Honam Mathematical J. **46** (2024), No. 2, pp. 291–306
https://doi.org/10.5831/HMJ.2024.46.2.291

ON TRIPLE SEQUENCES IN GRADUAL 2-NORMED LINEAR SPACES

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Abstract. The concept of lacunary statistical convergence of triple sequences with respect to gradual 2-normed linear spaces is introduced in this research. We learn about its link to some inclusion and fundamental properties. The notion of lacunary statistical Cauchy triple sequences is introduced in the conclusion, and it is demonstrated that it is equivalent to the idea of lacunary statistical convergence.

1. Introduction

Fast [12] independently created the idea of statistical convergence in 1951 using the concept of natural density. It is then analyzed in more detail in relation to summability theory and sequence space in ([15], [21], [29], [36], [37], [38]). For a comprehensive analysis of statistical convergence, one can reference several studies of other mathematicians from different countries ([1], [23], [30]). The notion of lacunary statistical convergence was first developed by Fridy [16] in 1993 as one of the extensions of statistical convergence (for further information on statistical convergence, please refer to [15]).

A lacunary sequence is an increasing integer sequence $\theta = (k_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfying $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$, as $n \to \infty$. A real-valued sequence (x_k) is lacunary statistically convergent (abbreviated S_θ-convergent) to a real number l, if for any $\varepsilon > 0, \lim_n \frac{1}{h_n} |\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0$, when $I_n = (k_{n-1}, k_n]$. The symbol for l in this context is $S_\theta - \lim(x_k) = l$ or $x_k \to l\,(S_\theta)$, and it is known as the lacunary statistical limit of the sequence (x_k). Additionally, S_θ denotes the collection of every statistical convergent lacunary sequences associated with the lacunary sequence θ. Comparison of statistical convergence and lacunary statistical convergence was demonstrated by Fridy and Orhan in [17]. Freedman et al. [14] investigated the connection between

Received November 3, 2023. Accepted December 8, 2023.
2020 Mathematics Subject Classification. 03E72, 40A35, 40A05.
Key words and phrases. triple sequence, gradual normed linear space, lacunary sequence, lacunary statistical convergence, 2-normed space.
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the two sequence spaces $|\sigma_1|$ and N_θ in defined as follows:

$$|\sigma_1| = \left\{ (x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - l| = 0 \right\}$$

and

$$N_\theta = \left\{ (x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{h_n} \sum_{k \in I_n} |x_k - l| = 0 \right\},$$

In [27], [31], [40] numerous further references may be found for more details on lacunary convergence and its generalizations.

Gähler [18] introduced the 2-metric in 1963. In the progress of his research, Gähler [19] introduced the mathematical structure of 2-normed spaces, a generalization of normed linear spaces. Researchers have been studying this issue for decades and have discovered a variety of intriguing aspects of it (see, [20], [21], [22], [24], [26], [35]).

In 2008, as components of fuzzy intervals, gradual real numbers were initially introduced by Fortin et al. [13]. Gradual real numbers are essentially understood by the assignment function that corresponds to them, which is defined in the range $(0, 1]$. Therefore, it is possible to think of each real number as a gradual number with a constant assignment function. While maintaining all of the algebraic characteristics of classical real numbers, these gradual real numbers have applications in computation and optimization issues. The idea of gradual normed linear space first came up in 2011 by Sadeqi and Azari [33]. They examined a variety of features of the space from both topological and algebraic viewpoints. One may consult [6], [25], [41] for a thorough research of gradual real numbers. The exploration of sequences convergence within gradual normed linear spaces remains relatively unexplored, still in its nascent stages. The existing body of research, however, demonstrates a notable resemblance in the convergence behavior of sequences within gradual normed linear spaces.

Ettefagh and his colleagues [11] recently introduced the concept of sequence convergence in gradually normed linear spaces. Their work included the investigation of various topological properties (as detailed in [10]). In a separate work by Choudhury and Debnath ([2], [3]) the notion of sequence convergence in gradual normed linear spaces was extended to ideal convergence and lacunary statistical convergence. Consequently, we can state that the logical and natural next step of the research is the lacunary statistical convergence of triple sequences in gradual 2-normed linear spaces.

2. Preliminaries

Definition 2.1. [13] *An assignment function, represented as $A_{\widetilde{r}} : (0, 1] \to \mathbb{R}$, describes a gradual real number, denoted as $\widetilde{r}$. $G(\mathbb{R})$ denotes the set that contains all gradually increasing real numbers. A gradual real number $\widetilde{r}$ is* *considered non-negative when for every* $\xi \in (0,1], A_{\tilde{r}(\xi)} \geq 0$. $G^*(\mathbb{R})$ *stands for the set of all gradual real numbers that are not negative.*

The following definitions describe the gradual operations for elements in $G(\mathbb{R})$:

Definition 2.2. *Assuming* $*$ *represent an arbitrary operation within the real numbers set* $\mathbb{R}$, *and given* $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ *with assignment functions.* $A_{\tilde{r}_1}$ *and* $A_{\tilde{r}_2}$, *we define* $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ *using the assignment function* $A_{\tilde{r}_1 * \tilde{r}_2}$, *as specified by* $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi), \forall \xi \in (0,1]$.

The gradual summation of $\tilde{r}_1$ *and* $\tilde{r}_2, \tilde{r}_1 + \tilde{r}_2$, *is determined as* $A_{\tilde{r}_1 + \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi)$ *and the gradual scalar product with* $c\tilde{r}$ *(where* $c \in \mathbb{R}$*), is determined as* $A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi)$ *for* $\forall \xi \in (0,1]$ [13].

For each $\forall \xi \in (0,1]$, the constant gradual real number $\tilde{p}$ is defined. Its identity is represented by the constant assignment function $A_{\tilde{p}}(\xi) = p$, which represents a real number $p \in \mathbb{R}$.

Specifically, the constant gradual numbers $\tilde{0}$ and $\tilde{1}$ are defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$, respectively. It is simple to demonstrate that $G(\mathbb{R})$ becomes a true vector space when equipped with gradual addition and gradual scalar multiplication.

Definition 2.3. [33] *Consider* X *as a real vector space. We define the function* $\|.\|_G : X \to G^*(\mathbb{R})$ *as a gradual norm on* X *if, for any* $\xi \in (0,1]$, *for any* $x, y \in X$, *the first three requirements are true:*

$(G1)$ $A_{\|x\|_G}(\xi) = A_{\tilde{0}}(\xi)$ *if and only if* $x = 0$;

$(G2)$ $A_{\|\lambda x\|_G}(\xi) = |\lambda| A_{\|x\|_G}(\xi)$ *for any* $\lambda \in R$;

$(G3)$ $A_{\|x+y\|_G}(\xi) \leq A_{\|x\|_G}(\xi) + A_{\|y\|_G}(\xi)$.

Gradual normed linear space (GNLS) is a term used to describe the pair $(X, \|.\|_G)$.

