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SURFACES WITH CONSTANT GAUSSIAN AND MEAN CURVATURES N THE ANTI-DE SITTER SPACE $\mathbb{H}^3_1$

Uğur Dursun

Abstract. In this work, we study time-like and space-like surfaces invariant by a group of translation isometries of the half-space model $\mathcal{H}^3_1$ of the anti-de Sitter space $\mathbb{H}^3_1$. We determine all such surfaces with constant mean curvature and constant Gaussian curvature. We also obtain umbilical surfaces of $\mathcal{H}^3_1$.

1. Introduction

We study time-like and space-like surfaces invariant by a group of translation isometries in the half-space model $\mathcal{H}^3_1$ of the anti-de Sitter 3-space $\mathbb{H}^3_1$. Such surfaces are translation surfaces. A translation surface in the Euclidean is obtained by a translation of a curve along another curve, and it is locally written as a sum of two curves of the ambient space. Translations surfaces in the Euclidean 3-surface are well-known and recently have been studied in some other ambient spaces too, for instance, see [3, 4, 6, 10, 11, 13, 17].

Minimal translations surfaces in different ambient spaces such as in Heisenberg group, hyperbolic space, Sol_3 and $\mathbb{H}^2 \times \mathbb{R}$ space were studied in [6, 10, 11, 13, 17].

Translation surfaces in the Euclidean and Minkowski spaces having constant Gaussian curvature were studied in [8]. In [15], Šipuš investigated translation surfaces in a simply isotropic space having constant isotropic Gaussian or mean curvature, and in [1], Belarbi studied constant extrinsically Gaussian curvature translation surfaces invariant under the 1-parameter group of isometries in the 3-dimensional Heisenberg group. Also, in [9], Lopez described all parabolic surfaces in $\mathbb{H}^3$ with constant Gaussian curvature. A parabolic surface in the hyperbolic space $\mathbb{H}^3$ is a surface which is invariant by a group of parabolic isometries of $\mathbb{H}^3$, in particular, it is a translation surface in a direction.

In [7], J. P. Lambert studied Lorentzian invariant space-like surfaces of constant mean curvature in the anti-de Sitter space $\mathbb{H}^3_1(-c^2)$ of constant sectional

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curvature $-c^2$ by considering a flat-chat model of $\mathbb{H}^3_1$, and he constructed surfaces of revolution with constant mean curvature $H = c$ and maximal surfaces of revolution in $\mathbb{H}^3_1$.

In [14], A. Seppi and E. Trebeschi developed a half-space model $\mathcal{H}^{p,q}$ for the $(p+q)$-dimensional pseudo-hyperbolic space $\mathbb{H}^{p,q}$ of signature (p,q) with $q \geq 1$. They showed that there exists an isometric imbedding of this space into $\mathbb{H}^{p,q}$ and its image is the complement of a totally geodesic degenerate hyperplane in $\mathbb{H}^{p,q}$. They also described full isometry group of $\mathcal{H}^{p,q}$. The half-space model for the anti-de Sitter space $\mathbb{H}^{2,1}$ was used in [2], also the half-space model of the de Sitter space $\mathbb{S}^{p,1}$ was studied in [12].

In this work we consider the half-space model $\mathcal{H}^3_1 = \mathcal{H}^{2,1}$ of the anti-de Sitter space $\mathbb{H}^3_1$, and motivated by the work done in [9] we study time-like and space-like surfaces invariant by a group of translation isometries in $\mathcal{H}^3_1$ having constant mean curvature and constant Gaussian curvature. As the horizontal coordinate plane of $\mathcal{H}^3_1$ is the Minkowski 2-plane we consider only time-like and space-like horizontal directions for the translation surfaces in $\mathcal{H}^3_1$. We obtain the Gaussian and mean curvatures of these translation surfaces in $\mathcal{H}^3_1$, and then we determine the generating curve of surfaces by solving differential equations when the mean and Gaussian curvatures are constants. The surfaces that we obtained include flat surfaces and surfaces with zero mean curvatures. Also, we determine umbilical surfaces of $\mathcal{H}^3_1$.

2. Preliminaries

We consider the upper half-space model $\mathcal{H}^3_1 = \mathcal{H}^{2,1}$ for the anti-de Sitter space $\mathbb{H}^3_1$ which is defined as the open half-space $\{(x,y,z) \in \mathbb{R}^3_1 | z > 0\}$ equipped with the Lorentzian metric $g = \dfrac{dx^2 - dy^2 + dz^2}{z^2}$, of constant curvature -1. The boundary of $\mathcal{H}^3_1$, denoted by $\partial\mathcal{H}^3_1$, is the Minkowski 2-plane $\{z = 0\}$.

By following Lemma 2.6 and Theorem 7.1 in [14], the full isometry group of half-space model $\mathcal{H}^3_1$ is given by the group

$$G = \{(x,y,z) \mapsto \lambda(A(x,y) + (x_0,y_0), z) | \lambda > 0, A \in O(1,1), (x_0,y_0) \in \mathbb{R} \oplus \mathbb{R}\}$$

from which we can have the isometries of the form: $\hat{T}(x,y,z) = \lambda(x,y,z)$, (homothety) $\tilde{T}(x,y,z) = (A(x,y),z)$ (rotation), and $T(x,y,z) = (x + x_0, y + y_0, z)$ (translation).

Let $\xi \in \partial\mathcal{H}^3_1$ be a space-like vector, and consider the horizontal translation T in a horizontal space-like direction ξ. Then, the set $G_\xi = \{T_a(p) | a \in \mathbb{R}, T_a(p) = p + a\xi, p = (x,y,z) \in \mathcal{H}^3_1\}$ is a subgroup of isometries of $\mathcal{H}^3_1$, that is, it is a group of translation isometries, and the orbits are horizontal straight lines parallel to ξ. Without loss of generality, we assume that horizontal space-like vector ξ that defines the group G_ξ is the space-like vector $\xi = (1,0,0)$. Let $P_\xi = \{(0,y,z) | z > 0\}$ which is a vertical geodesic time-like plane orthogonal to ξ. A surface M_q with the index q, $q = 0, 1$, invariant by G_ξ intersects P_ξ in a curve α which is called the generating curve of M_q. The index q depends on the curve α, i.e. $q = 0$ if α is space-like, and $q = 1$ if α is time-like. Let $\alpha(s) = (0, y(s), z(s))$ be parameterized by arc length parameter s with respect to the Minkowski metric whose domain J is an open interval including zero. Then, we have ${z'}^2 - {y'}^2 = \varepsilon = \pm 1$, and a parameterization of M_q is

$$\varphi(s, x) = (x, y(s), z(s)), \quad s \in J,\ x \in \mathbb{R}. \tag{1}$$

Similarly, let $\eta \in \partial \mathcal{H}^3_1$ be a time-like vector. We consider the horizontal translation in a horizontal time-like direction η. Then, the set $G_\eta = \{T_a(p) | a \in \mathbb{R}, T_a(p) = p + a\eta, p = (x, y, z) \in \mathcal{H}^3_1\}$ is a subgroup of translation isometries of $\mathcal{H}^3_1$. Without loss of generality, we assume that horizontal time-like vector η that defines the group G_η is the time-like vector $\eta = (0, 1, 0)$. Let $P_\eta = \{(x, 0, z) | z > 0\}$ which is a vertical geodesic space-like plane orthogonal to η. A time-like surface $\widetilde{M}_1$ invariant by G_η intersects P_η in a space-like curve γ which is the generating curve of $\widetilde{M}_1$. Let $\gamma(s) = (x(s), 0, z(s))$ be parameterized by arc length parameter s with respect to the Euclidean metric whose domain of definition J is an open interval including zero. Then, a parameterization of $\widetilde{M}_1$ is

