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A STUDY OF THE TUBULAR SURFACES ACCORDING TO MODIFIED ORTHOGONAL FRAME WITH TORSION

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Abstract. In this study, tubular surfaces were introduced according to the modified orthogonal frame defined at the points where the torsion is different from zero in the 3-dimensional Euclidean space. First, the relations between the Frenet frame and the modified orthogonal frame with torsion are given. Then, the singularity, Gaussian curvature, mean curvature and basic forms of the tubular surface given according to the modified orthogonal frame with torsion were calculated. In addition, the conditions for the parameter curves of the tubular surface to be geodesic, asymptotic and line of curvature were examined. Finally, tubular surface examples based on both the Frenet frame and the modified orthogonal frame with torsion were given to support the study.

1. Introduction

Tubular surfaces are a special type of canal surfaces, first described by Gaspard Monge in 1850. In fact, it is the case that the radius in question on canal surfaces is constant. Tubular surfaces have an important role in engineering, the automotive industry, art and architecture, sports equipment, aircraft and spacecraft. In this sense, studies on tubular surfaces and their characterizations have been carried out in various spaces such as Euclidean, Minkowski and Galilean spaces [1-9]. As it is known, there are various frames that can be installed on a curve, and the one that is most frequently studied is the Frenet frame. Although the Frenet frame is a frame that characterizes the curve, one of its disadvantages is that this frame cannot be defined if the curvature of the curve is zero. In 1975, Bishop eliminated this disadvantage and defined a new frame, the Bishop frame [10]. In addition, Sasai defined the modified orthogonal frame at points where the curvature is different from zero [11]. Bükçü and Karacan expressed Sasai's work in 3-dimensional Minkowski space. Additionally, they gave a new version of the modified orthogonal frame with torsion in three dimensional Euclidean and Minkowski space [12]. Recently, there have been various studies involving special curve pairs and special surfaces based

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on the modified orthogonal frame [13-17]. In this study, we characterized the tubular surface according to the modified orthogonal frame with torsion. Singularity, Gaussian curvature, mean curvature and basic forms of the tubular surface given according to this frame were calculated. In addition, necessary and sufficient conditions were given for the parameter curves of the tubular surface to be geodesic, asymptotic and line of curvature. Finally, the study was supported with various examples.

2. Preliminaries

This section provides a concise overview of modified orthogonal frames with non-zero torsion and their properties.

Let α (s) be a C^3 space curve of arc-length parameter s in the Euclidean 3-space. Then, the Frenet frame $\{t,\ n\ , b\}$ of the curve α (s) is given by

$$\frac{d}{ds}\begin{pmatrix} t\,(s) \\ n\,(s) \\ b\,(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa\,(s) & 0 \\ -\kappa\,(s) & 0 & \tau\,(s) \\ 0 & -\tau\,(s) & 0 \end{pmatrix}\begin{pmatrix} t\,(s) \\ n\,(s) \\ b\,(s) \end{pmatrix} \tag{1}$$

where t, n, b are tangent, principal normal, binormal vectors and the function $\kappa\,(s) = \|\alpha''(s)\|$ and $\tau\,(s) = \langle B'(s)\,,\, N(s)\rangle$ are called the curvature and torsion of the curve α, respectively .

Let α (s) be an analytical curve . Then, this curve can be reparametrized by its arc length s. For this curve, we will assume that the curvature function $\tau\,(s)$ is not identically zero. Thus, an orthogonal frame $\{T,\ N\ , B\}$ can be defined as follows:

$$\left\{ \begin{array}{l} T = \alpha'\,(s)\ , \\ N =\ T'\,(s)\ , \\ B = T \wedge N. \end{array} \right. \tag{2}$$

Considering the aforementioned equations and Frenet equations, the obtained relations linking the Frenet frame and a new frame at non-zero points of are as follows:

$$\{\ T = t, N = \tau\ n, B = \tau\ b \tag{3}$$

A new frame facilitates the incorporation of the following equations by means of the Frenet frame

$$T \cdot N = N \cdot B = T \cdot B = 0, T \cdot T = 1, N \cdot N = B \cdot B = \tau^2$$

where " $\cdot$" denotes the standard inner product in IR^3 [12].

