P_k^{k-1}\right)\boldsymbol{v}, \left(Id_k - I_{k-1}^k P_k^{k-1}\right)\boldsymbol{v}\right) \leq \frac{C_\dagger}{\eta} a\left((Id_k - R_k)\boldsymbol{v}, \boldsymbol{v}\right),$$

where $C_\dagger = C_{E,\dagger}$ (resp. $C_{V,\dagger}$) and $\eta = \eta_E$ (resp. η_V) if $M_k = M_{E,k}$ (resp. $M_{V,k}$).

Proof. Let $\boldsymbol{w} = \left(Id_k - I_{k-1}^k P_k^{k-1}\right)\boldsymbol{v}$. By (83), Lemma 4.9 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} a(\boldsymbol{w}, \boldsymbol{w}) = a(\boldsymbol{w}, \boldsymbol{v}) &= \left\langle M_k \left(M_k^{-1}\right) A_k \boldsymbol{v}, \boldsymbol{w}\right\rangle \\ &\leq \left\langle M_k \left(M_k^{-1} A_k\right)\boldsymbol{v}, \left(M_k^{-1} A_k\right)\boldsymbol{v}\right\rangle^{1/2} \left\langle M_k \boldsymbol{w}, \boldsymbol{w}\right\rangle^{1/2} \\ &\leq a\left(\left(M_k^{-1} A_k\right)\boldsymbol{v}, \boldsymbol{v}\right)^{1/2} \left(\frac{C^\dagger}{\eta}\right)^{1/2} a(\boldsymbol{w}, \boldsymbol{w})^{1/2} \\ &= a\left(\left(Id_k - R_k\right)\boldsymbol{v}, \boldsymbol{v}\right)^{1/2} \left(\frac{C^\dagger}{\eta}\right)^{1/2} a(\boldsymbol{w}, \boldsymbol{w})^{1/2}. \end{aligned}$$

Hence, we obtain

$$a(\left(Id_k - I_{k-1}^k P_k^{k-1}\right)\boldsymbol{v}, \left(Id_k - I_{k-1}^k P_k^{k-1}\right)\boldsymbol{v}) \leq \frac{C^\dagger}{\eta} a\left(\left(Id_k - R_k\right)\boldsymbol{v}, \boldsymbol{v}\right). \qquad (87)$$ □

Finally, we establish our main result, the uniform convergence of the V–cycle multigrid methods.

Theorem 4.11. *Let $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$. We then have*

$$\|E_k \boldsymbol{w}\|_a \leq \frac{(C_\dagger/\eta)}{(C_\dagger/\eta) + 2m} \|\boldsymbol{w}\|_a \qquad \forall\, \boldsymbol{w} \in N_k,\ k \geq 1,$$

where $C_\dagger = C_{E,\dagger}$ (resp. $C_{V,\dagger}$) and $\eta = \eta_E$ (resp. η_V) if $M_k = M_{E,k}$ (resp. $M_{V,k}$).

Proof. Due to the fact that E_k is symmetric positive semidefinite, it is enough to show that

$$a(E_k \boldsymbol{w}, \boldsymbol{w}) \leq \frac{C_*}{C_* + 2m} a(\boldsymbol{w}, \boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in V_k,\ k \geq 1, \qquad (88)$$

where $C_* = C_\dagger/\eta$.

We will prove (88) by induction. Obviously, the case for $k = 0$ holds automatically since $E_0 = 0$. Let $\delta = C_*/(C_* + 2m)$ and assume that the estimate (88) is satisfied for $k - 1$. We then have

