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ON C-BICONSERVATIVE HYPERSURFACES OF NON-FLAT RIEMANNIAN 4-SPACE FORMS

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Abstract. In this manuscript, the hypersurfaces of non-flat Riemannian 4-space forms are considered. A hypersurface of a 4-dimensional Riemannian space form defined by an isometric immersion $\mathbf{x} : M^3 \to \mathbb{M}^4(c)$ is said to be biconservative if it satisfies the equation $(\Delta^2\mathbf{x})^\top = 0$, where Δ is the Laplace operator on M^3 and $\top$ stands for the tangent component of vectors. We study an extended version of biconservativity condition on the hypersurfaces of the Riemannian standard 4-space forms. The C-biconservativity condition is obtained by substituting the Cheng-Yau operator C instead of Δ. We prove that C-biconservative hypersurfaces of Riemannian 4-space forms (with some additional conditions) have constant scalar curvature.

1. Introduction

The subject of biconservative submanifolds is an interesting research topic in mathematical physics, which has been started by Eells and Sampson and followed by Jiang ([5, 10]). From the physical points of view, we deal with the bienergy functional and its critical points arisen form the tension field. In geometric context, the subject of biconservative submanifolds has received much attentions. In 1995, Hasanis and Vlachos have classified the biconservative hypersurfaces (namely, H-hypersurfaces) of 3 and 4 dimensional Euclidean spaces ([9]). The notion of biconservative submanifold in an arbitrary manifold (not only in Euclidean spaces) has been introduced for the first time in [3]. The full classification of biconservative surfaces in 3-dimensional space forms was done in [12].

In 2015, Turgay has studied the H-hypersurfaces with 3 distinct principal curvatures in the Euclidean spaces ([16]). Also, the constant mean curvature biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ has been studied in [6]. In 2019, Gupta studied the biconservative hypersurfaces in Euclidean 5-space ([8]). Also, the biconservative hypersurfaces in Riemannian 4-space forms have been

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classified by Turgay and Upadhyay ([17]). In this paper, we study the C-biconservativity condition on the hypersurfaces of Riemannian 4-space forms. A hypersurface M^3 of $\mathbb{M}^4(c)$ is said to be C-biconservative if it satisfies the condition

$$\mathrm{N}_2(\nabla H_2) - c\mathrm{N}_1(\nabla H_1) = \frac{9}{2}H_2\nabla H_2. \tag{1}$$

Here, N_1 and N_2 are the first and second Newton transformations (respectively),and H_1 and H_2 are the ordinary and second mean curvatures on M^3 defined by $H_1 = \frac{1}{3}(\kappa_1+\kappa_2+\kappa_3)$ and $H_2 = \frac{1}{3}(\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3)$ (respectively), where κ_1, κ_2 and κ_3 are the principal curvatures of M^3.

We show that the C-biconservative hypersurfaces of $\mathbb{M}^4(c)$ with constant ordinary mean curvature have constant scalar curvature.

2. Preliminaries

We recall some notations and formulae from [1, 2, 11, 13, 19]. The 4-dimensional Riemannian standard space form $\mathbb{M}^4(c)$ of curvature c is

$$\mathbb{M}^4(c) = \begin{cases} \mathbb{S}^4 = \mathbb{S}^4(1) \subset \mathbb{E}^5 & (\text{if } c = 1) \\ \mathbb{E}^4 & (\text{if } c = 0) \\ \mathbb{H}^4 = \mathbb{H}^4(-1) \subset \mathbb{L}^5 & (\text{if } c = -1). \end{cases}$$

As usual, $\mathbb{E}^k$ is the Euclidean k-space (for each natural number k) with dot product $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^k v_i w_i$. The Euclidean k-space equipped with the Lorentz product defined by $\langle \mathbf{v}, \mathbf{w} \rangle = -v_1 w_1 + \sum_{i=2}^k v_i w_i$ (for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$) gives the Lorentz-Minkowski k-space $\mathbb{L}^k$. For $r > 0$,

$$\mathbb{S}^k(r) = \{\mathbf{v} \in \mathbb{E}^{k+1} | \langle \mathbf{v}, \mathbf{v} \rangle = r^2\}$$

denotes the Euclidean k-sphere of radius r and curvature $1/r^2$, and

$$\mathbb{H}^k(-r) = \{\mathbf{v} \in \mathbb{L}^{k+1} | \langle \mathbf{v}, \mathbf{v} \rangle = -r^2, v_1 > 0\}$$

denotes the hyperbolic k-space of radius $-r$ and curvature $-1/r^2$.

We consider a hypersurface M^3 in $\mathbb{M}^4(c)$ as a 3-dimensional submanifold isometrically immersed by a map $\mathbf{x} : M^3 \to \mathbb{M}^4(c)$. The notation $\chi(M^3)$ stands for the set of smooth tangent vector fields on M^3. The symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^3 and $\mathbb{M}^4(c)$, respectively. Also, ∇^0 denotes the Levi-Civita connection on $\mathbb{E}^5$ or $\mathbb{L}^5$. The Weingarten formula on M^3 is

$$\bar{\nabla}_V W = \nabla_V W + \langle SV, W \rangle \mathbf{n},$$

for each $V, W \in \chi(M^3)$, where S is the shape operator of M^3 associated to a unit normal vector field $\mathbf{n}$ on M^3. Furthermore, in the case $|c| = 1$, $\mathbb{M}^4(c)$ is a 4-hyperquadric in $\mathbb{E}^5$ or $\mathbb{L}^5$, with the unit normal vector field $\mathbf{x}$ and the Gauss formula $\nabla^0_V W = \bar{\nabla}_V W - c\langle V, W \rangle \mathbf{x}$.