The derivative equations of the new orthogonal frame are derived from these fundamental equations.

$$(4)\qquad \begin{cases} T'(s) = N(s), \\ N'(s) = -\tau^2 T(s) + \frac{\tau'}{\tau} N(s) + \tau B(s), \\ B'(s) = -\tau N(s) + \frac{\tau'}{\tau} B(s), \end{cases}$$

where $\tau(s) = \frac{det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of the curve α.

Thus, the orthogonal frame $\{T(s), N(s), B(s)\}$ is called the modified orthogonal frame with non-zero torsion [12].

A surface in denotes by P(s,v). The unit normal vector field of this surface is defined as follows:

$$U = \frac{P_s \times P_v}{\| P_s \times P_v \|}.$$

The coefficients of the first end second fundamental form of this surface, respectively, is found by

$$E = \| P_s \|^2 \ , \ F = P_s \cdot P_v \ , \ G = \| P_v \|^2,$$
$$e = U \cdot P_{ss} \ , \ f = U \cdot P_{sv} \ , \ g = U \cdot P_{vv}.$$

The Gaussian curvature and the mean curvature of the surface P(s,v) are given as follows:

$$K = \frac{eg - f^2}{EG - F^2} \ , \ H = \frac{eG - 2Ff + Ge}{2(EG - F^2)}.$$

3. Properties of Tubular Surfaces According To Modified Orthogonal Frame With Torsion

A canal surface is termed as the encompassing boundary of a sphere in motion with a varying radius. When the radius remains constant, this surface is referred to as the tubular surface. The parametric equation delineating the tubular surface is presented below

$$P(s, v) = \alpha(s) + r[\cos(v)N(s) + \sin(v)B(s)]$$

where $v \ \epsilon \ [0, 2\pi\ )$, r is the radius of the tubular surfaced and the curve α (s) is the center curve of the tubular surface. Also, the vectors N and B are perpendicular to the curve at the point α (s) of the curve α.

The derivatives according to parameters s and v of the tubular surface P(s,v) are, respectively,

$$P_s = (1 - r\kappa\tau \cos v)T + (r\frac{\tau^{'}}{\tau} \cos v - r\tau \sin v)N + (r\tau \cos v + r\frac{\tau^{'}}{\tau} \sin v)B,$$
$$P_v = r(-\sin v N + \cos v B).$$

From equations (1), the coefficients of the first fundamental form are found as

$$E = (1 - r\kappa\tau \cos v)^2 + r^2\tau^4 + (r^2\tau^{'})^2 \ , \ F = r^2\tau^3 \ , \ G = r^2\tau^2.$$

Moreover, considering equations of (1), the unit normal vector field of the tubular surface P(s,v) is obtained as

$$U=\frac{(r^2\frac{\tau'}{\tau})T+r\cos v(1-r\kappa\tau\cos v)N+r\sin v(1-r\kappa\tau\cos v)B}{A},$$

where

$$A=r\sqrt{(r^2\frac{\tau'}{\tau})^2+(\tau(-1+r\kappa\tau\cos v))^2}\neq 0$$

If the unit normal vector at any point of a surface P(s,v) vanishes, i.e. $P_s\times P_v=0$ at any points, then these points are called the singular points of the surface. So the following result is obvious.

Corollary 3.1. *P(s,v) is a regular tubular surfaces if and only if $r\kappa\tau\cos(v)\neq 1$ or τ is not a non-zero constant..*

The second partial derivatives of tubular surface are found by

$$P_{ss}=\left(r\kappa'\tau\cos v-2r\kappa\tau'\cos v+r\kappa\tau^2\sin v\right)T+$$
$$\left(\tfrac{\kappa}{\tau}v-r\kappa^2\cos v-r\tau^2\cos v\right)N+\left(3r\tau'\cos v+r\tfrac{\tau'}{\tau}\sin v-r\tau^2\sin v\right)B,$$

$$P_{sv}=(r\kappa\tau\sin v)\,t+\left(-r\tfrac{\tau'}{\tau}\sin v-r\tau\cos v\right)N+\left(r\tfrac{\tau'}{\tau}\cos v-r\tau\sin v\right)B,$$