$$\begin{aligned} a\left(E_k \boldsymbol{w}, \boldsymbol{w}\right) &= a\left(R_k^m \left(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}\right) R_k^m \boldsymbol{w}, \boldsymbol{w}\right) \\ &\leq a\left(\left(Id_k - I_{k-1}^k P_k^{k-1}\right) R_k^m \boldsymbol{w}, \left(Id_k - I_{k-1}^k P_k^{k-1}\right) R_k^m \boldsymbol{w}\right) \\ &\quad + \delta a\left(P_k^{k-1} R_k^m \boldsymbol{w}, P_k^{k-1} R_k^m \boldsymbol{w}\right) \\ &= (1 - \delta)\, a\left(\left(Id_k - I_{k-1}^k P_k^{k-1}\right) R_k^m \boldsymbol{w}, \left(Id_k - I_{k-1}^k P_k^{k-1}\right) R_k^m \boldsymbol{w}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \delta a\left(R_k^m \boldsymbol{w}, R_k^m \boldsymbol{w}\right) \\
&\leq (1-\delta)\, C_* a\left(\left(Id_k - R_k\right) R_k^m \boldsymbol{w}, R_k^m \boldsymbol{w}\right) + \delta a\left(R_k^m \boldsymbol{w}, R_k^m \boldsymbol{w}\right) \\
&\leq (1-\delta)\, \frac{C_*}{2m} a\left(\left(Id_k - R_k^{2m}\right) \boldsymbol{w}, \boldsymbol{w}\right) + \delta a\left(R_k^m \boldsymbol{w}, R_k^m \boldsymbol{w}\right) \\
&= \delta a\left(\boldsymbol{w}, \boldsymbol{w}\right)
\end{aligned}
$$

from the induction hypothesis, (82), (83), Lemma 4.8 and Lemma 4.10. □

5. Numerical experiments

In this section, we report the numerical results that support the theoretical estimates and demonstrate the performance of the V–cycle multigrid methods. We use the computational domain $\Omega = (-1,1)^3$. As the initial triangulation $\mathcal{T}_0$, we use eight identical unit cubes.

In the first set of experiments, we carry out the $k-$th level multigrid algorithm with the edge-based smoother introduced in Section 3.2 with m smoothing steps and the damping factor $\eta_E = 1/13$. We compute the contraction numbers for $k = 1, \ldots, 4$ and $m = 1, \ldots, 5$. We perform the experiments five times with the coefficient $\alpha = 0.01, 0.1, 1.0, 10.0, 100.0$. The results are reported in Table 1. As we see the result, the V–cycle multigrid methods provide uniform convergence.

We next perform similar experiments to the first set of experiments. The only differences are the smoother, the vertex-based smoother, and the damping factor $\eta_V = 1/9$. Other general settings are identical. The contraction numbers are reported in Table 2. The results are compatible with our theory and the uniform convergence of the methods is observed.

In the last round of experiments, we perform numerical tests to compare the computation times of multigrid methods with the nonoverlapping smoothers suggested in this paper and the overlapping smoother proposed in [4]. In each experiment, we consider the multigrid methods as iterative solvers and check the elapsed CPU time in seconds and the iteration counts. We use the parameters $k = 4, \alpha = 1.0$, $m = 1, \ldots, 5$, and the tolerance 10^{-5} for the stopping criterion, a relative reduction of the ℓ^2-norm. All tests were conducted on a desktop system equipped with an Intel Core i9 3.6GHz CPU. The results are presented in Table 3. As we see the results, the vertex-based method outperforms in terms of total CPU elapsed time, although it takes longer time in one multigrid sweep than any other methods. The edge-based method has the least elapsed time per one iteration and a total time comparable to the overlapping method, but 1.17 times faster on average.

We note that a part of implementations is based on the MFEM library; see [1, 25] for more details. The implemented codes are available at `https://github.com/duksoon-open/MG_ND`.