$$\psi(s, y) = (x(s), y, z(s)), \quad s \in J,\ y \in \mathbb{R}. \tag{2}$$

Let $\widetilde{\nabla}$ denote the connection of $\mathcal{H}^3_1$. Let $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_z = \frac{\partial}{\partial z}$ denote coordinate vector fields on $\mathcal{H}^3_1$. Then, the vectors $E_1 = z\partial_x, E_2 = z\partial_y, E_3 = z\partial_z$ with $g(E_1, E_1) = g(E_3, E_3) = 1,\ g(E_2, E_2) = -1$ form a pseudo-orthonormal frame on $\mathcal{H}^3_1$. In this frame, non-zero covariant derivatives of $\mathcal{H}^3_1$ are obtained as follows

$$\widetilde{\nabla}_{E_1} E_1 = E_3,\ \widetilde{\nabla}_{E_1} E_3 = -E_1,\ \widetilde{\nabla}_{E_2} E_2 = -E_3,\ \widetilde{\nabla}_{E_2} E_3 = -E_2.$$

2.1. Translation in a space-like direction in $\mathcal{H}^3_1$

Let us consider the surface $M_q : \varphi(s, x)$ in $\mathcal{H}^3_1$ defined by (1). Then, the coordinate vector fields of the surface M_q are $\varphi_s(s, x) = \frac{y'(s)}{z(s)} E_2 + \frac{z'(s)}{z(s)} E_3$ and $\varphi_x(s, x) = \frac{1}{z(s)} E_1$, and the coefficients of the first fundamental form induced by φ are $E = g(\varphi_s, \varphi_s) = \frac{{z'}^2 - {y'}^2}{z^2} = \frac{\varepsilon}{z^2},\ F = g(\varphi_s, \varphi_x) = 0,\ G = g(\varphi_x, \varphi_x) = \frac{1}{z^2}$ from which M_q is a regular surface if $z' \neq y'$ on J. We can choose a unit normal vector to M_q in $\mathcal{H}^3_1$ as $n = z'E_2 + y'E_3$, and the signature of n is $\varepsilon_n = -\varepsilon$.

By using the Gauss-Weingarden formulas it is seen that the vectors

$$e_1 = y'(s)E_2 + z'(s)E_3 \quad \text{and} \quad e_2 = E_1$$

with the signatures $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 1$, respectively, are the principal directions of the shape operator A_n for M_q, that is,

$$A_n(e_1) = [y' + \varepsilon z(y'z'' - z'y'')]e_1 \quad \text{and} \quad A_n(e_2) = y'e_2.$$

From the Gauss equation we have the Gaussian curvature K as follows

$$
\begin{aligned}
K =& -1+\varepsilon_n \frac{g(A_n(e_1),e_1)g(A_n(e_2),e_2)-g(A_n(e_1),e_2)^2)}{g(e_1,e_1)g(e_2,e_2)-g(e_1,e_2)^2} \\
=& -1-\varepsilon[{y'}^2+\varepsilon z y'(y'z''-z'y'')] \\
=& -1-\varepsilon {y'}^2 - z y'(y'z''-z'y'') \\
=& \varepsilon(zz''-{z'}^2). \qquad (3)
\end{aligned}
$$

We introduce the function $\theta(s)$ such that

$$y'(s)=\sinh\theta(s), \quad z'(s)=\cosh\theta(s) \quad \text{if} \quad \varepsilon=1$$

and

$$y'(s)=\cosh\theta(s), \quad z'(s)=\sinh\theta(s) \quad \text{if} \quad \varepsilon=-1.$$

Without loss of generality we may assume that

$$y(0)=0,\ z(0)=1,\ \theta(0)=\theta_0. \qquad (4)$$

Hence, $z'(0)=\cosh\theta_0$ when $\varepsilon=1$ and $z'(0)=\sinh\theta_0$ when $\varepsilon=-1$. In the case $\varepsilon=1$, $\theta(s)$ is the angle between the space-like vectors $\alpha'(s)$ and ∂_z. If $\theta\equiv 0$, then $\alpha(s)$ is a part of vertical straight line in the yz-plane, and the surface M_0 is the vertical space-like geodesic plane $y=0$ with $z>0$. If $\varepsilon=-1$, then $\theta(s)$ is the angle between the time-like vectors $\alpha'(s)$ and ∂_y. If $\theta\equiv 0$, then $\alpha(s)$ is the time-like horizontal line passing through $(0,0,1)$ with the direction $\alpha'=(0,1,0)$, and the surface M_1 is the horizontal time-like plane $z=1$ which is a horosphere in $\mathcal{H}^3_1$.

2.2. Umbilical Surfaces M_q in $\mathcal{H}^3_1$

In this section we study umbilical surfaces invariant by translation isometries in $\mathcal{H}^3_1$. A surface M_q in $\mathcal{H}^3_1$ is umbilical if $\varepsilon_1 g(A_n(e_1),e_1)=\varepsilon_2 g(A_n(e_2),e_2)$, that is,

$$\varepsilon z(y'z''-z'y'')+y'=y'$$

which yields $\theta'(s)=0$ for $\varepsilon=\mp 1$, i.e. $\theta=\theta_0$ is a constant.

Therefore, for $\varepsilon=1$ and using the initial condition (4) we have

$$y(s)=(\sinh(\theta_0))s, \quad z(s)=(\cosh(\theta_0))s+1$$

with $s>-1/\cosh(\theta_0)$. Without loss of generality we may take $\theta_0\geq 0$. These are the parametric equations of the space-like straight line $\alpha(s)=(0,(\sinh(\theta_0))s,(\cosh(\theta_0))s+1)$. For this generating curve the Gaussian curvature of the space-like umbilical surface M_0 is $K=-\cosh^2(\theta_0)$ which is a nonzero constant. Considering the initial condition (4) the space-like umbilical surfaces are the vertical space-like plane $y=0$ with $K=-1$ if $\theta_0=0$, or a space-like surface with $K=-\cosh^2(\theta_0)<-1$ if $\theta_0>0$.

Now, for $\varepsilon=-1$ and using the initial condition (4) we have

$$y(s)=(\cosh(\theta_0))s, \quad z(s)=(\sinh(\theta_0))s+1$$

with $s > -1/\sinh(\theta_0)$, that is, the time-like straight line

$$\alpha(s) = (0, (\cosh(\theta_0))s, (\sinh(\theta_0))s + 1).$$

For this generating curve the Gaussian curvature of the time-like umbilical surface M_1 is $K = \sinh^2(\theta_0)$ which is a constant, and it is zero only for $\theta_0 = 0$. Having the initial condition (4) the time-like umbilical surfaces are the horosphere $z = 1$ with $K = 0$ if $\theta_0 = 0$, or a time-like surface with $K = \sinh^2(\theta_0) > 0$ if $\theta_0 > 0$.