$$P_{vv}=-r\cos vN-r\sin vB.$$

From the equations (3) and (4), the coefficients of the second fundamental form are

$$e=\frac{1}{A}\left[(1-r\kappa\tau\cos v)\left(r^2\tau\left(\tau^3-\tau''\right)-r\kappa\tau\cos v(1-r\kappa\tau\cos v)\right)\right]$$
$$-r^3\kappa'\tau'\cos v+r^3\kappa'\tau\tau'\sin v-2r^3\kappa\frac{\left(\tau'\right)^2}{\tau}\cos v),$$

$$f=\frac{1}{A}\left(r^2\tau^3-r^3\kappa\tau^4\cos v+r^3\kappa\tau'\sin v\right),$$
$$g=\frac{1}{A}\left(r^2\tau^2-r^3\kappa\tau^3\cos v\right).$$

The Gaussian and mean curvatures of the tubular surface P(s,v) with help of equations (2) and (5) are obtained as

$$K=\frac{\begin{pmatrix}(1-r\kappa\tau\cos v)\tau^3\left(\tau^3-\tau''\right)-r\kappa\tau\cos v(1-r\kappa\tau\cos v)^3-r^3\tau^2\kappa'\tau'\cos v\\+r^4\kappa\kappa'\tau^3\tau'\cos^2 v+r^3\tau^3\kappa\tau'\sin v-r^4\kappa^2\left(\tau'\right)^2\sin^2 v\\-r^4\kappa^2\tau^4\left(\tau'\right)\cos v\sin v-2r^3\kappa\kappa'\tau'\cos v\\+2r^4\kappa^2\tau^2\left(\tau'\right)^2\cos^2 v-r^2\tau^6(1-r\kappa\tau\cos v)^2-2r^3\kappa\tau^2\sin v(1-r\kappa\tau\cos v)\end{pmatrix}}{A^2\left((1-r\kappa\tau\cos v)^2+\left(\tau'\right)^2\right)},$$

$$H=\frac{\left(\begin{array}{c}(1-r\kappa\tau\cos v)^2+r^2\tau^4+r^2\left(\tau'\right)^2)\tau^3(r^2\tau^2-r^3\tau^3\kappa\cos v)\\ -2r^2\tau^3\left(r^2\tau^3-r^3\kappa\tau^4\cos v+r^3\kappa\tau'\sin v\right)\\ +r^2\tau^2\left((1-r\kappa\tau\cos v)r^2\tau\left(\tau^3-\tau^{''}\right)-r\kappa\tau\cos v(1-r\kappa\tau\cos v)\right)\\ -r^3\kappa^{'}\tau'\cos v+r^3\kappa\tau\tau'\sin v\\ -2r^3\kappa\frac{\left(\tau'\right)^2}{\tau}\cos v)-r^2\tau^6(1-r\kappa\tau\cos v)^2-2r^3\kappa\tau^2\tau'\sin v(1-r\kappa\tau\cos v)\end{array}\right)}{2Ar^2\left((1-r\kappa\tau\cos v)^2+(\tau')^2\right)}$$

respectively.

Let's give some theorems about geometric interpretation of parametric curves of the tubular surface P(s,v).

Theorem 3.2. *i) The s-parameter curves of the tubular surface P(s,v) are geodesic curves if and only if*

$$r=\frac{\kappa}{\tau\left(3\tau'\sin v-\frac{\tau''}{\tau}\cos v+\kappa^2\cos v+\tau^2\cos v\right)},\ 3\tau'\sin v-\frac{\tau''}{\tau}\cos v+\kappa^2\cos v+\tau^2\cos v\neq 0.$$

ii) The v-parameter curves of the tubular surface P(s,v) are geodesic curves if and only if τ is non-zero constant.