Table 1. Edge-Based Methods

		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$\alpha = 0.01$	$k = 1$	7.88E-01	6.27E-01	4.44E-01	3.25E-01	3.11E-01
	$k = 2$	8.81E-01	7.79E-01	6.99E-01	5.90E-01	5.62E-01
	$k = 3$	9.24E-01	8.56E-01	7.92E-01	7.36E-01	6.77E-01
	$k = 4$	9.40E-01	8.90E-01	8.41E-01	7.98E-01	7.56E-01
$\alpha = 0.1$	$k = 1$	8.83E-01	7.85E-01	7.03E-01	6.33E-01	5.73E-01
	$k = 2$	9.30E-01	8.70E-01	8.18E-01	7.55E-01	7.25E-01
	$k = 3$	9.53E-01	9.19E-01	8.88E-01	8.52E-01	8.19E-01
	$k = 4$	9.72E-01	9.53E-01	9.35E-01	9.18E-01	9.01E-01
$\alpha = 1.0$	$k = 1$	9.07E-01	8.31E-01	7.69E-01	7.19E-01	6.77E-01
	$k = 2$	9.44E-01	9.17E-01	8.85E-01	8.58E-01	8.30E-01
	$k = 3$	9.70E-01	9.59E-01	9.44E-01	9.30E-01	9.17E-01
	$k = 4$	9.81E-01	9.72E-01	9.65E-01	9.63E-01	9.56E-01
$\alpha = 10.0$	$k = 1$	9.09E-01	8.36E-01	7.77E-01	7.30E-01	6.91E-01
	$k = 2$	9.49E-01	9.25E-01	8.97E-01	8.74E-01	8.55E-01
	$k = 3$	9.72E-01	9.65E-01	9.53E-01	9.42E-01	9.33E-01
	$k = 4$	9.82E-01	9.76E-01	9.73E-01	9.71E-01	9.66E-01
$\alpha = 100.0$	$k = 1$	9.10E-01	8.37E-01	7.78E-01	7.31E-01	6.93E-01
	$k = 2$	9.49E-01	9.26E-01	8.98E-01	8.76E-01	8.57E-01
	$k = 3$	9.73E-01	9.66E-01	9.54E-01	9.43E-01	9.34E-01
	$k = 4$	9.82E-01	9.76E-01	9.73E-01	9.72E-01	9.67E-01

Table 2. Vertex-Based Methods

		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$\alpha = 0.01$	$k = 1$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 2$	7.91E-01	6.26E-01	4.94E-01	3.92E-01	3.12E-01
	$k = 3$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 4$	7.90E-01	6.25E-01	4.94E-01	3.91E-01	3.09E-01
$\alpha = 0.1$	$k = 1$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 2$	7.91E-01	6.25E-01	4.94E-01	3.91E-01	3.10E-01
	$k = 3$	7.91E-01	6.26E-01	4.95E-01	3.91E-01	3.10E-01
	$k = 4$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
$\alpha = 1.0$	$k = 1$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 2$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.10E-01
	$k = 3$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
	$k = 4$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
$\alpha = 10.0$	$k = 1$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 2$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.10E-01
	$k = 3$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
	$k = 4$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
$\alpha = 100.0$	$k = 1$	7.90E-01	6.24E-01	4.93E-01	3.90E-01	3.08E-01
	$k = 2$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.10E-01
	$k = 3$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01
	$k = 4$	7.91E-01	6.26E-01	4.95E-01	3.92E-01	3.11E-01

TABLE 3. CPU time (in seconds), iteration counts, and average CPU time per iteration (in seconds) for multigrid methods based on edge-based, vertex-based, and overlapping smoothers

		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
edge-based	total time	148.83	145.71	160.81	165.90	185.11
	iters	389	204	149	118	105
	time per iter	0.383	0.714	1.079	1.406	1.763
vertex-based	total time	39.71	38.43	40.67	40.57	42.20
	iters	45	22	15	11	9
	time per iter	0.882	1.747	2.711	3.688	4.689
overlapping	total time	188.12	188.76	186.55	191.07	187.60
	iters	264	132	88	66	52
	time per iter	0.713	1.430	2.120	2.895	3.608

6. Concluding remarks

In this work, new multigrid methods based on nonoverlapping domain decomposition smoothers for vector field problems posed in $H(\mathbf{curl})$ have been developed and analyzed. The suggested methods provide uniform convergence and the numerical experiments are consistent with the theoretical results. We note that it is possible to extend our results to problems with nonhomogeneous boundary conditions with only minor changes.

There are a few challenges. In our convergence analysis, we assumed that the coefficients are constants and the domain is convex. The numerical results in [29] show that the V–cycle multigrid methods work well without the assumptions, i.e. constant coefficients and convex domain. Our theory can therefore be extended to coefficients with jumps or nonconvex domains. We believe that the results in [17, 20] would be good ingredients for establishing the stronger convergence analysis. We are also interested in the extension of our results with the use of the tetrahedral Nédélec finite element of the lowest order in order to handle more general convex polyhedral domains.

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