Example 2.4. [33] *Consider a space* $X = \mathbb{R}^m$; *so* $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, \xi \in (0,1]$, *define* $\|.\|_G$ *by*

$$A_{\|x\|_G}(\xi) = e^{\xi} \sum_{i=1}^{m} |x_i|.$$

In this context on $\mathbb{R}^m, \|.\|_G$ *is a gradual norm and* $(\mathbb{R}^m, \|.\|_G)$ *is a GNLS.*

Definition 2.5. *Suppose we have a sequence* (x_k) *within the GNLS* $(X, \|.\|_G)$. *We say that* (x_k) *is gradually bounded if, for any* $\xi \in (0,1]$, *there is* $B = B(\xi) > 0$ *such that for all* $k \in \mathbb{N}$, $A_{\|x_k\|_G} \leq B$ *holds true.*

Definition 2.6. *Consider a sequence* (x_k) *within the GNLS* $(X, \|.\|_G)$. *We can state that* (x_k) *gradually converges to* $x \in X$, *for all* $\xi \in (0,1]$ *and* $\varepsilon > 0$, *by asserting that for* $A_{\|x_k - x\|_G}(\xi) < \varepsilon, \forall k \geq N$, *there exists* $N(= N_\varepsilon(\xi)) \in \mathbb{N}$.

Definition 2.7. *Consider a sequence* (x_k) *within the GNLS* $(X, \|.\|_G)$*. We can state that* (x_k) *is gradually Cauchy, for all* $\forall \xi \in (0,1]$ *and* $\varepsilon > 0$*, by asserting that for* $A_{\|x_k - x_j\|_G}(\xi) < \varepsilon, \forall k, j \geq N$*, there exists* $N (= N_\varepsilon(\xi)) \in \mathbb{N}$.

Theorem 2.8. [11] *Any sequence that converges gradually in* X *is by definition a gradually Cauchy sequence if we consider* $(X, \|.\|_G)$ *to be a GNLS.*

Definition 2.9. [3] *Take* (x_k) *stands for a real-valued sequence and* $\theta = (k_n)$ *for a lacunary sequence. We express* (x_k) *as lacunary statistically Cauchy or* S_θ*-Cauchy if there exists a* $(x_{k\prime(n)})$ *subsequence of* (x_k) *satisfying the following three conditions:*
(i) $k\prime(n) \in I_n$*, for every* n;
(ii) $\left(x_{k\prime(n)}\right) \to x (n \to \infty)$;
(iii) *For every* $\varepsilon > 0, \lim_n \frac{1}{h_n} \left|\left\{k \in I_n : \left|x_k - x_{k\prime(n)}\right| \geq \varepsilon\right\}\right| = 0.$

Theorem 2.10. S_θ*-convergence is a prerequisite for the real-valued sequence* (x_k) *and sufficient condition for it to be* S_θ*-Cauchy.*

Definition 2.11.

$$|\sigma_1(G)| = \left\{(x_k) : \text{ for some } x \in X \text{ and } \forall \xi \in (0,1], \lim_n \frac{1}{n} \left(\sum_{k=1}^{n} A_{\|x_k - x\|_G}(\xi)\right) = 0\right\}$$

and

$$N_\theta(G) = \left\{(x_k) : \text{ for some } x \in X \text{ and } \forall \xi \in (0,1], \lim_n \frac{1}{h_n} \left(\sum_{k \in I_n} A_{\|x_k - x\|_G}(\xi)\right) = 0\right\}.$$

$(X, \|.\|_G)$ *might be any GNLS. As mentioned above, the new sequence spaces* $|\sigma_1(G)|$ *and* $N_\theta(G)$ *are defined.*

Definition 2.12. *Consider a sequence* (x_k) *in GNLS* $(X, \|.\|_G)$*. For every* $\xi \in (0,1]$ *and* $\varepsilon > 0$*, when the natural density of the set* $\left\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\xi) \geq \varepsilon\right\} = 0$*, we specify that the sequence* (x_k) *is gradually statistically convergent to* $x \in X$*, so we denote it as* $S(G)$ *convergent. Representing,* $S(G) - \lim x_k = x$ *or* $x_k \to x (S(G))$ *is written. Additionally, the collection of all gradually statistically convergent sequences in* X *is denoted by* $S(G)$.

Definition 2.13. X *is a vector space on the real space* $\mathbb{R}$ *with dimension greater than* 1*, when the following conditions are satisfied on the function* $\|\cdot,\cdot\| : X \times X \to \mathbb{R}_{\geq 0}$:
(i) *For every* $x, y \in X, \|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent,*
(ii) $\|y, x\| = \|x, y\|$ *for all* $x, y \in X$,
(iii) $|\alpha| \|x, y\| = \|\alpha x, y\|$, *whenever* $\alpha \in \mathbb{R}$ *and* $x, y \in X$,
(iv) $\|y + x, z\| \leq \|y, z\| + \|x, z\|$ *for all* $x, y, z \in X$.
The pair $(X, \|\cdot,\cdot\|)$ *is called a 2-normed space* [18], [19].

Example 2.14. *Let us take X as the Euclidean plane $\mathbb{R}^2$, where the metric used is the area of the parallelogram formed by the 2-norm $\|x,y\| :=$ x and y vectors. This norm can be explicitly defined by the formula:*

$$\|x,y\| = |x_1y_2 - x_2y_1|, \quad \text{where } x = (x_1,x_2), y = (y_1,y_2).$$

Pringsheim [32] provided the definition of the convergence of double sequences in 1900. The other researchers continued Şahiner et al. [34] initially suggested the idea of triple sequences (see [5], [7], [8], [9], [28]). Recall the concept of statistical convergence for triple sequences:

Definition 2.15. *Let $K(n,m,o)$ be the number of (j,k,l) in K such that $j \leq n$, $k \leq m$ and $l \leq o$, and let $K \subset \mathbb{N}^3$ be a three-dimensional set of positive integers. Hence, the three-dimensional analog of the natural density is defined as follows:*

A set $K \subset \mathbb{N}^3$'s lower asymptotic density is denoted by the symbol $\underline{\delta_3}(K) = P - \liminf_{nmo} \frac{K(n,m,o)}{nmo}$. So when $\left(\frac{K(n,m,o)}{nmo}\right)$ has a limit in the sense of Pringsheim, K is said to have a triple natural density and is defined as $\delta_3(K) = P - \lim_{nmo} \frac{K(n,m,o)}{nmo}$.

