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UNIQUENESS OF A MEROMORPHIC FUNCTION WITH DIFFERENCE POLYNOMIAL OF DIFFERENCE OPERATOR SHARING TWO VALUES CM

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Abstract. In this paper, we investigate the uniqueness of a meromorphic function $f(z)$ and its difference polynomial of difference operator with two sharing values counting multiplicities. Our two results improve and generalize the recent results of Barki Mahesh, Dyavanal Renukadevi S and Bhoosnurmath Subhas S [4] and for the case $q \geq 2$, this allows for a highly unique generalization. To further demonstrate the validity of our main result, we provide an example.

1. Introduction

Throughout the paper, we denote the set of all complex numbers and natural numbers by $\mathbb{C}$ and $\mathbb{N}$, respectively. We mean $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By any meromorphic function f we always mean that it is defined on $\mathbb{C}$. For any non-constant meromorphic function $h(z)$ we define $S(r,h) = o\{(T(r,h)\}$, $(r \longrightarrow \infty$, $r \notin E)$ where, E denotes any set of positive real numbers having finite linear measure. We assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [10], [21], [22]). We recall that $T(r,f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N\left(r, \dfrac{1}{f-a}\right) = N(r,a;f)$ $\left(\overline{N}\left(r, \dfrac{1}{f-a}\right) = \overline{N}(r,a;f)\right)$ denotes the counting function (reduced counting function) of a-points of f.

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a

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CM (counting multiplicities). If we do not consider multiplicities, then f and g are said to share the value a IM (ignoring multiplicities).

In 1926, Nevanlinna [17] investigated the following two theorems for shared values.

Theorem 1.1. *If two meromorphic functions f and g share five distinct values IM, then $f \equiv g$.*

Theorem 1.2. *If two meromorphic functions f and g share four distinct values CM, then $f \equiv g$ or $f \equiv Tog$, where T is a mobious transformation.*

In 1977, Rubel and Yang [19] investigated uniqueness of entire function f sharing values with its derivatives $f^{'}$. Mues and Steinmetz [[15], [16]] Gunder Sen [7] improved their results.

Recently Nevanlinna theory has been established for difference operators [[6], [5], [12], [18], [9], [1], [3]] Many authors [[8], [11], [12], [2], [14], [20]] have considered shared values of meromorphic function with their difference operators or shifts.

The following two theorems, due to Heittokangas Et. Al. [[11], [12]] give the idea of shared values of meromorphic function with their shifts.

Theorem 1.3. *Let f be a meromorphic function of finite order and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions a_1, a_2, a_3 $\in \tilde{S}(f) = S(f) \cup \infty$ with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem 1.4. *Let f be a meromorphic function of finite order and let $c \in \mathbb{C}$ and let a_1, a_2, $a_3 \in \tilde{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

In 2013, Jiang and Chen [13] proved the following theorem for two distinct shared CM with some conditions.

Theorem 1.5. *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let η be a constant such that $f(z+\eta) - f(z) \not\equiv 0$ and let a, b be two non zero distinct finite complex constants. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share a, b CM then $f(z+\eta) = 2f(z)$.*

In 2022, Barki Mahesh, Dyavanal Renukadevi S and Bhoosnurmath Subhas S [4] proved the following theorem for two distinct shared CM with some operator.

Theorem 1.6. *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $c \in \mathbb{C}$ be a constant such that $\Delta_c^n f(z) \not\equiv 0$ and let a, b be two non zero distinct finite complex constants. If $\Delta_c^n f(z)$ and $f(z)$ share a, b CM then $\Delta_c^n f(z) \equiv f(z)$.*

Let η be a non-zero complex constant, $k \geq 2$ be a natural number and $f(z)$ be a meromorphic function. $f(z+\eta)$ denotes the shift operator of $f(z)$. The difference and k-th difference operators are denoted by $\Delta_\eta f(z)$ and $\Delta_\eta^k f(z)$ respectively and defined as follows:

$$\Delta_\eta f(z) = f(z+\eta) - f(z) \text{ and } \Delta_\eta^k f(z) = \Delta_\eta^{k-1}(\Delta_\eta f(z)).$$

For a meromorphic function $f(z)$ and a nonzero complex constant η, for $q \in \mathbb{N}$, we define its Shift by $f(z+\eta)$ and q-th order difference operator $\Delta_\eta^q f(z)$ is defined by $\Delta_\eta^q f(z) = \Delta_\eta^{q-1} f(z)(\Delta_\eta f(z))$, where $q(\geq 2)$ and $\eta \in \mathbb{C}$, while the difference polynomial of difference operator is given by $L(\Delta_\eta f(z)) = \sum\limits_{i=1}^{q} a_i \Delta_\eta^i f(z)$, where $a_i(i = 1, 2, ..., q)$ are nonzero constants.