Therefore we state

Theorem 2.1. *A surface M_q invariant by a group of translation in $\mathcal{H}_1^3$ defined by* (1) *with a generating curve $\alpha(s) = (0, y(s), z(s))$ is umbilical if and only if*
$\alpha(s) = (0, (\sinh(\theta_0))s, (\cosh(\theta_0))s + 1)$ *when $q = 0$, or*
$\alpha(s) = (0, (\cosh(\theta_0))s, (\sinh(\theta_0))s + 1)$ *when $q = 1$.*
Moreover,

1. *M_0 is the vertical totally geodesic space-like plane $y = 0$ with $K = -1$ when $\theta_0 = 0$, or it is a space-like surface with $K = -\cosh^2\theta_0$ for $\theta \neq 0$;*
2. *M_1 is the horosphere $z = 1$ with $K = 0$ when $\theta_0 = 0$, or it is a time-like surface with $K = \sinh^2\theta_0$ for $\theta \neq 0$.*

2.3. Surfaces M_q in $\mathcal{H}_1^3$ with constant mean curvature

In this section we study surfaces M_q invariant by translation isometries in $\mathcal{H}_1^3$ with constant mean curvature.

The mean curvature H of the surface M_q defined by (1) in $\mathcal{H}_1^3$ is given by

$$H = \frac{\varepsilon_n}{2}[\varepsilon_1 g(A_n(e_1), e_1) + \varepsilon_2 g(A_n(e_2), e_2)],$$

that is,

$$H = -\frac{\varepsilon}{2}[\varepsilon z(y'z'' - z'y'') + 2y'].$$

Thus, M_q has a constant mean curvature $H = H_0$ if and only if the functions $y(s)$ and $z(s)$ satisfy the differential equation

$$\varepsilon z(y'z'' - z'y'') + 2y' = -2\varepsilon H_0. \tag{5}$$

Using ${z'}^2 - {y'}^2 = \varepsilon$, we have

$$zz'' - 2({z'}^2 - \varepsilon) = 2\varepsilon H_0\sqrt{{z'}^2 - \varepsilon}.$$

If we put $U(z) = z'(s)$, then we have $z'' = U(z)\frac{dU(z)}{dz}$, and hence the above differential equation turns to

$$zU\frac{dU}{dz} = 2\sqrt{U^2 - \varepsilon}(\sqrt{U^2 - \varepsilon} + \varepsilon H_0)$$

and its solution yields

$$U^2 = {z'}^2 = \varepsilon + (cz^2 - \varepsilon H_0)^2, \tag{6}$$

where c is a constant. If $c = 0$, then, by an appropriate isometry of $\mathcal{H}_1^3$ we can write $\alpha(s) = (0, sH_0, 1 + s\sqrt{H_0^2 + \varepsilon})$, where $|H_0| \geq 1$ when $\varepsilon = -1$. Therefore, the surface M_q defined by this generating curve α has constant mean curvature H_0, and it is umbilical.

Now, let $c \neq 0$. From(6) and $z'^2 - y'^2 = \varepsilon$ we obtain that

$$z'(s) = \mp\sqrt{\varepsilon + (cz^2 - \varepsilon H_0)^2} \quad \text{and} \quad y'(s) = \mp(cz^2 - \varepsilon H_0).$$

By an appropriate isometry of $\mathcal{H}_1^3$, we can consider only plus sign in the above equations. Hence, we write

$$\frac{dy}{dz} = \frac{cz^2 - \varepsilon H_0}{\sqrt{\varepsilon + (cz^2 - \varepsilon H_0)^2}}$$

from which we have

$$y(z) = \int_1^z \frac{ct^2 - \varepsilon H_0}{\sqrt{\varepsilon + (ct^2 - \varepsilon H_0)^2}} dt$$

for $z > 0$ in some open interval containing $z = 1$. Thus, we parameterize the generating curve α as

$$\alpha(z) = \Big(0, \int_1^z \frac{ct^2 - \varepsilon H_0}{\sqrt{\varepsilon + (ct^2 - \varepsilon H_0)^2}} dt, z\Big)$$

with the initial condition $\alpha(1) = (0, 0, 1)$ for $z > 0$ in some open interval. For $\varepsilon = 1$, the function $y(z)$ is defined for all $z > 0$, but for $\varepsilon = -1$ we determine the domain of definition for α depending on H_0 and c such that it contains $z = 1$.

Case 1. $H_0 < 0$. It follows from $\alpha'(z)$ that $|cz^2 + H_0| > 1$, that is, $cz^2 + H_0 < -1$ or $cz^2 + H_0 > 1$:

1.1) For $cz^2 + H_0 < -1$, we have

1.1-a) $\sqrt{\frac{-1-H_0}{c}} < z < \infty$ when $c < -1 - H_0$ and $-1 \leq H_0 < 0$, or

1.1-b) $0 < z < \sqrt{\frac{-1-H_0}{c}}$ when $0 < c < -1 - H_0$ and $H_0 < -1$, or

1.1-c) $0 < z < \infty$ when $c = 0$ and $H_0 < -1$;

1.2) For $cz^2 + H > 1$, we have $\sqrt{\frac{1-H_0}{c}} < z < \infty$ when $c > 1 - H_0$.

Case 2. $H_0 = 0$. From $\alpha'(z)$ we have $|cz^2| > 1$ that gives $\sqrt{\frac{1}{|c|}} < z < \infty$ when $|c| > 1$.

Case 3. $H_0 > 0$. From $\alpha'(z)$ we have that $|cz^2 + H_0| > 1$, that is, $cz^2 + H_0 < -1$ or $cz^2 + H_0 > 1$:

3.1) For $cz^2 + H_0 < -1$, we have $\sqrt{\frac{-1-H_0}{c}} < z < \infty$ when $c < -(1 + H_0) < 0$;

3.2) For $cz^2 + H > 1$, we have

3.2-a) $0 < z < \infty$ when $c = 0$ and $H_0 > 1$, or when $c > 0$ and $H_0 \geq 1$, or

3.2-b) $0 < z < \sqrt{\frac{1-H_0}{c}}$ when $1 - H_0 < c < 0$ and $H_0 > 1$, or

3.2-c) $\sqrt{\frac{1-H_0}{c}} < z < \infty$ when $c > 1 - H_0$ and $0 < H_0 \leq 1$.

For the infinite intervals, when $z \to \infty$, the second component of α' approaches to -1 or 1 according to c is negative or positive, respectively. Therefore, α is asymptotic to a line of the form $\beta(t) = (0, \pm t + b, t)$ which has a null direction in the zy-plane, where b is a non-zero constant.

Theorem 2.2. *Let M_0 be a space-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by* (1) *with the generating curve α. Then, M_0 has constant mean curvature H_0 if and only if α is given by*

$$\alpha(z) = \Big(0, \int_1^z \frac{ct^2 - H_0}{\sqrt{1 + (ct^2 - H_0)^2}} dt, z\Big)$$

for $0 < z < \infty$, where c is a zero constant.