Proof. i) For the s parameter curve of P(s,v) to be geodesic curves, necessary and sufficient condition is that $U\times P_{ss}=0$. In this case, from the equations (3) and (4), we obtain the following relations for the s parameter curve

$$\begin{aligned} U\times P_{ss} &= \left[\begin{array}{c} r\cos v\left(-1+r\kappa\tau\cos v\right)\left(3r\tau^{'}\cos v+r\frac{\tau'}{\tau}\sin v-r\tau^2\sin v\right)\\ -r\sin v\left(-1+r\kappa\tau\cos v\right)\\ \left(\frac{\kappa}{\tau}-3r\tau^{'}\sin v+r\frac{\tau^{''}}{\tau}\cos v-r\kappa^2\cos v-r\tau^2\cos v\right).\\ \left(-r\kappa^{'}\tau\cos v-2r\kappa\tau^{'}\cos v+r\kappa\tau^2\sin v\right)\end{array}\right]T\\ &+\left[\begin{array}{c} r\sin v\left(-1+r\kappa\tau\cos v\right)\left(-r\kappa^{'}\tau\cos v-2r\kappa\tau^{'}\cos v+r\kappa\tau^2\sin v\right)\\ -r^2\frac{\tau'}{\tau}\left(3r\tau^{'}\cos v+r\frac{\tau'}{\tau}\sin v-r\tau^2\sin v\right)\end{array}\right]N\\ &+\left[\begin{array}{c} r^2\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}-3r\tau^{'}\sin v+r\frac{\tau^{''}}{\tau}\cos v-r\kappa^2\cos v-r\tau^2\cos v\right)\\ -r\cos v\left(-1+r\kappa\tau\cos v\right)\left(-r\kappa^{'}\tau\cos v-2r\kappa\tau^{'}\cos v+r\kappa\tau^2\sin v\right)\end{array}\right]B. \end{aligned}$$

Because of T, N and B are linear independent, we have the following equalities

$$r\cos v\left(-1+r\kappa\tau\cos v\right)\left[\begin{array}{c}\cos v\left(3r\tau^{'}\cos v+r\frac{\tau'}{\tau}\sin v-r\tau^2\sin v\right)\\ -\sin v\left(\frac{\kappa}{\tau}-3r\tau^{'}\sin v+r\frac{\tau^{''}}{\tau}\cos v-r\kappa^2\cos v-r\tau^2\cos v\right)\end{array}\right]=0,$$

$$\begin{array}{c} r\sin v\left(-1+r\kappa\tau\cos v\right)\left(-r\kappa^{'}\tau\cos v-2r\kappa\tau^{'}\cos v+r\kappa\tau^2\sin v\right)\\ -r^2\frac{\tau'}{\tau}\left(3r\tau^{'}\cos v+r\frac{\tau'}{\tau}\sin v-r\tau^2\sin v\right)=0,\end{array}$$

$$r^2\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}-3r\tau'\sin v+r\frac{\tau''}{\tau}\cos v-r\kappa^2\cos v-r\tau^2\cos v\right)$$
$$-r\cos v\left(-1+r\kappa\tau\cos v\right)\left(-r\kappa'\tau\cos v-2r\kappa\tau'\cos v+r\kappa\tau^2\sin v\right)=0.$$

If necessary operations are done from these equations, we get
$r=\dfrac{\kappa}{\tau\left(3\tau'\sin v-\frac{\tau''}{\tau}\cos v+\kappa^2\cos v+\tau^2\cos v\right)}$, $3\tau'\sin v-\frac{\tau''}{\tau}\cos v+\kappa^2\cos v+\tau^2\cos v\neq 0$.

ii)From the equations (3) and (4), we have:

$$U\times P_{vv}=\frac{1}{A}\left(r^3\frac{\tau'}{\tau}\left(\sin vN+\cos vB\right)\right).$$

Since N and B are linearly independent, this means that $U\times P_{vv}=0$ if and only if τ is a non-zero constant. As a result, the v-parameter curves are geodesic curves.

