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EQUIVALENT CONDITIONS FOR A NOETHERIAN MODULE

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Abstract. In this paper, we present useful characterizations for Noetherian modules using the theory of prime submodules.

1. Introduction

In this paper, all rings are commutative with identity $1 \neq 0$ and all modules are unitary. Throughout, R will denote an arbitrary ring and M an arbitrary R-module. Also the notation $\mathbb{Z}$ (resp. $\mathbb{Q}$) will denote the ring of integers (resp. the field of fractions of $\mathbb{Z}$). If N is a subset of M we write $N \leq M$ (resp. $N \lneqq M$) to indicate that N is a submodule (resp. a proper submodule) of M. Let N and K be two submodules of M. Then the *colon ideal of N into K* is defined to be $(N :_R K) = \{r \in R : rK \subseteq N\}$. Particularly, we use $Ann_R(M)$ instead of $(0 :_R M)$ and $(N :_R m)$ instead of $(N :_R Rm)$, where Rm is the cyclic submodule of M generated by an element $m \in M$. A submodule P of M is called *prime or p-prime* if $P \neq M$ and for $p = (P : M)$, whenever $re \in P$ for $r \in R$ and $e \in M$, we have $r \in p$ or $e \in P$. If P is a prime submodule of M, then $p = (P :_R M)$ is a prime ideal of $V(Ann_R(M))$ and for every $e \in M \backslash P$, $(P :_R e) = p$. The set of all prime submodules of M is called the *prime spectrum* of M and denoted by $Spec(M)$ ([3]). The saturation of a submodule N with respect to p is the contraction of N_p in M and denoted by $S_p(N)$. It is known that $S_p(N) = N^{ec} = \{x \in M | \; tx \in N \text{ for some } t \in R - p\}$ ([6]). In particular, $S_p(pM)$ is denoted by $M(p)$ ([9]).

Karakas, in 1972 ([2]), Smith, in 1994 ([11]), and Parkash and Kour, in 2021 ([9]), obtained useful criteria for the Noetherianness of a finitely generated module by proving fundamental theorems. The following result is a brief summary of this investigation: Let M be a finitely generated R-module, then M is Noetherian iff for every prime ideal p of R with $Ann_R(M) \subseteq p$, there exists a finitely generated submodule N_p of M such that $pM \subseteq N_p \subseteq M(p)$ ([9]) iff every prime submodule of M is finitely generated ([2]). Lu, in 1983

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([3]), Jothilingam, in 2000 ([1]), and Naghipour, in 2005 ([8]), provided the other proofs of these results.

In this paper, we extend these results from finitely generated modules to other classes of modules using the theory of prime submodules by providing simpler proof (Theorem 2.2 and Corollary 2.5).

2. Main results

Remark 2.1. *Let M be an R-module.*

(a) *Let M be finitely generated and p be a prime ideal of R containing $Ann_R(M)$. Then there exists a prime submodule N of M with $(N :_R M) = p$ (*[4, Lemma, p.3746]*).*
(b) *Let N be a submodule of M and p be a prime ideal of R. Suppose that $S_p(N)$ is a p-prime submodule of M. Then any p-prime submodule of M, which contains N, must contain $S_p(N)$ (*[6, Result 3 (2)]*).*
(c) *Let N be a submodule of M such that $(N :_R M)$ is a maximal ideal of R. Then N is a prime submodule of M (*[3, Proposition 2]*).*
(d) *Let M be finitely generated and p be a prime ideal of R. Then $S_p(pM) = M(p)$ is a p-prime submodule of M if and only if $p \in V(Ann_R(M))$ (*[6, Corollary 3.8]*).*
(e) *A submodule N of M is called extended submodule, if there exists an ideal I of R such that $N = IM$ (*[1, Definition, p. 4864]*).*
(f) *If $Spec(M) \neq \emptyset$, the mapping $\psi : Spec(M) \to Spec(R/Ann_R(M))$ such that $\psi(P) = (P :_R M)/Ann_R(M)$ for every $P \in Spec(M)$, is called the natural map of $Spec(M)$. An R-module M is said to be primeful if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the natural map of $Spec(M)$ is surjective (*[5] *and* [7]*).*

In the next theorem, we extend Cohen's theorems for modules.

Theorem 2.2. *Let M be an R-module. If there exists a maximal ideal q of R such that qM is finitely generated, then the following statements are equivalent:*

(a) *Every proper submodule of M is finitely generated;*
(b) *For every prime ideal p of $V(Ann_R(M))$ and every prime submodule N of M with $(N :_R M) = p$, there exist two finitely generated submodules L and H of M such that $pM \subseteq L \subseteq N \subseteq H$.*