Definition 2.16. *A real (complex) triple sequence $x = (x_{jkl})$ is the name given to the function $x : \mathbb{N}^3 \to \mathbb{R}$(or $\mathbb{C}$). Given that $|x_{jkl} - L| < \varepsilon$ whenever $j,k,l > N$, there is $N \in \mathbb{N}$ such that for all $\varepsilon > 0$. Thus a triple sequence $x = (x_{jkl})$ converges to a number L in Pringsheim's sense.*

Definition 2.17. *The space of all P-convergent sequences will be represented by the symbol c^3. A bounded triple sequence is one for which $|x_{jkl}| < M$ for all (j,k,l) exists as a positive number M and denotes such bounded triple sequences by $\|x\|_{(\infty,3)} = \sup_{jkl} |x_{jkl}| < \infty$. We will also use the symbol l^3_∞ to indicate the set of all bounded triple sequences. A P-convergent triple sequence need not be bounded, in contrast to the situation for a single sequence.*

Definition 2.18. *If for every $\varepsilon > 0$, $\delta_3\left(\left\{(j,k,l) \in \mathbb{N}^3 : |x_{jkl} - L| \geq \varepsilon\right\}\right) = 0$, then a real triple sequence $x = (x_{jkl})$ is statistically convergent to L in Pringsheim's sense.*

Definition 2.19. *By triple lacunary sequence, we mean an increasing sequence $\theta_3 = \theta_{rst} = \{(j_r,k_s,l_t)\}$ of positive integers satisfying: $j_0 = 0, h_r = j_r - j_{r-1} \to \infty$ as $r \to \infty, k_0 = 0, h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$ and $l_0 = 0, h_t = l_t - l_{t-1} \to \infty$ as $t \to \infty$.*

Note that $k_{rst} = j_rk_sl_t, h_{rst} = h_rh_sh_t$ and θ_{rst} we denote the intervals as follows: $I_{rst} = \{(j,k,l) : j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s, l_{t-1} < l \leq l_t\}, q_r = \frac{j_r}{j_{r-1}}, q_s = \frac{k_s}{k_{s-1}}, q_t = \frac{l_t}{l_{t-1}}$ and $q_{rst} = q_rq_sq_t$.

3. Main results

Throughout the article, $X, x = \{x_{jkl}\}$ and $\theta_3 = \{\theta_{nop}\}$ will be taken as a real vector space, a triple sequence and any triple lacunary sequence in the G2-NLS $(X, \|\cdot,\cdot\|_G)$, respectively. To keep things simple, we refer to the m-tuple $(0, 0, ..., 0, 0)$ as $\mathbf{0}$.

Definition 3.1. *If for every $\xi \in (0, 1]$ and for any $x, y, z \in X$, the function $\|\cdot,\cdot\|_G : X \times X \to G^* (\mathbb{R})$ is determined to be a gradual 2-norm on X. These conditions are met:*
$(G2_1)$ $A_{\|x,z\|_G} (\xi) = A_{\tilde{0}} (\xi)$ *iff x and z are linear dependent;*
$(G2_2)$ $|\lambda| A_{\|x,z\|_G} (\xi) = A_{\|\lambda x,z\|_G} (\xi)$ *whenever $\lambda \in \mathbb{R}$;*
$(G2_3)$ $A_{\|y+x,z\|_G} (\xi) \leq A_{\|y,z\|_G} (\xi) + A_{\|x,z\|_G} (\xi)$.
When the pair is $(X, \|\cdot,\cdot\|_G)$ it is called a gradual 2-normed linear space (G2-NLS).

Definition 3.2. *Suppose that X is a d-dimensional space with $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\|_G : X \times X \to G^* (\mathbb{R})$. Then, if for any $z \in X, \varepsilon > 0$ and $\xi \in (0, 1]$, it says that $\{x_{jkl}\}$ is statistically gradually convergent to $R \in X$.*

$$\delta_3 \left(\left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G} (\xi) \geq \varepsilon \right\} \right) = 0.$$

Thus, we can write $st_3 (G2) - \lim x_{jkl} = R$ or $x_{jkl} \to R \{st_3 (G2)\}$.

Definition 3.3. *Let X be a d-dimensional space with $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\|_G : X \times X \to G^* (\mathbb{R})$. For each $z \in X, \varepsilon > 0$ and $\xi \in (0, 1]$, it is said that $\{x_{jkl}\}$ is gradually S_{θ_3}-convergent to $R \in X$*

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G} (\xi) \geq \varepsilon \right\} \right| = 0.$$

In this case, we write $S_{\theta_3}(G2) - \lim x_{jkl} = R$ or $x_{jkl} \to R \{S_{\theta_3} (G2)\}$. Additionally, let the collection of all S_{θ_3}-convergent triple sequences in X be known as $S_{\theta_3} (G2)$.

Example 3.4. *Take $X = \mathbb{R}^m$ and $\|\cdot,\cdot\|_G$ to be the 2-norm as defined in Example 2.1.*

Defined by $\theta_{nop} = \begin{cases} 0, & nop = 0 \\ 3^{nop}, & nop \geq 1 \end{cases}$. The triple sequence $\{x_{jkl}\} \in \mathbb{R}^m$ was then defined as

$$x_{jkl} = \begin{cases} (0, 0, \ldots, 0, m), & \text{if } j = r^2, k = s^2, l = t^2; \quad (r, s, t) \in \mathbb{N}^3 \\ (0, 0, \ldots, 0, 0), & \text{otherwise} \end{cases}$$

is gradually S_{θ_3}-convergent $\mathbf{0}$ in $\mathbb{R}^m$.

We have, for all $z \in X$

$$\begin{aligned}
& \lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-\mathbf{0},z\|_G}(\xi) \geq \varepsilon \right\} \right| \\
= \;& 3 \times \lim_{nop} \frac{1}{3^{nop}} \left\{ j \in \left(3^{n-1},3^{n}\right], k \in \left(3^{o-1},3^{o}\right], l \in \left(3^{p-1},3^{p}\right] : \right. \\
& \left. A_{\|x_{jkl}-\mathbf{0},z\|_G}(\xi) \geq \varepsilon \right\} \Big| \\
\leq \;& 3 \times \lim_{nop} \frac{1}{3^{nop}} \left| \left\{ j \leq 3^{n}, k \leq 3^{o}, l \leq 3^{p} : A_{\|x_{jkl}-\mathbf{0},z\|_G}(\xi) \geq \varepsilon \right\} \right| \\
\leq \;& 3 \times \lim_{nop} \frac{\left[\sqrt{nop}\right]}{nop} = 0
\end{aligned}$$

where the greatest integer " $\leq rst$" is denoted by $[rts]$.

Hence we conclude that $x_{jkl} \to \mathbf{0}$ $\{S_{\theta_3}(G2)\}$.

Example 3.5. *Take $X = \mathbb{R}$ and also take $\|\cdot\|_G$ be the norm defined as $A_{\|R\|_G} = e^{\xi}|R|$ for any $R \in \mathbb{R}$. Consider the triple sequence $\theta_3 = \{\theta_{nop}\}$ defined in Example* 3.1. *Then x defined as $x_{jkl} = j^2k^2l^2$ is not $S_{\theta_3}(G2)$-convergent.*

Rationale: For every $R \in \mathbb{R}$, take $R \leq 0$ or $R > 0$. In all of the ensuing circumstances, x will not $S_{\theta_3}(G2)$-converge to R.