□

Theorem 3.3. *i) The s-parameter curves of the tubular surface P(s,v) are asymptotic curves if and only if*
$r^3\left(\kappa'\tau'\cos v-\kappa\tau\tau'\sin v+2\kappa\frac{\left(\tau'\right)^2}{\tau}\cos v\right)=(1-r\kappa\tau\cos v)(r^2\tau\left(\tau^3-\tau''\right)-r\kappa\tau\cos v(1-r\kappa\tau\cos v)).$

ii) The v-parameter curves of the tubular surface P(s,v) are asymptotic curves if and only if $r\kappa\tau\cos(v)=1$ *or* τ *is not a non-zero constant.*

Proof. i) For the s parameter curve of P(s,v) to be asymptotic curves, necessary and sufficient condition is that . From equations (5), we have

$$e=U\cdot P_{ss}=\frac{1}{A}[(1-r\kappa\tau\cos v)r^2\tau\left(\tau^3-\tau''\right)-r\kappa\tau\cos v(1-r\kappa\tau\cos v)$$
$$-r^3\kappa'\tau'\cos v+r^3\kappa\tau\tau'\sin v-2r^3\kappa\frac{\left(\tau'\right)^2}{\tau}\cos v].$$

s-parameter curves of the tubular surface are asymptotic curves if and only if e=0. In this case, when necessary operations are taken in the equation above, we get

$$r^3\left(\kappa'\tau'\cos v-\kappa\tau\tau'\sin v+2\kappa\frac{\left(\tau'\right)^2}{\tau}\cos v\right)=(1-r\kappa\tau\cos v)(r^2\tau\left(\tau^3-\tau''\right)$$
$$-r\kappa\tau\cos v(1-r\kappa\tau\cos v)).$$

ii)From equation (5), we know that

$$g=U\cdot P_{vv}=\frac{1}{A}r^2\tau^2(1-r\kappa\tau\cos v)$$

If v-parameter curves of the tubular surface P(s,v) are asymptotic curves. In that case, τ has not to be constant. But if τ is also constant , the normal vector of the tubular surfaces does not vanish.

□

Theorem 3.4. *The s and v-parameter curves of the tubular surface P(s,v) can not also be a line of curvature.*

Proof. If the parameter curves of surface are lines of curvature, then F = f= 0 . In that case, from equations (2) and (5), we get

$$r^2\tau^3 = 0$$

and

$$r^2\tau^2(1 - r\kappa\tau\cos v) + r^3\kappa\tau'\sin v = 0.$$

If τ=0 , F = 0, but this is not possible since τ will be different from zero for the modified orthogonal frame with torsion to be defined.So, the s and v-parameter curves of the tubular surface P(s,v) can not also be a line of curvature. □

4. Examples

Example 4.1. *Let* $\alpha(s) = (\frac{3}{5}\sin(s), -\frac{3}{5}\cos(s), \frac{4}{5}s)$ *be a unit speed curve. It is obvious that the Frenet frame of* $\alpha(s)$ *curve is*

$$t\,(s) = (\tfrac{3}{5}\cos(s), \tfrac{3}{5}\sin(s), \tfrac{4}{5}),$$

$$n(s) = (-\sin(s), -\cos(s),\ 0),$$

$$b(s) = (\tfrac{4}{5}\cos(s), -\tfrac{4}{5}\sin(s), -\tfrac{3}{5}).$$

The torsion of the unit speed curve $\alpha(s)$ *can be determined as follows:*
$\tau\,(s) = -\frac{4}{5}$

The modified orthogonal frame with torsion of the unit speed curve $\alpha(s)$ *the derived elements as follows:*

$$T\,(s) = \left(\tfrac{3}{5}\cos(s), -\tfrac{3}{5}\cos\,(s)\,, \tfrac{4}{5}\right),$$

$$N\,(s) = \left(\tfrac{4}{5}\sin(s), \tfrac{4}{5}\cos\,(s)\,,\ 0\right),$$

$$B\,(s) = \left(-\tfrac{16}{25}\cos\,(s)\,, \tfrac{16}{25}\sin\,(s)\,,\ \tfrac{12}{25}\right).$$