We can also deduce that,

$$\Delta_\eta^q f(z) = \sum_{i=0}^{q} \binom{q}{i} f(z + (q-i)\eta),$$

$$L(\Delta_\eta f(z)) = \sum_{i=1}^{q} a_i \Delta_\eta^i f(z) = a_1 \Delta_\eta^1 f(z) + ... + a_q \Delta_\eta^q f(z).$$

In this article we generalize Theorem 1.6 to the case of difference polynomial of difference operators defined above.

Theorem 1.7. *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $\eta \in \mathbb{C}$ be a constant such that $L(\Delta_\eta f(z)) \not\equiv 0$ and let e_1, e_2 be two non zero distinct finite complex constants. If $L(\Delta_\eta f(z))$ and $f(z)$ share e_1, e_2 CM then $L(\Delta_\eta f(z)) \equiv f(z)$.*

Corollary 1.8. *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $\eta \in \mathbb{C}$ be a constant such that $L(\Delta_\eta f(z)) \not\equiv 0$ and let e_1, e_2 be two non zero distinct finite complex constants. If $L(\Delta_\eta f(z))$ and $f(z)$ share e_1, e_2 CM then $f(z + q\eta) = a_q 2^q f(z)$.*

Remark 1.9. *The corollary 1.8 generalizes Theorem 1.6.*

Example 1.10. *Let $Q(z)$ be a periodic function of period 1 such that $\sigma(Q) = 2$ and $f(z) = \dfrac{Q(z)e^{zlog2}}{sin2\pi z}$ then $L(\Delta_\eta f(z)) = \sum\limits_{i=1}^{q} a_i \Delta_\eta^i f(z) = a_1 \Delta_\eta^1 f(z) +$ $... + a_q \Delta_\eta^q f(z)$ and $f(z)$ share $1, 2$ CM and $N(r, f) = S(r, f)$, hence using corollary 1.8, we get $L(\Delta_\eta f(z)) \equiv f(z)$. Also $f(z + q\eta) = a_q 2^q f(z)$.*

2. Lemma

In this section, we state some lemmas which will be needed in the sequel.

Lemma 2.1. [22] *Let $f(z)$ be a non constant meromorphic function, $b_i(i = 1, ..., n)$ be m distinct complex numbers. Then we have*

$$m\left(r, \sum_{i=1}^{n} \frac{1}{f - b_i}\right) = \sum_{i=1}^{n} \left(r, \frac{1}{f - b_i}\right) + O(1).$$

In 2013, Chen and Feng [6] and in 2006, Halburd and Korhonen [13] investigated the Value distribution theory of meromorphic function for difference operators. Analogous result of the Lemma on logarithmic derivatives is as follows

Lemma 2.2. [8] *Let $f(z)$ be a meromorphic function of finite order and let η be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = S(r, f).$$

Lemma 2.3. *Let $f(z)$ be a non constant meromorphic function in $\mathbb{C}$. Let $b(i = 1, ..., n)$ be m distinct complex numbers. Then we have*

$$\sum_{i=1}^{n} m\left(r, \frac{1}{f - b_i}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

$$\sum_{i=1}^{n} m\left(r, \frac{1}{f - b_i}\right) \leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + S(r, f).$$

Proof. By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\sum_{i=1}^{n} m\left(r, \frac{1}{f - b_i}\right) &= m\left(r, \sum_{i=1}^{n} \frac{1}{f - b_i}\right) + O(1) \\
&= m\left(r, \sum_{i=1}^{n} \frac{f^{(k)}}{f^{(k)}(f - b_i)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{f^{(k)}}\right) + m\left(r, \sum_{i=1}^{n} \frac{f^{(k)}}{f - b_i}\right) + O(1) \\
&= m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).
\end{aligned}$$