Situation-I : Whenever $R \leq 0$, get $\varepsilon = \frac{1}{2}e^{\xi}$. Next, we've got

$$\begin{aligned}
& \lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon \right\} \right| \\
= \;& \lim_{nop} \frac{3}{nop} \left| \left\{ j \in \left(3^{n-1},3^{n}\right], k \in \left(3^{o-1},3^{o}\right], l \in \left(3^{p-1},3^{p}\right] : A_{\|j^2k^2l^2-R,z\|_G}(\xi) \geq \frac{1}{2}e^{\xi} \right\} \right| \\
= \;& 1,
\end{aligned}$$

for all $z \in X$.

Situation-II : If $R > 0$, then there are $(j_0,k_0,l_0) \in \mathbb{N}^3$ such that $x_{j_0k_0l_0-1} \leq R \leq x_{j_0k_0l_0}$.

Subsituation-I : Whenever $0 < R < 1$, get $\varepsilon = \frac{e^{\xi}}{2}\min\{R, 1-R\}$. The ability to illustrate the following is therefore easy:

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon \right\} \right| = 1, \quad \text{for all } z \in X.$$

Subsituation-II : If $R \geq 1$, then choose $\varepsilon = \frac{e^{\xi}}{2}\min\{R - x_{j_0k_0l_0-1}, x_{j_0k_0l_0} - R\}$. The ability to illustrate the following is therefore easy:

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon \right\} \right| = 1, \quad \text{for all } z \in X.$$

Definition 3.6. *We define the new sequence spaces* $|\sigma_{1,1,1}(G2)|$ *and* $N_{\theta_3}(G2)$ *as follows:*

$$\begin{aligned} |\sigma_{1,1,1}(G2)| &= \Big\{\{x_{jkl}\}: \text{ for some } R \in X \text{ and } \forall \xi \in (0,1], \\ &\qquad \lim_{nop} \frac{1}{nop}\left(\sum_{j,k,l=1,1,1}^{n,o,p} A_{\|x_{jkl}-R,z\|_G}(\xi)\right) = 0\Big\} \end{aligned}$$

and

$$\begin{aligned} N_{\theta_3}(G2) &= \Big\{\{x_{jkl}\}: \text{ for some } R \in X \text{ and } \forall \xi \in (0,1], \\ &\qquad \lim_{nop} \frac{1}{h_{nop}}\left(\sum_{(j,k,l)\in I_n} A_{\|x_{jkl}-R,z\|_G}(\xi)\right) = 0\Big\}, \end{aligned}$$

for all $z \in X$.

Theorem 3.7. *The following applies:*

(i) *If* $x_{jkl} \to R\{N_{\theta_3}(G2)\}$, *then* $x_{jkl} \to R\{S_{\theta_3}(G2)\}$ *but the reverse is not true.*

(ii) *If* $\{x_{jkl}\}$ *is gradually bounded, the reverse of* (i) *holds.*

Proof. (i) Assume $\varepsilon > 0$ be arbitrary and $x_{jkl} \to R\{N_{\theta_3}(G2)\}$. Then, we shall write

$$\begin{aligned} &\sum_{(j,k,l)\in I_{nop}} A_{\|x_{jkl}-R,z\|_G}(\xi) \\ \geq &\sum_{\substack{(j,k,l)\in I_{nop} \\ A_{\|x_{jkl}-R,z\|_G}(\xi)\geq\varepsilon}} A_{\|x_{jkl}-R,z\|_G}(\xi) \\ \geq &\ \varepsilon \left|\left\{(j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon\right\}\right|, \qquad \text{for all } z \in X. \end{aligned}$$

Next, we create a counterexample while taking into account the gradual 2-normed space $(\mathbb{R}^m, \|\cdot,\cdot\|_G)$, where $\|\cdot,\cdot\|_G$ is the 2-norm defined in Example 2.1. Let θ_3 be given and x be
$$\begin{pmatrix} 1 & 2 & 3 & \cdots & \left[\sqrt[3]{h_{nop}}\right] & 0 & \cdots \\ 2 & 2 & 3 & \cdots & \left[\sqrt[3]{h_{nop}}\right] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & \left[\sqrt[3]{h_{nop}}\right] & \cdots & \cdots & \left[\sqrt[3]{h_{nop}}\right] & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
in [39]. Then, we have for any $\varepsilon > 0$ with $0 < \varepsilon e^{\xi} \leq 1$,

$$\begin{aligned} \lim_{nop} \frac{1}{h_{nop}} \left|\left\{(j,k,l) \in I_{nop} : A_{\|x_{jkl}-\mathbf{0},z\|_G}(\xi) \geq \varepsilon\right\}\right| &= \lim_{nop} \frac{\left[\sqrt[3]{h_{nop}}\right]}{h_{nop}} \\ &= 0, \end{aligned}$$

for all $z \in X$.

So, $x_{jkl} \to \mathbf{0}\left\{S_{\theta_3}(G2)\right\}$. And on the other side,

$$\begin{aligned} & \lim_{nop} \frac{1}{h_{nop}} \left(\sum_{(j,k,l)\in I_{nop}} A_{\|x_{jkl}-\mathbf{0},z\|_G}(\xi) \right) \\ = & \lim_{nop} \frac{\left[\sqrt[3]{h_{nop}}\right]\left(\left[\sqrt[3]{h_{nop}}\right]\left(\left[\sqrt[3]{h_{nop}}\right]+1\right)\right)}{2h_{nop}} \\ = & \frac{1}{2} \neq 0, \end{aligned}$$

for all $z \in X$, therefore $x_{jkl} \nrightarrow \mathbf{0}\left\{S_{\theta_3}(G2)\right\}$.

(ii) Assume $x_{jkl} \to R\left\{S_{\theta_3}(G2)\right\}$ and x is gradually bounded, say thus, if x is gradual bounded, it signifies the presence of a positive constant $B = B(\xi) > 0$, satisfying $A_{\|x_{jkl}-R,z\|_G} \leq B$ for every $z \in X$ and $(j,k,l) \in \mathbb{N}^3$. Then for any $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{h_{nop}} \left(\sum_{(j,k,l)\in I_{nop}} A_{\|x_{jkl}-R,z\|_G}(\xi) \right) = & \frac{1}{h_{nop}} \left(\sum_{\substack{(j,k,l)\in I_{nop} \\ A_{\|x_{jkl}-R,z\|_G}(\xi)\geq\varepsilon}} A_{\|x_{jkl}-R,z\|_G}(\xi) \right) \\ & + \frac{1}{h_{nop}} \left(\sum_{\substack{(j,k,l)\in I_{nop} \\ A_{\|x_{jkl}-R,z\|_G}(\xi)<\varepsilon}} A_{\|x_{jkl}-R,z\|_G}(\xi) \right) \\ \leq & \frac{B}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon \right\} \right| \\ & + \varepsilon, \end{aligned}$$

for all $z \in X$, which consequently implies that $x_{jkl} \to R\left\{N_{\theta_3}(G2)\right\}$. □