Theorem 2.3. *Let M_1 be a time-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by* (1) *with the generating curve α. Then, M_1 has constant mean curvature H_0 if and only if α is given by*

$$\alpha(z) = \Big(0, \int_1^z \frac{ct^2 + H_0}{\sqrt{(ct^2 + H_0)^2 - 1}} dt, z\Big),$$

where c is a constant, and

1. *for $H_0 < 0$, $\alpha(z)$ is defined on the interval $\sqrt{\frac{-1-H_0}{c}} < z < \infty$ when $c < -1 - H_0$ and $-1 \leq H_0 < 0$, or $0 < z < \sqrt{\frac{-1-H_0}{c}}$ when $0 < c < -1 - H_0$ and $H_0 < -1$, or $0 < z < \infty$ when $c = 0$ and $H_0 < -1$, or $\sqrt{\frac{1-H_0}{c}} < z < \infty$ when $c > 1 - H_0$;*
2. *for $H_0 = 0$, $\alpha(z)$ is defined on the interval $\sqrt{\frac{1}{|c|}} < z < \infty$ when $|c| > 1$;*
3. *for $H_0 > 0$, $\alpha(z)$ is defined on the interval $\sqrt{\frac{-1-H_0}{c}} < z < \infty$ when $c < -(1+H_0) < 0$, or $0 < z < \infty$ when $c = 0$ and $H_0 > 1$, or when $c > 0$ and $H_0 \geq 1$, or $0 < z < \sqrt{\frac{1-H_0}{c}}$ when $1 - H_0 < c < 0$ and $H_0 > 1$, or $\sqrt{\frac{1-H_0}{c}} < z < \infty$ when $c > 1 - H_0$ and $0 < H_0 \leq 1$.*

All these intervals contain $z = 1$.

2.4. Surfaces M_q in $\mathcal{H}_1^3$ with constant Gaussian curvature

In this section we study surfaces M_q invariant by a group of translation isometries in $\mathcal{H}_1^3$ with constant Gaussian curvature.

By considering (3), a surface M_q in $\mathcal{H}_1^3$ defined by (1) with the generating curve α has constant Gaussian curvature $K = K_0$ if the $z(s)$ component of the profile curve $\alpha(s)$ satisfies the equation

$$\varepsilon(zz'' - z'^2) = K_0, \tag{7}$$

and the $y(s)$ component of $\alpha(s)$ is given by $y(s) = \int_0^s \sqrt{z'^2(\tau) - \varepsilon d\tau}$. The first integral of the above differential equation is obtained as

$$z'^2 = Cz^2 - \varepsilon K_0, \tag{8}$$

where C is a constant. By taking the derivative of this with respect to s, we get

$$z'' = Cz. \tag{9}$$

We investigate the solution of (9) according the values of ε, C, and K_0.

Let $\varepsilon = 1$. For $s = 0$, we have $C = \cosh^2\theta_0 + K_0$ from (4) and (8). Thus, we have the followings:

Case 1.1. $C = K_0 + \cosh^2\theta_0 < 0$, that is, $K_0 < -\cosh^2\theta_0$. Then, by using (4), the solution of (9) yields

$$z(s) = \sqrt{\frac{K_0}{C}}\cos(\sqrt{-C}s - B),$$

where $B = \arccos\sqrt{\frac{C}{K_0}}$, and hence $y(s) = \int_0^s \sqrt{-K_0\sin^2(\sqrt{-C}\tau - B) - 1 d\tau}$, where $s_1 = \frac{1}{\sqrt{-C}}\left(\arcsin\frac{1}{\sqrt{-K_0}} + \arccos\sqrt{\frac{C}{K_0}}\right) \leq s < \frac{1}{\sqrt{-C}}\left(\frac{\pi}{2} + \arccos\sqrt{\frac{C}{K_0}}\right)$. The curve α is decreasing on the interval and has a vertical tangent at s_1.

Case 1.2. C=0, that is, $K_0 = -\cosh^2\theta_0$. Then, the solution of (8) by using (4) gives $z(s) = (\cosh\theta_0)s + 1$, and hence $y(s) = (\sinh\theta_0)s$. The space-like surface M_0 with the generating curve $\alpha(s) = (0, (\sinh\theta_0)s, (\cosh\theta_0)s + 1)$ is umbilical that was studied in Section 2.2.

Case 1.3. $C = K_0 + \cosh^2\theta_0 > 0$ and $-1 \leq K_0 < 0$. Then, by using (4), the solution of (9) gives

$$z(s) = \sqrt{\frac{-K_0}{C}}\sinh(\sqrt{C}s + B),$$

where $B = \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}$, and hence $y(s) = \int_0^s \sqrt{-K_0\cosh^2(\sqrt{C}\tau + B) - 1 d\tau}$ which is defined if $\sqrt{C}s + B \geq \operatorname{arccosh}(\frac{1}{\sqrt{-K_0}})$ that holds when $-1 \leq K_0 < 0$. Hence we have $s_2 = \frac{1}{\sqrt{C}}\left(\operatorname{arccosh}\frac{1}{\sqrt{-K_0}} - \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}\right) \leq s < \infty$. It can be seen that $\lim_{s\to\infty}\frac{dz}{dy} = 1$ which implies that the graph of α is asymptotic to a line with the direction $\vec{t} = (0, 1, 1)$, and also it has a vertical tangent when s approaches to s_2 from the right.

Case 1.4. $C = \cosh^2\theta_0 > 0$ and $K_0 = 0$. Then, by using (4), the solution of (8) gives

$$z(s) = e^{(\cosh\theta_0)s},$$

and hence

$$
\begin{aligned}
y(s) = &\int_0^s \sqrt{(\cosh^2\theta_0)e^{2(\cosh\theta_0)\tau} - 1}d\tau \\
= &\frac{1}{\cosh\theta_0}\Big(\sqrt{(\cosh^2\theta_0)e^{2(\cosh\theta_0)s} - 1} - \arctan\sqrt{(\cosh^2\theta_0)e^{2(\cosh\theta_0)s} - 1}\Big) \\
&- \frac{1}{\cosh\theta_0}(\sinh\theta_0 - \arctan(\sinh\theta_0))
\end{aligned}
$$

for $s_3 = -\frac{\ln(\cosh\theta_0)}{\cosh\theta_0} \le s < \infty$. For this case we again have $\lim_{s\to\infty}\frac{dy}{dz} = 1$ which implies that the graph of α is asymptotic to a line with the direction $\vec{t} = (0,1,1)$, and also it has a vertical tangent when s approaches to s_3 from the right.

Case 1.5. $C = K_0 + \cosh^2\theta_0 > 0$ and $K_0 > 0$. Then, by using (4), the solution of (9) yields

$$
z(s) = \sqrt{\frac{K_0}{C}}\cosh(\sqrt{C}s + B),
$$

where $B = \operatorname{arccosh}\sqrt{\frac{C}{K_0}}$, and hence $y(s) = \int_0^s \sqrt{K_0\sinh^2(\sqrt{C}\tau + B) - 1}d\tau$ for $s_4 \le s < \infty$, where $s_4 = \frac{1}{\sqrt{C}}\Big(\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} - \operatorname{arccosh}\sqrt{\frac{C}{K_0}}\Big) \le 0$. The graph of α is also asymptotic to a line with the direction $\vec{t} = (0,1,1)$, and also it has a vertical tangent when s approaches to s_4 from the right.