Proof. (a) $\Rightarrow$ (b): is clear.
(b) $\Rightarrow$ (a): We assume that M has a proper submodule such that it is not finitely generated. Put $\Omega = \{K \mid K \subsetneqq M$ *and* K *is not finitely generated*$\}$. Using Zorn's lemma, we show that Ω has a maximal element. To see this, let $\{K_\alpha\}_{\alpha\in\Lambda}$ be an ascending chain of elements of Ω. If $\bigcup_{\alpha\in\Lambda} K_\alpha = M$, then $q(\bigcup_{\alpha\in\Lambda} K_\alpha) = \bigcup_{\alpha\in\Lambda} qK_\alpha = qM$. Since qM is finitely generated, then there

exists an element α^* of Λ such that $qK_{\alpha^*} = qM$. But $qK_{\alpha^*} \neq M$, so $qM \neq M$ and hence $(K_{\alpha^*} :_R M) = (qM :_R M) = q$. This implies that K_{α^*} is a q-prime submodule of M by Remark 2.1 (c). Now by assumption, there exists a finitely generated submodule H_* of M such that $qM \subseteq K_{\alpha^*} \subseteq H_*$. In this case, H_*/qM is a finite-dimensional R/q-vector space and, therefore, the R/q-subspace K_{α^*}/qM of H_*/qM is also finite-dimensional. So K_{α^*}/qM is a finitely generated R-module. Consequently, K_{α^*} is a finitely generated R-module, a contradiction. Therefore, we conclude that $\bigcup_{\alpha\in\Lambda} K_\alpha \neq M$ and hence according to the definition of Ω, we have $\bigcup_{\alpha\in\Lambda} K_\alpha \in \Omega$. It follows that Ω has a maximal element, say N. We claim that N is a prime submodule of M. To see this, assume the contrary. Then there exists $r \in R$ and $e \in M$ such that $re \in N$, $r \notin (N :_R M)$, and $e \notin N$. Hence $N \subsetneqq (N :_M r)$ and $N \subsetneqq N + rM$. This implies that both $(N :_M r)$ and $N + rM$ are two finitely generated submodules of M by the maximal property of N. On the other hand, we have the following R-module isomorphisms,

$$(N + rM)/rM \cong N/(N \cap rM),\ N \cap rM \cong (N :_M r)/(0 :_M r).$$

Combining this relations, we conclude that N is finitely generated, which is a contradiction. Therefore, N is a prime submodule of M. Put $(N :_R M) = p$. By assumption, there exist two finitely generated submodules L and H of M such that $pM \subseteq L \subseteq N \subseteq H$. Hence $(pM :_R M) = (L :_R M) = (N :_R M) = p$. But N is not finitely generated, so we may choose an element e of $H\backslash N$. Because N is prime, $(N :_R e) = p$ and, therefore, $(N/L :_{R/p} \bar{e}) = 0_{R/p}$, where $\bar{e} = e + L$. Also we have $N/L + \bar{e}R/p$ is a finitely generated R/p-submodule of M by the maximal property of N. Then N/L is a finitely generated R/p-submodule of M by using the following R/p-module isomorphisms,

$$\frac{N/L}{(N/L) \cap \bar{e}R/p} \cong \frac{(N/L) + \bar{e}R/p}{\bar{e}R/p},\ (N/L) \cap \bar{e}R/p \cong \frac{(N/L :_{R/p} \bar{e})}{(0 :_{R/p} \bar{e})} = 0.$$

Hence N/L is finitely generated R-module and thus N is finitely generated R-module, which is a contradiction. This completes the proof. □

Corollary 2.3. *Let M be an R-module. Then the following are equivalent:*

(a) *M is a Noetherian R-module;*
(b) *For every submodule N of M, there exists a finitely generated submodule K of N such that $(N :_R M) = (K :_R M)$;*
(c) *For every submodule N of M, $(N :_R M)M$ is finitely generated;*
(d) *For every ideal I of R, IM is finitely generated, in other words, every extended submodule of M is finitely generated.*

Proof. (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (d), and (d) $\Rightarrow$ (c) are clear.
(b) $\Rightarrow$ (a): By hypothesis, there exists a finitely generated submodule K of M such that $(K :_R M) = (M :_R M)$. This implies that $M = K$ is finitely generated. Let $q \in Max(R) \cap V(Ann_R(M))$. Then $qM \neq M$ by Remark 2.1 (a). According to the assumption, there exists a finitely generated submodule

L of qM such that $(L :_R M) = (qM :_R M) = q$, whence $qM = L$ is finitely generated. Let p be an element of $V(Ann_R(M))$ and N be a prime submodule of M with $(N :_R M) = p$. Then by assumption, there exists a finitely generated submodule K of N such that $(N :_R M) = (K :_R M) = p$. This implies that $pM \subseteq K \subseteq N \subseteq M$. It follows that every proper submodule of M is finitely generated by Theorem 2.2. Therefore, M is Noetherian. (c) $\Rightarrow$ (b): Let N be a submodule of M. Set $K = (N :_R M)M$. By assumption, K is finitely generated. On the other hand, it is easy to see that $K \subseteq N$ and $(K :_R M) = (N :_R M)$. □

Corollary 2.4. *Let M be a finitely generated R-module. Then the following are equivalent:*

(a) *M is a Noetherian R-module;*
(b) *For every prime submodule N of M, there exists a finitely generated submodule K of N such that $(N :_R M) = (K :_R M)$;*
(c) *For every prime ideal p of $V(Ann_R(M))$, there exists a finitely generated submodule N of M such that $pM \subseteq N \subseteq M(p)$;*
(d) *For every prime ideal p of $V(Ann_R(M))$, we have pM or $M(p)$ is finitely generated;*
(e) *For every prime submodule N of M, we have N or $(N :_R M)M$ is finitely generated.*