Theorem 3.8. *$x_{jkl} \to R_1\left\{S_{\theta_3}(G2)\right\}$ for a fixed θ_3. Then R_1 is unique.*

Proof. If possible suppose $x_{jkl} \to R_1\left\{S_{\theta_3}(G2)\right\}$ and $x_{jkl} \to R_2\left\{S_{\theta_3}(G2)\right\}$ $(R_1 \neq R_2)$ in X. It follows that

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R_1,z\|_G}(\xi) \geq \varepsilon \right\} \right| = 0$$

and

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R_2,z\|_G}(\xi) \geq \varepsilon \right\} \right| = 0,$$

for all $\xi \in (0,1]$, $\varepsilon > 0$ and $z \in X$. Therefore,

$$M = \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R_1,z\|_G}(\xi) < \varepsilon \right\} \cap \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl}-R_2,z\|_G}(\xi) < \varepsilon \right\} \neq \varnothing.$$

Choose $\varepsilon = A_{\left\| \frac{R_1 - R_2}{2}, z \right\|_G} (\xi)$. Then, for these $r, s, t \in M$,

$$\begin{aligned} 2\varepsilon &= A_{\|R_1 - R_2, z\|_G} (\xi) \\ &\leq A_{\|x_{rst} - R_1, z\|_G} (\xi) + A_{\|x_{rst} - R_2, z\|_G} (\xi) \\ &< \varepsilon + \varepsilon = 2\varepsilon \qquad \text{for all } z \in X, \end{aligned}$$

we have a contradiction. So, $R_1 = R_2$ must be. □

Theorem 3.9. *Take $\{x_{jkl}\}$ be a triple sequence and other $\{y_{jkl}\}$ be a triple sequence in the G2-NLS $(X, \|\cdot, \cdot\|_G)$. Then,*
(i) $x_{jkl} + y_{jkl} \to R_1 + R_2 \{S_{\theta_3} (G2)\}$ and
(ii) For any $c \in \mathbb{R}$, $cx_{jkl} \to cR_1 \{S_{\theta_3} (G2)\}$.

Proof. $i)$ When $x_{jkl} \to R_1 \{S_{\theta_3} (G2)\}$ and $y_{jkl} \to R_2 \{S_{\theta_3} (G2)\}$, so for every $\xi \in (0, 1]$ and $\varepsilon > 0$,

$$\lim_{nop} \frac{1}{h_{nop}} |C_1| = 0 \text{ and } \lim_{nop} \frac{1}{h_{nop}} |C_2| = 0, \tag{3.1}$$

where

$$C_1 = \left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl} - R_1, z\|_G} (\xi) \geq \frac{\varepsilon}{2} \right\}$$

and

$$C_2 = \left\{ (j, k, l) \in I_{nop} : A_{\|y_{jkl} - R_2, z\|_G} (\xi) \geq \frac{\varepsilon}{2} \right\}, \qquad z \in X.$$

Now since the inclusion

$$\left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl} + y_{jkl} - R_1 - R_2, z\|_G} (\xi) \geq \varepsilon \right\} \subseteq C_1 \cup C_2$$

holds, we must have

$$\frac{1}{h_{nop}} \left| \left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl} + y_{jkl} - R_1 - R_2, z\|_G} (\xi) \geq \varepsilon \right\} \right| \leq \frac{1}{h_{nop}} |C_1| + \frac{1}{h_{nop}} |C_2|$$

and consequently from (3.1) we have,

$$\lim_{nop} \tfrac{1}{h_{nop}} \left| \left\{ (j, k, l) \in I_{nop} : A_{\|x_{jkl} + y_{jkl} - R_1 - R_2, z\|_G} (\xi) \geq \varepsilon \right\} \right| = 0, \quad z \in X.$$

And this concludes the evidence.

$ii)$ This section is skipped because it is easy to prove. □

We shall investigate the inclusion relationships between the sets $S(G2)$ and $S_{\theta_3}(G2)$ subject to certain restrictions on θ_3 in the following lemmas. We will use $q_{nop} = q_n q_o q_p$.

Lemma 3.10. *$S(G2) - \lim x_{jkl} = R$ implies $S_{\theta_3}(G2) - \lim x_{jkl} = R$ if and only if $\liminf_{nop} q_{nop} > 1$.*

Moreover, if $\liminf_{nop} q_{nop} = 1$, in that case a triple sequence that is $S(G2)$-convergent but not $S_{\theta_3}(G2)$-convergent to any limit exists.

Proof. Take $\liminf_{nop} q_{nop} > 1$. Under the condition of sufficiently large n, o, p; there exists a $v > 0$, ensuring that $q_n > 1 + \nu, q_o > 1 + \nu$ and $q_p > 1 + \nu$ which means $(j_n/h_n) \leq (1+v)/v, (k_o/h_o) \leq (1+v)/v$ and $(l_p/h_p) \leq (1+v)/v$. Now as $S(G2) - \lim x_{jkl} = R$, so for any $\varepsilon > 0$ and for sufficiently large n, o, p; the next inequation

$$\begin{aligned} & \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl} - R, z\|} (\xi) \geq \varepsilon \right\} \right| \\ = \;& \frac{k_{nop}}{h_{nop}} \frac{1}{k_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl} - R, z\|_G} (\xi) \geq \varepsilon \right\} \right| \\ \leq \;& \left(\frac{1+v}{v} \right)^3 \frac{1}{k_{nop}} \left| \left\{ j \leq j_n, k \leq k_o, l \leq l_p : A_{\|x_{jkl} - R, z\|_G} (\xi) \geq \varepsilon \right\} \right|, \end{aligned}$$

for all $z \in X$, yields that $S_{\theta_3}(G2) - \lim x_{jkl} = R$.