Let us now proceed with illustrating the graphs of tubular surfaces, for which the equations are as follows:

$$\Psi_F(s,v) = \alpha(s) + r[\cos(v)n(s) + \sin(v)b(s)] \tag{5}$$

and

$$\Psi_M(s,v) = \alpha(s) + r[\cos(v)N(s) + \sin(v)B(s)] \tag{6}$$

as per the Frenet frame and modified orthogonal frame with torsion, respectively. For r = 5*, these surfaces are*

$$\Psi_F(s,v) = \begin{pmatrix} \frac{3}{5}\sin(s) + 5sin(s)\cos(v) + 4sin(v)\cos(s), \\ \frac{3}{5}\cos(s) - 5cos(s)\cos(v) - 4sin(v)\sin(s), \\ \frac{4}{5}\ s\ -\ 3sin(v) \end{pmatrix}$$

$$\Psi_M(s,v) = \begin{pmatrix} \frac{3}{5}\sin(s) + 4sin(s)\cos(v) - \frac{16}{5}\sin(v)\cos(s), \\ \frac{3}{5}\cos(s) + 4cos(s)\cos(v) + \frac{16}{5}\sin(v)\sin(s), \\ \frac{4}{5}\ s\ + \frac{12}{5}\sin(v) \end{pmatrix}$$

The figure of these tubular surface is indicated in the Figure 1 and Figure 2 for the values $-2\pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

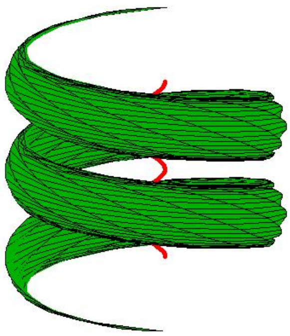

FIGURE 1. Tubular surface obtained by Frenet frame

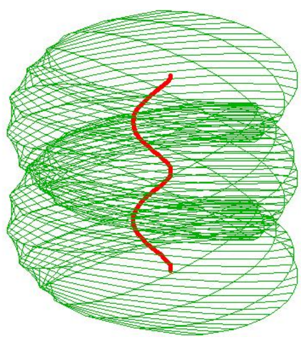

FIGURE 2. Tubular surface obtained by modified orthogonal frame with torsion

Example 4.2. *Let $\alpha(s) = (\cos s, \sin s, s)$ be a curve. Then it is easy to show that from Eqn. (4),*

$$t\,(s) = \tfrac{\sqrt{2}}{2}(-\sin s, \cos s, 1),$$

$$n\,(s) = (\cos s, \sin s, 0),$$

$$b\,(s) = \tfrac{\sqrt{2}}{2}(\sin s, -\cos s, 1).$$

The torsion of the curve $\alpha(s)$ can be determined as follows:
$\tau\,(s) = \frac{1}{2}$

The modified orthogonal frame with torsion of the curve $\alpha(s)$ the derived elements as follows:

$$T\,(s) = \tfrac{\sqrt{2}}{2}(-\sin s, \cos s, 1),$$

$$N\,(s) = \tfrac{1}{2}(\sin s, \cos s, 0),$$

$$B\,(s) = \tfrac{\sqrt{2}}{4}(\sin s, -\cos s, 1).$$

From equations (8) and (9) , these surfaces for $r = \sqrt{2}$

$$\Psi_F(s,v) = \begin{pmatrix} \frac{3}{5}\sin(s) + 5\sin(s)\cos(v) + 4\sin(v)\cos(s), \\ \frac{3}{5}\cos(s) - 5\cos(s)\cos(v) - 4\sin(v)\sin(s), \\ \frac{4}{5}\ s\ - 3\sin(v) \end{pmatrix}$$

$$\Psi_M(s,v) = \begin{pmatrix} \frac{3}{5}\sin(s) + 4sin(s)\cos(v) - \frac{16}{5}\sin(v)\cos(s), \\ \frac{3}{5}\cos(s) + 4cos(s)\cos(v) + \frac{16}{5}\sin(v)\sin(s), \\ \frac{4}{5}\ s\ + \frac{12}{5}\sin(v) \end{pmatrix}$$

is obtained as. The figure of these tubular surface is indicated in the Figure 3 and Figure 4 for the values $-2\pi \le s \le 2\pi$, $-2\pi \le v \le 2\pi$.