Proof. (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (e), and (d) $\Rightarrow$ (c) are clear.
(b) $\Rightarrow$ (c): Let $p \in V(Ann_R(M))$. Then by Remark 2.1 (d), $M(p)$ is a p-prime submodule of M. By assumption, there exists a finitely generated submodule K of $M(p)$ such that $(K :_R M) = (M(p) :_R M) = p$. Therefore, $pM \subseteq K$. This completes the proof.
(c) $\Rightarrow$ (a): Let $q \in Max(R) \cap V(Ann_R(M))$. Then there exists a finitely generated submodule L of M such that $qM \subseteq L \subseteq M(q)$. It is easy to see that $qM = L = M(q)$ by Remark 2.1 (b) and (c). Hence qM is finitely generated. Let p be an element of $V(Ann_R(M))$ and K be a prime submodule of M with $(K :_R M) = p$. By hypothesis, there exists a finitely generated submodule N of M such that $pM \subseteq N \subseteq M(p)$. But by Remark 2.1 (b), $M(p) \subseteq K$. Thus $pM \subseteq N \subseteq K \subseteq M$. Now we conclude that M is Noetherian by Theorem 2.2.
(e) $\Rightarrow$ (a): It is easy to see that there exists a maximal ideal q of R such that qM is a finitely generated submodule of M. Now the claim is immediate by using Theorem 2.2. □

In the next Corollary, we extend Cohen's theorem in [2].

Corollary 2.5. *Let M be an R module. Then M is Noetherian if and only if every prime submodule of M is finitely generated, if M belongs to any of the following classes of modules:*

(a) *M is primeful.*
(b) *M is finitely generated.*

(c) *R is integral domain and M is projective over R.*
(d) *M is faithfully flat.*

Proof. (a): Let M be a non-zero primeful R-module. The necessity is clear. To prove the sufficiency, let $q \in Max(R) \cap V(Ann_R(M))$. Then $qM \neq M$ by Remark 2.1 (f). Thus $(qM :_R M) = q$ and hence qM is prime by Remark 2.1 (c). By assumption, qM is finitely generated. Therefore, every proper submodule of M is finitely generated by Theorem 2.2. To show that M is Noetherian, it is suffices to prove that M is finitely generated. Put $\bar{M} = M/qM$ and $\bar{R} = R/q$. By the above arguments, every proper subspace of $\bar{M}$ is finite-dimensional. We prove that $\bar{M}$ is a finite-dimensional vector space. Assume the contrary and let $\bar{M} = \bigoplus_{\alpha\in\Lambda} \bar{R}_\alpha$, where Λ is infinite set and $\bar{R}_\alpha$ is isomorphic to $\bar{R}$ for every $\alpha \in \Lambda$. Let $\alpha_0 \in \Lambda$. Then $\bigoplus_{\alpha_0\neq\alpha\in\Lambda} \bar{R}_\alpha$ is a proper subspace of $\bar{M}$ which is not finite-dimensional, a contradiction. Hence $\bar{M}$ is a finitely generated $\bar{R}$-module. Therefore $\bar{M}$ is a finitely generated R-module. On the other hand, qM is finitely generated, whence M is a finitely generated R-module. It follows that M is Noetherian. The parts (b), (c), and (d) follows from [7, Theorem 2.2], [7, Corollary 4.3], and [7, Proposition 2.4], respectively. □

Example 2.6. (a) *Put $M = \mathbb{Q}$ and $R = \mathbb{Z}$. Then $Spec_R(M) = \{(0)\}$ by* [4, Corollary p. 3746]. *Hence every prime submodule of M is finitely generated, while M is not Noetherian.*

(b) *Let M be an infinite-dimensional vector space over the field R. Clearly, for every prime ideal p of $V(Ann_R(M))$, pM is finitely generated. Also there exists a maximal ideal q of R such that qM is finitely generated. However, every proper submodule of M is not necessarily finitely generated.*

(c) *Let R be a discrete valuation domain with maximal ideal p. If M is the field of quotients of R, then we have the following:*
- *$Spec_R(M) = \{(0)\}$ by* [4, Theorem 1].
- *$V(Ann_R(M)) = \{(0), p\}$ and $pM = M$.*
- *M is not finitely generated.*
- *Every proper submodule of M is finitely generated, by* [10, Proposition 2.2].

In view of Example 2.6 above, we obtain the following results.

Result 2.7. (a) *The condition "M is primeful" in Corollary 2.5 is a necessary condition and can not be omitted, see Example 2.6 (a).*

(b) *The condition "M is finitely generated" in Corollary 2.4 is a necessary condition and can not be omitted, see Example 2.6 (b).*

(c) *The condition "there exists a maximal ideal q of R such that qM is a finitely generated submodule of M" in Theorem 2.2 is not necessarily a necessary condition, see Example 2.6 (a) and (c).*

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