Hence

$$\sum_{i=1}^{n} m\left(r, \frac{1}{f - b_i}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Now by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\sum_{i=1}^{n} m\left(r, \frac{1}{f-b_i}\right) &= m\left(r, \sum_{i=1}^{n} \frac{1}{f-b_i}\right) + O(1)\\
&= m\left(r, \sum_{i=1}^{n} \frac{L(\Delta_\eta f)}{L(\Delta_\eta f)(f-b_i)}\right) + O(1)\\
&\leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + m\left(r, \sum_{i=1}^{n} \frac{L(\Delta_\eta f)}{f-b_i}\right) + O(1)\\
&\leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + \sum_{i=1}^{n} m\left(r, \frac{L(\Delta_\eta f)}{f-b_i}\right) + O(1)\\
&\leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + S(r,f).
\end{aligned}$$

Hence

$$\sum_{i=1}^{n} m\left(r, \frac{1}{f-b_i}\right) \leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + S(r,f).$$

□

Lemma 2.4. [6] *Let $f(z)$ be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and η be a fixed non-zero complex number, then for each $\epsilon > 0$, we have*

$$T(r,(f+\eta)) = T(r,f) + O(r^{\sigma-1+\epsilon}) + S(r,f).$$

3. Proof of the Theorem

Theorem 1.7

Proof. By Nevanlinna's second fundamental theorem and the assumption that $L(\Delta_\eta f(z))$ and $f(z)$ share e_1, e_2 CM, we have

$$\begin{aligned}
T(r,f) &\leq N(r,f) + N\left(r, \frac{1}{f-e_1}\right) + N\left(r, \frac{1}{f-e_2}\right) + S(r,f)\\
&= N\left(r, \frac{1}{f-e_1}\right) + N\left(r, \frac{1}{f-e_2}\right) + S(r,f)\\
&= N\left(r, \frac{1}{L(\Delta_\eta f - e_1}\right) + N\left(r, \frac{1}{L(\Delta_\eta f - e_2}\right) + S(r,f)\\
&\leq N\left(r, \frac{1}{L(\Delta_\eta f) - f}\right) + S(r,f).
\end{aligned}$$

That is

$$T(r,f) \leq N\left(r, \frac{1}{L(\Delta_\eta f) - f}\right) + S(r,f). \tag{1}$$

Since $N(r,f)=S(r,f)$ and using first fundamental theorem, we have

$$\begin{aligned}
N\left(r,\frac{1}{L(\Delta_\eta f)-f}\right)+S(r,f) =& N\left(r,\frac{f}{L(\Delta_\eta f)-f}\right)+S(r,f)\\
=& N\left(r,\frac{1}{\frac{L(\Delta_\eta f)}{f}-1}\right)+S(r,f)\\
\leq& T\left(r,\frac{1}{\frac{L(\Delta_\eta f)}{f}-1}\right)+S(r,f)\\
=& T\left(r,\frac{L(\Delta_\eta f)}{f}\right)+S(r,f)\\
=& m\left(r,\frac{L(\Delta_\eta f)}{f}\right)+N\left(r,\frac{L(\Delta_\eta f)}{f}\right)\\
&+S(r,f)\\
=& N\left(r,\frac{L(\Delta_\eta f)}{f}\right)+S(r,f)\\
\leq& N\left(r,\frac{1}{f}\right)+N(r,L(\Delta_\eta f))+S(r,f).
\end{aligned}$$

Hence by using the Lemma 2.3, we have

$$\begin{aligned}
N\left(r,\frac{1}{L(\Delta_\eta f)-f}\right)+S(r,f) &\leq N\left(r,\frac{1}{f}\right)+a_q 2^q N(r,f)+S(r,f)\\
&= N\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq T\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned}$$

That is

$$N\left(r,\frac{1}{L(\Delta_\eta f)-f}\right)+S(r,f)\leq T(r,f)+S(r,f). \tag{2}$$

From (1) and (2), we obtain

$$T(r,f)=N\left(r,\frac{1}{f}\right)+S(r,f)=N\left(r,\frac{1}{f-e_1}\right)+N\left(r,\frac{1}{f-e_2}\right)+S(r,f). \tag{3}$$

Let $\psi=\frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f)-e_1}-\frac{f'}{f-e_1}$. Using the logarithmic derivative theorem, we have

$$\begin{aligned}
m(r,\psi) &\leq m\left(r,\frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f)-e_1}\right)+m\left(r,\frac{f'}{f-e_1}\right)+O(1)\\
&\leq S(r,L(\Delta_\eta f))+S(r,f).
\end{aligned} \tag{4}$$