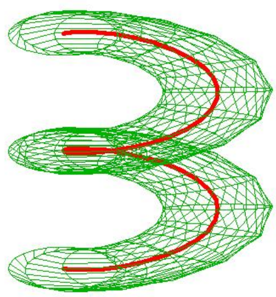

Figure 3. Tubular surface obtained by Frenet frame

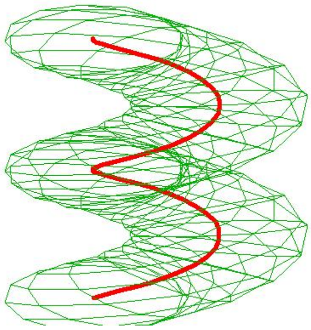

FIGURE 4. Tubular surface obtained by modified orthogonal frame with torsion

Example 4.3. *Let* $\alpha(s) = (\frac{\sqrt{3}}{3}s^{\frac{3}{2}}, \frac{\sqrt{3}}{3}(1-s)^{\frac{3}{2}}, \frac{s}{2})$ *be a curve. Then it is easy to show that,*

$$t\,(s) = (\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, -\frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}),$$

$$n\,(s) = ((1-s)^{\frac{1}{2}}, s^{\frac{1}{2}}, 0),$$

$$b\,(s) = (-\frac{1}{2}s^{\frac{1}{2}}, \frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{\sqrt{3}}{2}).$$

The torsion of the curve $\alpha(s)$ *can be determined as follows:*
$\tau\,(s) = \frac{1}{2}$

The modified orthogonal frame with torsion of the curve $\alpha(s)$ *the derived elements as follows:*

$$T\,(s) = (\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, -\frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}),$$

$$N\,(s) = \frac{1}{4}(\frac{1}{\sqrt{s}}, \frac{1}{\sqrt{1-s}}, 0),$$

$$B\,(s) = \frac{1}{8}(-\frac{1}{\sqrt{1-s}}, \frac{1}{\sqrt{s}}, \frac{\sqrt{3}}{\sqrt{s}\sqrt{1-s}}).$$

From equations (8) and (9) , these surfaces for r = 1

$$\Psi_F(s,v) = \begin{pmatrix} \frac{\sqrt{3}}{3}s^{\frac{3}{2}} + cos(v)(1-s)^{\frac{1}{2}} - \frac{1}{2}sin(v)s^{\frac{1}{2}}, \\ \frac{\sqrt{3}}{3}(1-s)^{\frac{3}{2}} + cos(v)s^{\frac{1}{2}} + \frac{1}{2}sin(v)(1-s)^{\frac{1}{2}}, \\ \frac{s}{2} + \ \frac{\sqrt{3}}{2}sin(v) \end{pmatrix}$$

$$\Psi_M(s,v) = \begin{pmatrix} \frac{\sqrt{3}}{3}s^{\frac{3}{2}} + cos(v)\frac{1}{4\sqrt{s}} - \frac{1}{8}sin(v)\frac{1}{\sqrt{1-s}}, \\ \frac{\sqrt{3}}{3}(1-s)^{\frac{3}{2}} + cos(v)\frac{1}{4\sqrt{1-s}} + \frac{1}{8\sqrt{s}}sin(v), \\ \frac{s}{2} + \ \frac{\sqrt{s}}{\sqrt{1-s}}\frac{\sqrt{3}}{8}sin(v) \end{pmatrix}$$

is obtained as. The figure of these tubular surface is indicated in the Figure 5 and Figure 6 for the values $1 < s \leq \pi$, $-2\pi \leq v \leq 2\pi$.

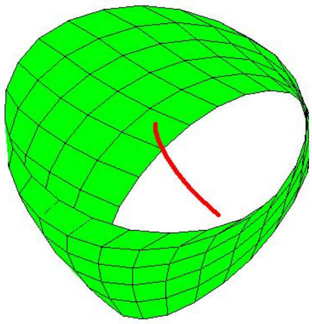

FIGURE 5. Tubular surface obtained by Frenet frame

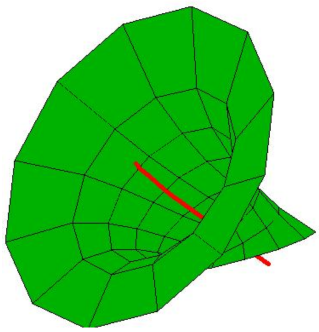

FIGURE 6. Tubular surface obtained by modified orthogonal frame with torsion

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