Therefore we have

Theorem 2.4. *Let M_0 be a space-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by* (1) *with the generating curve α. Then, M_0 has constant Gaussian curvature K_0 if and only if the y and z components of α are given by one of the followings:*

1. *For $K_0 < -\cosh^2\theta_0$,*

$$
y(s) = \int_0^s \sqrt{-K_0\sin^2(\sqrt{-C}\tau + B) - 1}d\tau,\ z(s) = \sqrt{\frac{K_0}{C}}\cos(\sqrt{-C}s + B),
$$

where $s_1 = \frac{1}{\sqrt{-C}}\Big(\arcsin\frac{1}{\sqrt{-K_0}} + B\Big) \le s < \frac{1}{\sqrt{-C}}\Big(\frac{\pi}{2} + B\Big)$, $C = K_0 + \cosh^2\theta_0 < 0$ and $B = \arccos\sqrt{\frac{C}{K_0}}$. The curve α is decreasing on the interval and has a vertical tangent when s approaches to s_1 from the right.

2. *For $K_0 = -\cosh^2\theta_0$, $y(s) = (\sinh\theta_0)s$, $z(s) = (\cosh\theta_0)s + 1$. For this generating curve, M_0 is umbilical.*

3. *For $-1 \le K_0 < 0$,*

$$
y(s) = \int_0^s \sqrt{-K_0\cosh^2(\sqrt{C}\tau + B) - 1}d\tau,\ z(s) = \sqrt{\frac{-K_0}{C}}\sinh(\sqrt{C}s + B)
$$

for $s_2 = \frac{1}{\sqrt{C}}\Big(\operatorname{arccosh}\frac{1}{\sqrt{-K_0}} - \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}\Big) \le s < \infty$, *where* $C = K_0 + \cosh^2\theta_0 > 0$ *and* $B = \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}$. *The graph of* α *is asymptotic to a line with the direction* $\vec{t} = (0,1,1)$, *and it has a vertical tangent when* s *approaches to* s_2 *from the right.*

4. *For* $K_0 = 0$,

$$y(s) = \frac{1}{\cosh\theta_0}\Big(\sqrt{(\cosh^2\theta_0)e^{2(\cosh\theta_0)s} - 1} - \arctan\sqrt{(\cosh^2\theta_0)e^{2(\cosh\theta_0)s} - 1}\Big)$$
$$- \frac{1}{\cosh\theta_0}(\sinh\theta_0 - \arctan(\sinh\theta_0)),$$
$$z(s) = e^{(\cosh\theta_0)s}$$

for $s_3 = -\frac{\ln(\cosh\theta_0)}{\cosh\theta_0} \le s < \infty$. *The graph of* α *is asymptotic to a line with the direction* $\vec{t} = (0,1,1)$, *and it has a vertical tangent when* s *approaches to* s_3 *from the right.*

5. *For* $K_0 > 0$,

$$y(s) = \int_0^s \sqrt{K_0\sinh^2(\sqrt{C}\tau + B) - 1}\,d\tau,\ z(s) = \sqrt{\frac{K_0}{C}}\cosh(\sqrt{C}s + B),$$

for $s_4 \le s < \infty$, *where* $C = K_0 + \cosh^2\theta_0$, $s_4 = \frac{1}{\sqrt{C}}\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} - \operatorname{arccosh}\sqrt{\frac{C}{K_0}}\Big) \le 0$ *and* $B = \operatorname{arccosh}\sqrt{\frac{C}{K_0}}$. *The graph of* α *is asymptotic to a line with the direction* $\vec{t} = (0,1,1)$, *and has a vertical tangent when* s *approaches to* s_4 *from the right.*

Let $\varepsilon = -1$, and consider the initial value condition (4). For $s = 0$, we have $C = \sinh^2\theta_0 - K_0$ from (8). Thus, we have the following cases:

Case 2.1. $C = \sinh^2\theta_0 - K_0 > 0$ and $K_0 < 0$. Then, the solution of (9) by using (4) yields

$$z(s) = \sqrt{\frac{-K_0}{C}}\cosh(\sqrt{C}s + B),$$

where $B = \operatorname{arccosh}\sqrt{\frac{C}{-K_0}}$, and hence $y(s) = \int_0^s \sqrt{1 - K_0\sinh^2(\sqrt{C}\tau + B)}d\tau$ for $s \in \mathbb{R}$. It follows that the curve α has the horizontal tangent $\alpha' = (0,1,0)$ at $\hat{s} = -B/\sqrt{C}$, and at that point, $z(\hat{s}) = \sqrt{\frac{-K_0}{C}}$ and $z(s) \ge z(\hat{s})$ for all s, that is, α has a minimum at the point $\hat{s}$. Also, it can be shown that the graph of α is symmetric across the line $y = y(\hat{s})$ in the yz-plane, and that $\lim_{s\to\mp\infty}\frac{dz}{dy} = \mp 1$ which implies that the graph of α is asymptotic to lines with the directions $\vec{t_1} = (0,-1,1)$ and $\vec{t_2} = (0,1,1)$.

Case 2.2. $C = \sinh^2\theta_0 > 0$ and $K_0 = 0$. Without loss of generality we assume that $\theta_0 > 0$. Then, by using (4), the solution of (8) yields $z(s) = e^{(\sinh\theta_0)s}$, and hence

$$
\begin{aligned}
y(s) =& \int_0^s \sqrt{1+(\sinh^2\theta_0)e^{2(\sinh\theta_0)\tau}}d\tau \\
=& \frac{1}{\sinh\theta_0}\Big(\sqrt{1+(\sinh^2\theta_0)e^{2(\sinh\theta_0)s}} - \operatorname{arccoth}\sqrt{1+(\cosh^2\theta_0)e^{2(\cosh\theta_0)s}}\Big) \\
&- \frac{1}{\sinh\theta_0}(\cosh\theta_0 - \operatorname{arccoth}(\cosh\theta_0))
\end{aligned}
$$

for $-\infty < s < \infty$. It is easily seen that $\lim_{s\to\infty}\frac{dz}{dy} = 1$ and $\lim_{s\to-\infty}\frac{dz}{dy} = 0$ which imply that the graph of α is asymptotic to a line with the direction $\vec{t} = (0,1,1)$ and to the y-axis, respectively. Also it is increasing on $\mathbb{R}$.

Case 2.3. $C = \sinh^2\theta_0 - K_0 > 0$ and $0 < K_0 < \sinh^2\theta_0$. Then, by using (4), the solution of (9) yields

$$
z(s) = \sqrt{\frac{K_0}{C}}\sinh(\sqrt{C}s + B),
$$

where $B = \operatorname{arcsinh}\sqrt{\frac{C}{K_0}}$, and hence $y(s) = \int_0^s \sqrt{1+K_0\cosh^2(\sqrt{C}\tau + B)}d\tau$ for $s > -\frac{B}{\sqrt{C}}$ as $z > 0$. It can be seen that $\lim_{s\to\infty}\frac{dz}{dy} = 1$ which implies that the graph of α is asymptotic to a line with the direction $\vec{t} = (0,1,1)$.

Case-2.4. $C = \sinh^2\theta_0 - K_0 < 0$, that is, $K_0 > \sinh^2\theta_0$. Then, by using (4), the solution of (9) yields

$$
z(s) = \sqrt{\frac{K_0}{-C}}\cos(\sqrt{-C}s - B),
$$

where $B = \arccos\sqrt{\frac{-C}{K_0}}$, and hence $y(s) = \int_0^s \sqrt{1+K_0\sin^2(\sqrt{-C}\tau - B)}d\tau$ for $\frac{-\pi+2B}{2\sqrt{-C}} < s < \frac{\pi+2B}{2\sqrt{-C}}$ as $z > 0$. It follows from $\frac{dz}{dy}$ that the graph of α has an absolute maximum at $s = \frac{B}{\sqrt{-C}}$ in the interval.