As for the opposite, assume $\liminf_{nop} q_{nop} = 1$. We want to build a triple sequence that is $S(G2)$-convergent but not to any limit $S_{\theta_3}(G2)$-convergent. Continuing as in ([14], [17]), we can choose a triple subsequence $(k_{n_a o_b p_c})$ of the lacunary triple sequence θ_3 satisfying:

$$\begin{array}{llll} (j_{n_a - 1}/j_{n_a}) > a/(a+1) & \text{and} & \left(j_{n_a - 1}/j_{n_{(a-1)}}\right) > a & \text{where} \quad n_a - n_{(a-1)} \geq 2; \\ (k_{o_b - 1}/k_{o_b}) > b/(b+1) & \text{and} & \left(k_{o_b - 1}/k_{o_{(b-1)}}\right) > b & \text{where} \quad o_b - o_{(b-1)} \geq 2; \\ (l_{p_c - 1}/l_{p_c}) > c/(c+1) & \text{and} & \left(l_{p_c - 1}/l_{p_{(c-1)}}\right) > c & \text{where} \quad p_c - p_{(c-1)} \geq 2. \end{array}$$

The following is how a gradually bounded triple sequence x in $(\mathbb{R}^m, \|\cdot,\cdot\|_G)$ is defined (in which $\|\cdot,\cdot\|_G$ is the 2-norm as described in Example 2.1):

$$x_{jkl} = \begin{cases} (0,0,\ldots,0,1), & (j,k,l) \in I_{n_a o_b p_c}; \quad a,b,c = 1,2,\ldots \\ \mathbf{0}, & \text{otherwise} \end{cases}.$$

Then, for every $R \in \mathbb{R}^m$, we have

$$(1/h_{n_a o_b p_c}) \left(\sum_{(j,k,l) \in I_{n_a o_b p_c}} A_{\|x_{jkl} - R, z\|_G} (\xi) \right) = A_{\|(0,0,\ldots,0,1) - R, z\|_G} (\xi); \; a,b,c = 1,2,\ldots$$

and

$$(1/h_{n_a o_b p_c}) \left(\sum_{(j,k,l) \in I_{n_a o_b p_c}} A_{\|x_{jkl} - R, z\|_G} (\xi) \right) = A_{\|R, z\|_G} (\xi)$$

for all $z \in X$ and $n \neq n_a, o \neq o_b, p \neq p_c$, which as a consequence gives

$$\lim_{nop} (1/h_{nop}) \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl} - R, z\|_G} (\xi) \geq \varepsilon \right\} \right| \neq 0,$$

for all $z \in X$, i.e., $x_{jkl} \nrightarrow R \{S_{\theta_3}(G2)\}$.

However $\{x_{jkl}\}$ is $S(G2)$-convergent, since if α, β, γ are any integers are big enough, that is sufficiently large, we can have just a, b, c supplying $j_{n_a - 1} < \alpha \leq$

$j_{n_a}, k_{o_b-1} < \beta \leq k_{o_b}, l_{p_c-1} < \gamma \leq l_{p_c}$, and then

$$\begin{aligned} & 1/\left(\alpha\beta\gamma\right)\left(\sum_{j=1,k=1,l=1}^{\alpha,\beta,\gamma} A_{\|x_{jkl},z\|_G}\left(\xi\right)\right) \\ \leq \quad & \left(\left(j_{n_a-1}k_{o_b-1}l_{p_c-1} + h_{n_a o_b p_c}\right)/\left(j_{n_a-1}k_{o_b-1}l_{p_c-1}\right)\right) \\ < \quad & 2/\left(abc\right), \qquad \text{for all } z \in X \end{aligned}$$

as $\alpha, \beta, \gamma \to \infty$, it follows that $a, b, c \to \infty$. Hence $\{x_{jkl}\} \in |\sigma_{1,1,1}(G2)|^0$. Hence, using the proof technique [4] of Theorem 2.1, it can be shown that $\{x_{jkl}\}$ is $S(G2)$-convergent. □

Lemma 3.11. *$S(G2) - \lim x_{jkl} = R$ implies $S_{\theta_3}(G2) - \lim x_{jkl} = R$ iff $\limsup_{nop} q < \infty$. Additionally, if $\limsup_{nop} q_{nop} = \infty$,, then a triple sequence that is $S_{\theta_3}(G2)$-convergent but not $S(G2)$-convergent to any limit exists.*

Proof. Let us assume first $\limsup_{nop} q_{nop} < \infty$ with $S_{\theta_3}(G2) - \lim x_{jkl} = R$. So that $q_{nop} < B$ takes for any n, o and p, such that is $\infty > B > 0$. Take N_{nop} denote the set's cardinal number, $\left\{(j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon\right\}$, for all $z \in X$. Then, by our assumption, for given $\eta > 0$, there are $n_0, o_0, p_0 \in \mathbb{N}$ such that $\forall n \geq n_0, o \geq o_0, p \geq p_0; (N_{nop}/h_{nop}) < \eta$.
Let $M = \max\{N_{nop} : 1 \leq n \leq n_0, 1 \leq o \leq o_0, 1 \leq p \leq p_0\}$ and let α, β, γ be three integers satisfying $j_{(n-1)} < \alpha < j_n, k_{(o-1)} < \beta < k_o, l_{(p-1)} < \gamma < l_p$. Then we have,

$$\begin{aligned} & (1/\alpha\beta\gamma)\left|\left\{j \leq \alpha, k \leq \beta, l \leq \gamma : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon\right\}\right| \\ \leq \quad & \left(1/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right)\right)\left|\left\{j \leq j_n, k \leq k_o, l \leq l_p : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon\right\}\right| \\ = \quad & \left(1/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right)\right)\{N_{111} + N_{222} + \cdots + N_{n_0o_0p_0} + N_{n_0+1o_0+1p_0+1} \\ & + \cdots + N_{nop}\} \\ \leq \quad & \left(M/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right)\right)n_0o_0p_0 \\ & + \left(1/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right)\right)\{\left(h_{n_0+1o_0+1p_0+1}N_{n_0+1o_0+1p_0+1}\right)/h_{n_0+1o_0+1p_0+1} \\ & + \cdots + \left(h_{nop}N_{nop}\right)/h_{nop}\} \\ \leq \quad & \left(n_0o_0p_0M/j_{(n-1)}k_{(o-1)}l_{(p-1)}\right) \\ & +1/j_{(n-1)}k_{(o-1)}l_{(p-1)} \sup_{n>n_0,o>o_0,p>p_0}\left(N_{nop}/h_{nop}\right)\{h_{n_0+1o_0+1p_0+1} + \cdots + h_{nop}\} \\ \leq \quad & \left(\left(n_0o_0p_0M\right)/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right)\right) \\ & +\eta\left(j_nk_ol_p - j_{n_0}k_{o_0}l_{p_0}\right)/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right) \\ \leq \quad & \left(n_0o_0p_0M\right)/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right) + \eta q_{nop} \\ \leq \quad & \left(n_0o_0p_0M\right)/\left(j_{(n-1)}k_{(o-1)}l_{(p-1)}\right) + \eta B, \qquad \text{for all } z \in X \end{aligned}$$

which immediately gives $S(G2) - \lim x_{jkl} = R$.