Since

$$
\begin{aligned}
T(r, L(\Delta_\eta f)) &= m(r, L(\Delta_\eta f)) + N(r, L(\Delta_\eta f)) \\
&\leq m\left(r, f\frac{L(\Delta_\eta f)}{f}\right) + \sum_{i=1}^{q} a_i \binom{q}{i} N(r,f) + S(r,f) \\
&\leq m(r,f) + m\left(r, \frac{L(\Delta_\eta f)}{f}\right) + a_q 2^q N(r,f) + S(r,f) \\
&\leq m(r,f) + S(r,f) \\
&\leq T(r,f) + S(r,f).
\end{aligned} \tag{5}
$$

Since $S(r, L(\Delta_\eta f)) = S(r,f)$, from (4), we have $m(r,\psi) = S(r,f)$. Since ψ is the logarithmic derivative of $\dfrac{L(\Delta_\eta f) - e_1}{f - e_1}$, the poles of ψ derive from the zeros and poles of $\dfrac{L(\Delta_\eta f) - e_1}{f - e_1}$. Since f, $L(\Delta_\eta f)$ share the value e_1 CM, then $\dfrac{L(\Delta_\eta f) - e_1}{f - e_1}$ has no zeros and has at most $N(r,f)$ poles.
Thus $N(r,\psi) = N(r,f) = S(r,f)$. Combining this and (4), we have $T(r,\psi) = S(r,f)$.

Suppose that $\psi \not\equiv 0$, then from

$$
\frac{\psi}{f - e_2} = \frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f) - e_1} . \frac{L(\Delta_\eta f)}{f - e_2} - \frac{f^1}{(f - e_1) - (f - e_2)},
$$

we obtain

$$
\begin{aligned}
m\left(r, \frac{1}{f - e_2}\right) &\leq m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{\psi}{f - e_2}\right) \\
&\leq m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f) - e_1}\right) + m\left(r, \frac{L(\Delta_\eta f)}{f - e_2}\right) \\
&+ m\left(r, \frac{f'}{(f - e_1)(f - e_2)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{(L(\Delta_\eta f))'}{e_1}\left(\frac{1}{L(\Delta_\eta f) - e_1} - \frac{1}{L(\Delta_\eta f)}\right)\right) \\
&+ m\left(r, \frac{L(\Delta_\eta f)}{f - e_2}\right) + m\left(r, \frac{f'}{(e_1 - e_2)}\left(\frac{1}{f - e_1}\frac{1}{f - e_2}\right)\right) + O(1) \\
&\leq T(r,\psi) + S(r, L(\Delta_\eta f)) + S(r,f) \\
&= S(r,f).
\end{aligned}
$$

Hence

$$
m\left(r, \frac{1}{f - e_2}\right) = S(r,f).
$$

By the first fundamental theorem we get

$$T(r,f) = N\left(r, \frac{1}{f-e_2}\right) + S(r,f). \tag{6}$$

Thus, by (3) and (6), we have

$$N\left(r, \frac{1}{f-e_1}\right) = S(r,f).$$

By the assumption that $L(\Delta_\eta f)$ and $f(z)$ share e_1 CM, we have

$$N\left(r, \frac{1}{L(\Delta_\eta f)-e_1}\right) = N\left(r, \frac{1}{f-e_1}\right) = S(r,f).$$

Hence

$$N\left(r, \frac{1}{L(\Delta_\eta f)-e_1}\right) = S(r,f)$$

Since $L(\Delta_\eta f)$ and $f(z)$ share e_2 CM, by equation (5) and (6) and first fundamental theorem, we have

$$\begin{aligned} m\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) + N\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) &= T(r, L(\Delta_\eta f)) + S(r,f) \\ &\leq T(r,f) + S(r,f) \\ &= N\left(r, \frac{1}{f-e_2}\right) + S(r,f) \\ &= N\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) + S(r,f). \end{aligned}$$

Thus,

$$m\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) = S(r, L(\Delta_\eta f)) = S(r,f).$$

By Lemma 2.3, we have that

$$\begin{aligned} &m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + m\left(r, \frac{1}{L(\Delta_\eta f)-e_1}\right) + m\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) \\ \leq &m\left(r, \frac{1}{(L(\Delta_\eta f))'}\right) + S(r,f), \end{aligned} \tag{7}$$

$$m\left(r, \frac{1}{f-e_1}\right) + m\left(r, \frac{1}{f-e_2}\right) \leq m\left(r, \frac{1}{L(\Delta_\eta f)}\right) + S(r,f). \tag{8}$$