Case 2.5. C=0, that is, $K_0 = \sinh^2\theta_0$. Then, the solution of (8) by using (4) gives $z(s) = (\sinh\theta_0)s + 1$ and hence $y(s) = (\cosh\theta_0)s$. The time-like surface M_1 with the generating curve $\alpha(s) = (0, (\cosh\theta_0)s, (\sinh\theta_0)s + 1)$ is umbilical that was studied in Section 2.2. For $\theta = 0$, that is, $K_0 = 0$, M_1 is a horosphere.

Therefore we have

Theorem 2.5. *Let M_1 be a time-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by (1) with the generating curve α. Then, M_1 has constant Gaussian curvature K_0 if and only if α is given by one of the followings:*

1. *For $K_0 < 0$,*

$$
y(s) = \int_0^s \sqrt{1-K_0\sinh^2(\sqrt{C}\tau + B)}d\tau, \quad z(s) = \sqrt{\frac{-K_0}{C}}\cosh(\sqrt{C}s + B)
$$

for $s \in \mathbb{R}$, *where* $C = \sinh^2\theta_0 - K_0 > 0$ *and* $B = \operatorname{arccosh}\sqrt{\frac{C}{-K_0}}$. *The graph of* α *has an absolute minimum at* $s = -B/\sqrt{C}$, *and it has oblique asymptotes with the directions* $\vec{t}_1 = (0,-1,1)$ *and* $\vec{t}_2 = (0,1,1)$.

2. *For* $K_0 = 0$,

$$
\begin{aligned}
y(s) &= \frac{1}{\sinh\theta_0}\Big(\sqrt{1+(\sinh^2\theta_0)e^{2(\sinh\theta_0)s}} - \operatorname{arccoth}\sqrt{1+(\cosh^2\theta_0)e^{2(\cosh\theta_0)s}}\Big)\\
&\quad - \frac{1}{\sinh\theta_0}(\cosh\theta_0 - \operatorname{arccoth}(\cosh\theta_0)),\\
z(s) &= e^{(\sinh\theta_0)s}
\end{aligned}
$$

for $-\infty < s < \infty$. *The graph of* α *is increasing and asymptotic to a line with the direction* $\vec{t} = (0,1,1)$ *and to the* y*-axis.*

3. *For* $0 < K_0 < \sinh^2\theta_0$,

$$
y(s) = \int_0^s \sqrt{1+K_0\cosh^2(\sqrt{C}\tau + B)}d\tau, \quad z(s) = \sqrt{\frac{K_0}{C}}\sinh(\sqrt{C}s + B)
$$

for $-\frac{B}{\sqrt{C}} < s < \infty$ *as* $z > 0$, *where* $C = \sinh^2\theta_0 - K_0$ *and* $B = \operatorname{arcsinh}\sqrt{\frac{C}{K_0}}$. *The graph of* α *is increasing and asymptotic to a line with the direction* $\vec{t} = (0,1,1)$.

4. *For* $K_0 > \sinh^2\theta_0$,

$$
y(s) = \int_0^s \sqrt{1+K_0\sin^2(\sqrt{-C}\tau - B)}d\tau, \quad z(s) = \sqrt{\frac{K_0}{-C}}\cos(\sqrt{-C}s - B)
$$

for $\frac{-\pi+2B}{2\sqrt{-C}} < s < \frac{\pi+2B}{2\sqrt{-C}}$ *as* $z > 0$, *where* $C = \sinh^2\theta_0 - K_0 < 0$ *and* $B = \arccos\sqrt{\frac{-C}{K_0}}$. *The graph of* α *has an absolute maximum at* $s = \frac{B}{\sqrt{-C}}$ *in the interval.*

5. *For* $K_0 = \sinh^2\theta_0$,

$$
y(s) = (\cosh\theta_0)s, \quad z(s) = (\sinh\theta_0)s + 1.
$$

The time-like surface M_1 *with this generating curve is umbilical, and for* $\theta = 0$, *i.e.* $K_0 = 0$, M_1 *is a horosphere.*

3. Translation in a time-like direction in $\mathcal{H}_1^3$

Now, we study time-like surfaces invariant by a group of translation isometries in a time-like direction in $\mathcal{H}_1^3$ defined by (2) with the generating curve $\gamma(s) = (x(s), 0, z(s))$. By similar calculations given in Section 2.4, we obtain the shape operator $A_{\tilde{n}}$ for the time-like surface $\widetilde{M}_1$ as

$$
A_{\tilde{n}}(\tilde{e}_1) = [x' + \varepsilon z(x'z'' - z'x'')]\tilde{e}_1 \quad \text{and} \quad A_{\tilde{n}}(\tilde{e}_2) = x'\tilde{e}_2,
$$

where the vectors $\tilde{e}_1 = z\frac{\partial}{\partial s} = x'(s)E_1 + z'(s)E_3$ and $\tilde{e}_2 = E_2$ with the signatures $\tilde{\varepsilon}_1 = 1$ and $\tilde{\varepsilon}_2 = -1$ are orthonormal tangent vectors, and $\tilde{n}$ is the unit normal vector of the immersion (2).

The Gaussian curvature K and mean curvature H of $\widetilde{M}_1$ are obtained, respectively, as follows:

$$K = zz'' - z'^2 \tag{10}$$

and

$$H = \frac{1}{2}[z(x'z'' - z'x'') + 2x']. \tag{11}$$

If the Gaussian curvature $K = K_0$ is constant, then the first integral of (10) is obtained as

$$z'^2 = Cz^2 - K_0 \tag{12}$$

from which we have

$$z'' = Cz, \tag{13}$$

where C is a constant.

We introduce the function $\tilde{\theta}(s)$ which is the angle between the vector $\gamma'(s)$ and ∂_z such that

$$x'(s) = \sin\tilde{\theta}(s), \quad z'(s) = \cos\tilde{\theta}(s),$$

and without loss of generality we may assume that

$$x(0) = 0,\ z(0) = 1,\ \tilde{\theta}(0) = \tilde{\theta}_0,\ 0 \le \tilde{\theta} < \pi. \tag{14}$$

Hence, $z'(0) = \cos\tilde{\theta}_0$.

3.1. Umbilical Surfaces $\widetilde{M}_1$ in $\mathcal{H}_1^3$

We study umbilical time-like surfaces $\widetilde{M}_1$ invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by (2) with the generating curve $\gamma(s)$.