Conversely, suppose $\limsup_{nop} q_{nop} = \infty$. The triple sequence we want to start with is $S_{\theta_3}(G2)$-convergent but not $S(G2)$-convergent to any limit. Following the idea in ([14], [17]), we might construct a triple subsequence $\{j_{n_a} k_{o_b} l_{p_c}\}$ of the triple lacunary sequence $\theta_3 = (j_n k_o l_p)$ such that $q_{n_a o_b p_c} > abc$. Now we define the following gradually bounded triple sequence $\{x_{jkl}\}$ in $(\mathbb{R}^m, \|\cdot,\cdot\|_G)$ (which in, the norm referenced in Example 2.1 is $\|\cdot,\cdot\|_G$):

$$x_{jkl} = \begin{cases} (0,0,\ldots,0,1) & j_{n_a-1} < j < 2j_{n_a-1}, \quad k_{o_b-1} < k < 2k_{o_b-1}, \\ & l_{p_c-1} < l < 2l_{p_c-1}; \qquad a,b,c = 1,2,\ldots \\ \mathbf{0}, & \text{otherwise} \end{cases}.$$

Proceeding as [14] one can show $x \in N_{\theta_3}(G2)$ but $x \notin |\sigma_1(G2)|$. Hence Theorem 1 of [17] implies that x is $S_{\theta_3}(G2)$-convergent but it can be easily shown that x is not $S(G2)$-convergent using a similar procedure of Theorem 2.1 of [4].

The two lemmas mentioned before combined allow us to arrive at the following theorem: □

Theorem 3.12. *$S(G2) - \lim x_{jkl} = S_{\theta_3}(G2) - \lim x_{jkl}$ if and only if $1 \leq \liminf_{nop} q_{nop} \leq \limsup_{nop} q_{nop} < \infty$.*

Definition 3.13. *Let $\theta_3 = \{j_n k_o l_p\}$ be a triple lacunary sequence. If x has a triple subsequence $\{x_{j\prime(n)k\prime(o)l\prime(p)}\}$ and all three of the following requirements are true, then x is said to be gradually lacunary statistical Cauchy (in the short $S_{\theta_3}(G2)$-Cauchy):*

(i) $(j\prime(n), k\prime(o), l\prime(p)) \in I_{nop}$ for any, n, o, p,

(ii) For some $R \in X$, $\{x_{j\prime(n)k\prime(o)l\prime(p)}\}$ is gradually convergent to $R\,(n,o,p \to \infty)$,

(iii)

$$\lim_{nop} \frac{1}{h_{nop}} \left| \left\{ (j,k,l) \in I_{nop} : A_{\|x_{jkl} - x_{j\prime(n)k\prime(o)l\prime(p)}, z\|_G}(\xi) \geq \varepsilon \right\} \right| = 0,$$

for any $\varepsilon > 0$, $\xi \in (0,1]$ and $z \in X$.

Theorem 3.14. *x is $S_{\theta_3}(G2)$-convergent if and only if x is $S_{\theta_3}(G2)$-Cauchy.*

Proof. For each $e \in \mathbb{N}$, suppose $x_{jkl} \to R\{S_{\theta_3}(G2)\}$ and

$$K(\xi, e) = \left\{ (j,k,l) \in \mathbb{N}^3 : A_{\|x_{jkl} - R, z\|_G}(\xi) < 1/e \right\}.$$

Hence, $K(\xi, e) \supseteq K(\xi, e+1)$ holds, for any $e \in \mathbb{N}$ and we have $\lim_{nop} \frac{|K(\xi,e) \cap I_{nop}|}{h_{nop}} = 1$. Choose s_1, t_1, w_1 such that $n \geq s_1, o \geq t_1, p \geq w_1$ implies $\frac{|K(\xi,e) \cap I_{nop}|}{h_{nop}} > 0$ i.e., $K(\xi, 1) \cap I_{nop} \neq \varnothing$. Next choose $s_2 > s_1, t_2 > t_1, w_2 > w_1$ such that $n \geq s_2, o \geq t_2, p \geq w_2$ implies $K(\xi, 2) \cap I_{nop} \neq \varnothing$. Then, for each n, o, p satisfying $s_1 < n \leq s_2, t_1 < o \leq t_2, w_1 < p \leq w_2$, choose $(j\prime(n), k\prime(o), l\prime(p)) \in I_{nop}$ such that $(j\prime(n), k\prime(o), l\prime(p)) \in I_{nop} \cap K(\xi, 1)$, i.e., $A_{\|x_{j\prime(n)k\prime(o)l\prime(p)} - R, z\|_G}(\xi) < 1$, for all $z \in X$. Proceeding like this, one can choose $s_{(e+1)} > s_e, t_{(e+1)} >$

$t_e, w_{(e+1)} > w_e$ such that $n > s_{(e+1)}, o > t_{(e+1)}, p > w_{(e+1)}$ refers $K(\xi, e+1) \cap I_{nop} \neq \varnothing$. Hence, $s_e \leq r < s_{(e+1)}, t_e \leq o < t_{(e+1)}, w_e \leq p < w_{(e+1)}$, for every s, t, w, choose $(j\prime(n), k\prime(o), l\prime(p)) \in I_{nop} \cap K(\xi, e)$, i.e.,

$$A_{\left\|x_{j\prime(n)k\prime(o)l\prime(p)}-R,z\right\|_G}(\xi) < 1/e, \text{ for all } z \in X. \tag{3.2}$$

Hence, we have $(j\prime(n), k\prime(o), l\prime(p)) \in I_{nop}$ for any s, t, w and 3.2 implies that $\left\{x_{j\prime(n)k\prime(o)l\prime(p)}\right\}$ is gradually convergent to $R(n, o, p \to \infty)$.

Therefore we get

$$\begin{aligned} & \frac{1}{h_{nop}}\left|\left\{(j,k,l) \in I_{nop} : A_{\left\|x_{jkl}-x_{j\prime(n)k\prime(o)l\prime(p)},z\right\|_G}(\xi) \geq \varepsilon\right\}\right| \\ \leq \; & \frac{1}{h_{nop}}\left|\left\{(j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon/2\right\}\right|, \end{aligned}$$

for any $\xi \in (0,1]$, $\varepsilon > 0$ and $z \in X$.

Using the following assumption, the consequence of the aforementioned inequation is given by $x_{jkl} \to x\{S_{\theta_3}(G2)\}$ and the fact that $\left\{x_{j\prime(n)k\prime(o)l\prime(p)}\right\}$ gradually converges to R.

For the converse, suppose $\{x_{jkl}\}$ is an S_{θ_3}-Cauchy triple sequence. So for any $\xi \in (0,1]$, $\varepsilon > 0$ and $z \in X$,

$$\begin{aligned} & \left|\left\{(j,k,l) \in I_{nop} : A_{\|x_{jkl}-R,z\|_G}(\xi) \geq \varepsilon\right\}\right| \\ \leq \; & \left|\left\{(j,k,l) \in I_{nop} : A_{\left\|x_{jkl}-x_{j\prime(n)k\prime(o)l\prime(p)},z\right\|_G}(\xi) \geq \varepsilon/2\right\}\right| \\ & + \left|\left\{(j,k,l) \in I_{nop} : A_{\left\|x_{j\prime(n)k\prime(o)l\prime(p)}-R,z\right\|_G}(\xi) \geq \varepsilon/2\right\}\right| \end{aligned}$$

which as a consequence implies that $x_{jkl} \to R\{S_{\theta_3}(G2)\}$. □

4. Conclusion

The aim of this paper is to investigate the notion of convergence of gradual lacunary statistical convergent sequences of ternary sequences in 2- normed linear spaces (G2-NLS). Furthermore, some algebraic and topological properties of this set of ternary sequences are obtained with the notion of gradual. The theorems are proved in the light of the G2-NLS theory approach. Important findings have been obtained that reveal various basic features of this concept.