By using (3) and (7)-(8), we have

$$N\left(r, \frac{1}{L(\Delta_\eta f)-e_1}\right) + N\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) \leq N\left(r, \frac{1}{L(\Delta_\eta f)-e_2}\right) + S(r,f) \tag{9}$$

$$N\left(r, \frac{1}{f-e_1}\right) + N\left(r, \frac{1}{f-e_2}\right) \leq T(r,f) + S(r,f). \tag{10}$$

By (9)-(10), we obtain

(11)
$$\begin{aligned}
T\left(r,\frac{1}{L(\Delta_\eta f)-e_1}\right)+T\left(r,\frac{1}{L(\Delta_\eta f)-e_2}\right)+T\left(r,\frac{1}{f-e_1}\right)+T\left(r,\frac{1}{f-e_2}\right)\\
\leq m\left(r,\frac{1}{(L(\Delta_\eta f))'}\right)+N\left(r,\frac{1}{L(\Delta_\eta f)-e_2}\right)+T(r,f)\\
+S(r,f)\\
\leq T\left(r,\frac{1}{(L(\Delta_\eta f))'}\right)+T\left(r,\frac{1}{L(\Delta_\eta f)-e_2}\right)+T(r,f)\\
+S(r,f).
\end{aligned}$$

Since $N(r,f)=S(r,f)$, we get

$$\begin{aligned}
T\left(r,\frac{1}{(L(\Delta_\eta f))'}\right) &= m(r,(L(\Delta_\eta f))')+N(r,(L(\Delta_\eta f))')+O(1)\\
&= m\left(r,L(\Delta_\eta f)\frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f)}\right)+N(r,L(\Delta_\eta f))+\overline{N}(r,L(\Delta_\eta f)\\
&\quad +O(1)\\
&\leq m(r,L(\Delta_\eta f))+m\left(r,\frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f)}\right)+S(r,f)\\
&\leq T(r,L(\Delta_\eta f))+S(r,f).
\end{aligned}$$

Hence

$$T\left(r,\frac{1}{(L(\Delta_\eta f))'}\right)\leq T(r,L(\Delta_\eta f))+S(r,f). \tag{12}$$

Considering (4), (11), (12) and first fundamental theorem, we get

$$T(r,f)=S(r,f),$$

which is a contradiction.

Therefore $\psi\equiv 0$, that is

$$\frac{(L(\Delta_\eta f))'}{L(\Delta_\eta f)-e_1}=\frac{f'}{f-e_1}. \tag{13}$$

By integrating (13), we obtain

$$\frac{L(\Delta_\eta f)-e_1}{f-e_1}\equiv C_1, \tag{14}$$

where C_1 is some non-zero constant.

Using the same procedure as above, by the assumption, $L(\Delta_\eta f)$ and $f(z)$ share e_2 CM, we get

$$\frac{L(\Delta_\eta f)-e_2}{f-e_2}\equiv C_2, \tag{15}$$

where C_2 is some non-zero constant.
If $C_1 \equiv 1$ or ($C_2 \equiv 1$), then by (14) and (15), we obtain

$$L(\Delta_\eta f) \equiv f(z).$$

Hence conclusion holds.
If $C_1 \neq 1$ and $C_2 \neq 1$, then by (14) and (15), we get

$$\begin{aligned}(L(\Delta_\eta f) - e_1) - (L(\Delta_\eta f) - e_2) &= (C_1 f - C_1 e_1) - (C_2 f - C_2 e_2)\\ e_1 - e_2 &= (C_1 - C_2) f + C_2 e_2 - C_1 e_1\end{aligned}$$

$$(C_1 - C_2) f = e_1 - e_2 + C_2 e_2 - C_1 e_1. \tag{16}$$

If $C_1 \neq C_2$, then f is a constant.
Which is contradiction.
Hence $C_1 = C_2$, thus by (16), we have

$$e_1 - e_2 = C_1(e_1 - e_2)$$

Therefore $C_1 = 1 = C_2$.
Which is contradiction.
This completes the proof. □

Corollary 1.8

Proof. Using Theorem 1.7, we have

$$L(\Delta_\eta f) \equiv f(z).$$

This implies that

$$\begin{aligned}f(z + q\eta) &= \left[\sum_{i=1}^{q-1} a_i \binom{q}{i} 2^{is} + 1\right] f(z)\\ &= a_q 2^q f(z)\end{aligned}$$

Hence the proof □

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