A surface $\widetilde{M}_1$ in $\mathcal{H}_1^3$ is umbilical if

$$x'z'' - z'x'' = 0,$$

which yields $\tilde{\theta}'(s) = 0$, i.e. $\tilde{\theta} = \tilde{\theta}_0$ is a constant. Therefore, using the initial condition (14), we have $x(s) = (\sin(\tilde{\theta}_0))s,\ z(s) = (\cos(\tilde{\theta}_0))s + 1$ with $s > -1/\cos(\tilde{\theta}_0)$ and $\cos(\tilde{\theta}_0) \neq 0$. These are the parametric equations of the time-like straight line $\gamma(s) = ((\sin(\tilde{\theta}_0))s, 0, (\cos(\tilde{\theta}_0))s + 1)$. For this generating curve the Gaussian curvature of the time-like umbilical surface $\widetilde{M}_1$ is $K = -\cos^2(\tilde{\theta}_0)$ which is a nonpositive constant. Considering the initial condition (14), the time-like umbilical surfaces are the vertical totally geodesic time-like plane $x = 0$ with $K = -1$ if $\theta_0 = 0$, or the horosphere $z = 1$ with $K = 0$ if $\tilde{\theta}_0 = \pi/2$, or a time-like surface with $K = -\cos^2(\tilde{\theta}_0)$ if $\tilde{\theta}_0 \in (0, \pi/2) \cup (\pi/2, \pi)$.

So, we state

Theorem 3.1. *A surface $\widetilde{M}_1$ invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by* (2) *with a generating curve $\gamma(s) = (x(s), 0, z(s))$ is umbilical if and only if $\gamma(s) = ((\sin(\tilde{\theta}_0))s, 0, 1 + (\cos(\tilde{\theta}_0))s)$. Moreover,*

1. *$\widetilde{M}_1$ is the vertical totally geodesic time-like plane $x = 0$ with $K = -1$ if $\tilde{\theta}_0 = 0$, or*
2. *$\widetilde{M}_1$ is a time-like surface with $K = -\cos^2(\tilde{\theta}_0)$ for $\tilde{\theta}_0 \in (0, \pi/2) \cup (\pi/2, \pi)$, or*
3. *$\widetilde{M}_1$ is the horosphere $z = 1$ with $K = 0$ if $\theta_0 = \pi/2$.*

3.2. Surfaces $\widetilde{M}_1$ in $\mathcal{H}_1^3$ with constant mean curvature

Here, we determine time-like surfaces $\widetilde{M}_1$ in $\mathcal{H}_1^3$ with constant mean curvature H_0. When we solve the differential equation (11) by using ${x'}^2 + {z'}^2 = 1$ similar to the solution of (5) we obtain that

$$ {z'}^2 = 1 - (cz^2 - \varepsilon H_0)^2, $$

where c is a constant. If $c = 0$, then, by an appropriate isometry of $\mathcal{H}_1^3$ we write $\gamma(s) = (0, sH_0, 1 + s\sqrt{1 - H_0^2})$, where $|H_0| \leq 1$. Therefore, the surface $\widetilde{M}_1$ defined by this generating curve γ has constant mean curvature H_0, and it is umbilical.

For $c \neq 0$, by an appropriate isometry of $\mathcal{H}_1^3$, we can only consider

$$ z'(s) = \sqrt{1 - (cz^2 - \varepsilon H_0)^2} \quad \text{and} \quad x'(s) = cz^2 - H_0 $$

from which we can have

$$ x(z) = \int_1^z \frac{ct^2 - H_0}{\sqrt{1 - (ct^2 - H_0)^2}} dt $$

that holds the initial condition (14). Thus, we can parameterize the generating curve γ as

$$ \gamma(z) = \Big(\int_1^z \frac{ct^2 - H_0}{\sqrt{1 - (ct^2 - \varepsilon H_0)^2}} dt, 0, z \Big) $$

for $z > 0$ with the initial condition $\gamma(1) = (0, 0, 1)$. By finding the domain of γ depending on H_0 and c, we have

Theorem 3.2. *Let $\widetilde{M}_1$ be a time-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by* (2) *with the generating curve γ. Then, $\widetilde{M}_1$ has constant mean curvature H_0 if and only if γ is given by*

$$ \gamma(z) = \Big(\int_1^z \frac{ct^2 - H_0}{\sqrt{1 - (ct^2 - H_0)^2}} dt, 0, z \Big), $$

where c is a constant, and the domain of γ is given by

1. *for $H_0 < 0$,*
   (a) $0 < z < \sqrt{\frac{1+H_0}{c}}$ *when $0 < c < 1 + H_0$ and $-1 < H_0 < 0$, or*

(b) $0 < z < \sqrt{\frac{H_0-1}{c}}$ *when* $H_0 - 1 < c < 0$ *and* $-1 \leq H_0 < 0$, *or*

(c) $\sqrt{\frac{1+H_0}{c}} < z < \sqrt{\frac{H_0-1}{c}}$ *when* $H_0 - 1 < c < H_0 + 1$ *and* $H_0 < -1$;

2. *for* $H_0 = 0$, $0 < z < \frac{1}{\sqrt{|c|}}$ *when* $|c| < 1$;
3. *for* $H_0 > 0$,

(a) $0 < z < \sqrt{\frac{1+H_0}{c}}$ *when* $0 < c < H_0 + 1$ *and* $0 < H_0 \leq 1$, *or*

(b) $\sqrt{\frac{H_0-1}{c}} < z < \sqrt{\frac{H_0+1}{c}}$ *when* $H_0 - 1 < c < H_0 + 1$ *and* $H_0 > 1$, *or*

(c) $0 < z < \sqrt{\frac{H_0-1}{c}}$ *when* $H_0 - 1 < c < 0$ *and* $0 < H_0 < 1$.

All the intervals contain $z = 1$.

3.3. Surfaces $\widetilde{M}_1$ in $\mathcal{H}_1^3$ with constant Gaussian curvature

Here we determine time-like surfaces $\widetilde{M}_1$ invariant by a group of translation isometries in $\mathcal{H}_1^3$ with constant Gaussian curvature K_0. When we solve the differential equation (10) by using $x'^2 + z'^2 = 1$ similar to the solution of (7) we have the followings.

For $s = 0$ we have $C = \cos^2\theta_0 + K_0$ from (12) and (14).

Case 1.1. $C = K_0 + \cos^2\theta_0 < 0$ with $K_0 \leq -1$, $(\theta_0 \in (0, \pi))$. Then, by using (14), the solution of (13) yields

$$z(s) = \sqrt{\frac{K_0}{C}}\cos(\sqrt{-C}s - B),$$

where $B = \arccos\sqrt{\frac{C}{K_0}}$, and hence $x(s) = \int_0^s \sqrt{1 - (-K_0)\sin^2(\sqrt{-C}\tau - B)}d\tau$, for $\frac{1}{\sqrt{-C}}\Big(-\arcsin\frac{1}{\sqrt{-K_0}} + B\Big) < s < \frac{1}{\sqrt{-C}}\Big(\arcsin\frac{1}{\sqrt{-K_0}} + B\Big)$. It follows from $\frac{dz}{dx}$ that γ has an absolute maximum at $s = \frac{B}{\sqrt{C}}$. For $K_0 = -1$ and $\theta_0 \neq 0$, γ is an open part of the upper semicircle in the xz-plane centered at $(-\frac{\cos\theta_0}{|\sin\theta_0|}, 0)$ with radius $\frac{1}{|\sin\theta_0|}$.

Case 1.2. C=0, that is, $K_0 = -\cos^2\theta_0$. Then, the solution of (12) with (14) gives $z(s) = (\cos\theta_0)s + 1$ and hence $x(s) = (\sin\theta_0)s$. The time-like surface $\widetilde{M}_1$ with the generating curve $\gamma(s) = ((\sin\theta_0)s, 0, (\cos\theta_0)s + 1)$ with $s > -\sec\theta_0$ is umbilical that was studied in Section 3.1.