Acknowledgement. The abstract of this work was presented in the "2nd International e-Conference on Mathematical and Statistical Sciences: A Selcuk Meeting (ICOMSS'23)" which was held between June 5–7, 2023 in Selçuk University, Konya, Turkey.

References

[1] B. Altay and F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309** (2005), no. 1, 70–90.

[2] C. Choudhury and S. Debnath, *On $\mathcal{I}$-convergence of sequences in gradual normed linear spaces*, Facta Univ. Ser. Math. Inform. **36** (2021), no. 3, 595–604.

[3] C. Choudhury and S. Debnath, *On lacunary statistical convergence of sequences in gradual normed linear spaces*, An. Univ. Craiova Ser. Mat. Inform. **49** (2022), no. 1, 110–119.

[4] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis **8** (1988), 47–63.

[5] I. A. Demirci and M. Gürdal, *Lacunary statistical convergence for sets of triple sequences via Orlicz function*, Theory Appl. Math. Comput. Sci. **11** (2021), no. 1, 1–13.

[6] D. Dubois and H. Prade, *Gradual elements in a fuzzy set*, Soft Comput. **12** (2007), 165–175.

[7] A. J. Dutta, A. Esi, and B. C. Tripathy, *Statistically convergent triple sequence spaces defined by Orlicz function*, J. Math. Anal. **4** (2013), no. 2, 16–22.

[8] A. Esi, *Statistical convergence of triple sequences in topological groups*, Ann. Univ. Craiova Ser. Math. Comput. Sci. Ser. **40** (2013), no. 1, 29–33.

[9] A. Esi and E. Savaş, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Appl. Math. Inf. Sci. **9** (2015), no. 5, 2529–2534.

[10] M. Ettefagh, F. Y. Azari, and S. Etemad, *On some topological properties in gradual normed spaces*, Facta Univ. Ser. Math. Inform. **35** (2020), no. 3, 549–559.

[11] M. Ettefagh, S. Etemad, and F. Y. Azari, *Some properties of sequences in gradual normed spaces*, Asian-Eur. J. Math. **13** (2020), no. 4, 1–9.

[12] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), no. (3-4), 241–244.

[13] J. Fortin, D. Dubois, and H. Fargier, *Gradual numbers and their application to fuzzy interval analysis*, IEEE Trans. Fuzzy Syst. **16** (2008), no. 2, 388–402.

[14] A. R. Freedman, J. J. Sember, and R. Raphael, *Some p-Cesáro-type summability spaces*, Proc. Lond. Math. Soc. **S3-37** (1978), 508–520.

[15] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[16] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), no. 4, 1187–1192.

[17] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pacific J. Math. **160** (1993), no. 1, 43–51.

[18] S. Gähler, 2*-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–118.

[19] S. Gähler, *Linear* 2*-normierte Räume*, Math. Nachr. **28** (1965), ,1–43.

[20] M. Gürdal and I. Açık, *On I-Cauchy sequences in* 2*-normed spaces*, Math. Inequal. Appl. **11** (2008), no. 2, 349–354.

[21] M. Gürdal and M. B. Huban, *On I-convergence of double sequences in the topology induced by random* 2*-norms*, Mat. Vesnik **66** (2014), no. 1, 73–83.

[22] M. Gürdal and S. Pehlivan, *The statistical convergence in* 2*-Banach spaces*, Thai J. Math. **2** (2004), no. 1, 107–113.

[23] M. Gürdal and A. Şahiner, *Extremal $\mathcal{I}$-limit points of double sequences*, Appl. Math. E-Notes **8** (2008), 131–137.

[24] M. Gürdal, A. Şahiner, and I. Açık, *Approximation theory in* 2*-Banach spaces*, Nonlinear Anal. **71** (2009), no. 5, 1654–1661.

[25] L. Lietard and D. Rocacher, *Conditions with aggregates evaluated using gradual numbers*, Control Cybernet. **38** (2009), no. 2, 395–417.

[26] S. A. Mohiuddine and M. Aiyub, *Lacunary statistical convergence in random* 2*-normed spaces*, Appl. Math. Inf. Sci. **6** (2012), no. 3, 518–585.

[27] M. Mursaleen and C. Belen, *On statistical lacunary summability of double sequences*, Filomat **28** (2014), no. 2, 231–239.
[28] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223–231.
[29] A. Nabiev, S. Pehlivan, and M. Gürdal, *On $\mathcal{I}$-Cauchy sequences*, Taiwanese J. Math. **11** (2007), no. 2, 569–576.
[30] F. Nuray, U. Ulusu, and E. Dündar, *Lacunary statistical convergence of double sequences of sets*, Soft Computing **20** (2016), no. 7, 2883–2888.
[31] S. Pehlivan, M. Gürdal, and B. Fisher, *Lacunary statistical cluster points of sequences*, Math. Commun. **11** (2006), 39–46.
[32] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), 289–321.
[33] I. Sadeqi and F. Y. Azari, *Gradual normed linear space*, Iran. J. Fuzzy Syst. **8** (2011), no. 5, 131–139.
[34] A. Şahiner, M. Gürdal, and F. K. Düden, *Triple sequences and their statistical convergence*, Selcuk J. Appl. Math. **8** (2007), no. 2, 49–55.
[35] A. Şahiner, M. Gürdal, and T. Yiğit, *Ideal convergence characterization of the completion of linear n-normed spaces*, Comput. Math. with Appl. **61** (2011), no. 3, 683–689.
[36] E. Savaş and M. Gürdal, *Generalized statistically convergent sequences of functions in fuzzy 2-normed spaces*, J. Intell. Fuzzy Syst. **27** (2014), no. 4, 2067–2075.
[37] E. Savaş and M. Gürdal, *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Scienceasia **41** (2015), no. 4, 289–294.
[38] E. Savaş and M. Gürdal, *I-statistical convergence in probabilistic normed spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), no. 4, 195–204.
[39] E. Savaş and R. F. Patterson, *Lacunary statistical convergence of multiple sequences*, Appl. Math. Lett. **19** (2006), no. 6, 527–534.
[40] H. Şengul and M. Et, *f-lacunary statistical convergence and strong f-lacunary summability of order α*, Filomat **32** (2018), no. 13, 4513–4521.
[41] E. A. Stock, *Gradual numbers and fuzzy optimization*, Appl. Math. Lett. Ph. D. thesis, University of Colorado Denver, Denver, America, 2010.

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