Case 1.3. $C = K_0 + \cos^2\theta_0 > 0$ with $-\cos^2\theta_0 < K_0 < 0$. Then, by using (14), the solution of (13) yields

$$z(s) = \sqrt{\frac{-K_0}{C}}\sinh(\sqrt{C}s + B),$$

where $B = \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}$, and hence $x(s) = \int_0^s \sqrt{1-(-K_0)\cosh^2(\sqrt{C}\tau + B)}d\tau$ for $-\frac{B}{\sqrt{C}} < s < \frac{1}{\sqrt{C}}\Big(\operatorname{arccosh}\frac{1}{\sqrt{-K_0}} - B\Big) = s_1$. It can be seen that γ is increasing on the interval, and it has a vertical tangent when s approaches to s_1 from the left.

Case 1.4. $C = \cos^2\theta_0 > 0$ and $K_0 = 0$. Then, by using (14), the solution of (12) yields

$$z(s) = e^{(\cos\theta_0)s},$$

and hence

$$\begin{aligned} x(s) =& \int_0^s \sqrt{1-(\cos^2\theta_0)e^{2(\cos\theta_0)\tau}}d\tau \\ =& \frac{1}{\cos\theta_0}\Big(\sqrt{1-(\cos^2\theta_0)e^{2(\cos\theta_0)s}} - \operatorname{arctanh}\sqrt{1-(\cos^2\theta_0)e^{2(\cos\theta_0)s}}\Big) \\ &- \frac{1}{\cos\theta_0}(\sin\theta_0 - \operatorname{arctanh}(\sin\theta_0)) \end{aligned}$$

for $-\infty < s < -\frac{\ln(\cos\theta_0)}{\cos\theta_0} = s_2$. From which it can be seen that γ is increasing and $\lim_{s\to-\infty}\frac{dz}{dx} = 0$ which implies that the graph of α is asymptotic to the x-axis, and also it has a vertical tangent when s approaches to s_2 from the left.

Case 1.5. $C = K_0 + \cos^2\theta_0 > 0$ with $K_0 > 0$. Then, by using (14), the solution of (13) yields

$$z(s) = \sqrt{\frac{K_0}{C}}\cosh(\sqrt{C}s + B),$$

where $B = \operatorname{arccosh}\sqrt{\frac{C}{K_0}}$, and hence $x(s) = \int_0^s \sqrt{1-K_0\sinh^2(\sqrt{C}\tau + B)}d\tau$ for $-\frac{1}{\sqrt{C}}\Big(\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} + B\Big) \le s \le \frac{1}{\sqrt{C}}\Big(\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} - B\Big)$. The curve γ is has a horizontal tangent at $s = \frac{-B}{\sqrt{C}}$, and it has vertical tangents at the end points. From $\frac{dz}{dx}$, it seen that γ has an absolute minimum at $s = \frac{-B}{\sqrt{C}}$.

As a summary we state

Theorem 3.3. *Let $\widetilde{M}_1$ be a time-like surface invariant by a group of translation isometries in $\mathcal{H}_1^3$ defined by (2) with the generating curve γ. Then, $\widetilde{M}_1$ has constant Gaussian curvature K_0 if and only if the x and z components of γ are given by one of the following:*

1. *For $K_0 \le -1$,*

$$x(s) = \int_0^s \sqrt{1-(-K_0)\sin^2(\sqrt{-C}\tau - B)}d\tau,\ z(s) = \sqrt{\frac{K_0}{C}}\cos(\sqrt{-C}s - B),$$

for $\frac{1}{\sqrt{-C}}\Big(-\arcsin\frac{1}{\sqrt{-K_0}} + B\Big) < s < \frac{1}{\sqrt{-C}}\Big(\arcsin\frac{1}{\sqrt{-K_0}} + B\Big)$, where $C = K_0 + \cos^2\theta_0 < 0$ and $B = \arccos\sqrt{\frac{C}{K_0}}$. The curve γ has an absolute maximum at $s = \frac{B}{\sqrt{C}}$. For $K_0 = -1$ with $\theta_0 \ne 0$, γ is an open part of *the upper semicircle in the xz-plane centered at* $(-\frac{\cos\theta_0}{|\sin\theta_0|}, 0)$ *with radius* $\frac{1}{|\sin\theta_0|}$.

2. *For* $K_0 = -\cos^2\theta_0$, $\gamma(s) = ((\sin\theta_0)s, 0, (\cos\theta_0)s + 1)$ *with* $s > -\sec\theta_0$. *For this generating curve* $\widetilde{M}_1$ *is umbilical.*
3. *For* $-\cos^2\theta_0 < K_0 < 0$,

$$x(s) = \int_0^s \sqrt{1 - (-K_0)\cosh^2(\sqrt{C}\tau + B)}d\tau,\ z(s) = \sqrt{\frac{-K_0}{C}}\sinh(\sqrt{C}s + B),$$

for $-\frac{B}{\sqrt{C}} < s < \frac{1}{\sqrt{C}}\Big(\operatorname{arccosh}\frac{1}{\sqrt{-K_0}} - B\Big) = s_1$, *where* $C = K_0 + \cos^2\theta_0$ *and* $B = \operatorname{arcsinh}\sqrt{\frac{C}{-K_0}}$. *The curve* γ *is increasing, and it has a vertical tangent when* s *approaches to* s_1 *from the left.*

4. *For* $K_0 = 0$,

$$x(s) = \frac{1}{\cos\theta_0}\Big(\sqrt{1 - (\cos^2\theta_0)e^{2(\cos\theta_0)s}} - \operatorname{arctanh}\sqrt{1 - (\cos^2\theta_0)e^{2(\cos\theta_0)s}}\Big)$$
$$- \frac{1}{\cos\theta_0}(\sin\theta_0 - \operatorname{arctanh}(\sin\theta_0)),$$
$$z(s) = e^{(\cos\theta_0)s}$$

for $-\infty < s < -\frac{\ln(\cos\theta_0)}{\cos\theta_0} = s_2$. *It can be seen that* γ *is asymptotic to the* x*-axis, increasing and has a vertical tangent when* s *approaches to* s_2 *from the left.*

5. *For* $K_0 > 0$,

$$x(s) = \int_0^s \sqrt{1 - K_0\sinh^2(\sqrt{C}\tau + B)}d\tau, \quad z(s) = \sqrt{\frac{K_0}{C}}\cosh(\sqrt{C}s + B)$$

for $-\frac{1}{\sqrt{C}}\Big(\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} + B\Big) \le s \le \frac{1}{\sqrt{C}}\Big(\operatorname{arcsinh}\frac{1}{\sqrt{K_0}} - B\Big)$, *where* $C = K_0 + \cos^2\theta_0$ *and* $B = \operatorname{arccosh}\sqrt{\frac{C}{K_0}}$. *The curve* γ *has an absolute minimum at* $s = \frac{-B}{\sqrt{C}}$, *and it has vertical tangents at the end points.*

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Uğur Dursun
Department of Mathematics, Işık University,
Meşrutiyet Mah., Şile, 34980, Istanbul, Türkiye
E-mail: ugur.dursun@isikun.edu.tr