Denoting the eigenvalues of S (i.e. the principal curvatures of M) by $\kappa_1, \kappa_2, \kappa_3$ on M, we define the jth elementary symmetric function as

$$s_j := \sum_{1\leq i_1<...<i_j\leq n} \kappa_{i_1}...\kappa_{i_j},$$

and the jth mean curvature of M as $\binom{3}{j}H_j = s_j$ (for instance, see [1] and [2]). In special case $j = 1$, H_1 is the ordinary mean curvature H. The second mean curvature H_2 and the normalized scalar curvature R satisfy the equality $H_2 := n(n-1)(1-R)$.

The hypersurface M^3 in $\mathbb{M}^4(c)$ is called *j-minimal* if its $(j+1)$th mean curvature H_{j+1} is identically zero.

Also, we apply the Newton map on M^3 by expression

$$\mathrm{N}_0 = I,\ \mathrm{N}_1 = -s_1 I + S,\ \mathrm{N}_2 = s_2 I - s_1 S + S^2, \tag{2}$$

where I is the identity map. If e_1, e_2 and e_3 are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p), \kappa_2(p)$ and $\kappa_3(p)$, respectively, then they are also the eigenvectors of $\mathrm{N}_j(p)$ with corresponding eigenvalues given by $\mu_{1,1} = -\kappa_2 - \kappa_3$, $\mu_{2,1} = -\kappa_1 - \kappa_3$, $\mu_{3,1} = -\kappa_1 - \kappa_2$, $\mu_{1,2} = \kappa_2\kappa_3$, $\mu_{2,2} = \kappa_1\kappa_3$, $\mu_{3,2} = \kappa_1\kappa_2$.

We have the following formulae for the Newton transformations:

$$\begin{aligned} &tr(\mathrm{N}_j) = c_j H_j,\ tr(S\circ \mathrm{N}_j) = c_j H_{j+1}, \\ &tr(S^2\circ \mathrm{N}_1) = 9H_1H_2 - 3H_3,\ tr(S^2\circ \mathrm{N}_2) = 3H_1H_3, \end{aligned} \tag{3}$$

where $j = 0, 1, 2$, $c_0 = c_2 = 3$ and $c_1 = 6$.

Now, we consider the second-order linear differential operator $\mathrm{C} : \mathcal{C}^\infty(M^3) \to \mathcal{C}^\infty(M^3)$ given by $\mathrm{C}(f) = tr(\mathrm{N}_1 \circ \nabla^2 f)$, where, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f which is given for every vector fields $X, Y \in \chi(M^3)$, by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle.$$

In other words, $\mathrm{C}(f)$ is given by $\mathrm{C}(f) = \sum_{i=1}^3 \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. So, we get

$$\mathrm{C}\mathbf{n} = -3grad(H_2) - (9H_1H_2 - 3H_3)\mathbf{n} + 6cH_2\mathbf{x},$$

and

$$\begin{aligned} \mathrm{C}^2\mathbf{x} = &-54H_2\nabla H_2 + 12\mathrm{N}_2\nabla H_2 - 12c\mathrm{N}_1\nabla H_1 \\ &+ 6\left(\mathrm{C}(H_2) - 9H_1H_2^2 + 3H_2H_3 - 6cH_1H_2\right)\mathbf{n} \\ &- 6c\left(\mathrm{C}(H_1) - 6H_2^2 - 6cH_1^2\right)\mathbf{x} \end{aligned} \tag{4}$$

By definition, M^3 is called *C-biconservative* if $\mathbf{x}$ satisfies $(\mathrm{C}^2\mathbf{x})^\top = 0$ (i.e the condition (1)).

According to (local) orthonormal tangent frame $\{e_m\}_{1\leq m\leq 4}$ in $\mathbb{R}^4$, and associated co-frame $\{\omega_m\}_{1\leq m\leq 4}$, where e_1, e_2, e_3 are tangent to M and e_4 is positively normal to M. The structure equations of $\mathbb{R}^4$ are

$$d\omega_A = \sum_{B=1}^{4} \omega_{AB} \wedge \omega_B,\ \omega_{AB} + \omega_{BA} = 0,\ d\omega_{AB} = \sum_{C=1}^{4} \omega_{AC} \wedge \omega_{CB}.$$

Of course, we have $\omega_4 = 0$ and $0 = d\omega_4 = \sum_{i=1}^{3} \omega_{4i} \wedge \omega_i$ on M.

Using the well-known Cartan Lemma, we have functions h_{ij} such that $h_{ij} = h_{ji}$ and

$$\omega_{4i} = \sum_{j=1}^{3} h_{ij}\omega_j. \tag{5}$$

Since the second fundamental form of M is $B = \sum_{i,j=1}^{4} h_{ij}\omega_i\omega_j e_4$, the mean curvature H has the simple form $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. Hence, from (5) we can get the structure equations as (see [19])

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j,\ \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l.$$

Also, we have the Gauss equation $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} stand for the components of the tensor of Riemannian curvature on M. Finally, we have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{6}$$

where h_{ijk} is the covariant derivative of h_{ij}. Thus, by exterior differentiation of (5), we obtain the Codazzi equation $h_{ijk} = h_{ikj}$. One can choose e_1, e_2, e_3 such that $h_{ij} = \kappa_i\delta_{ij}$. On the other hand, the Levi-Civita connection of M^3 satisfies

$$\nabla_{e_i} e_j = \sum_k \omega_{jk}(e_i)e_k,$$

and we have

$$\begin{aligned} &e_i(k_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j),\\ &\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l), \end{aligned} \tag{7}$$

whenever i, j, l are distinct.

3. Examples

In this section we see several examples of C-biconservative hypersurfaces in $\mathbb{S}^4$ and $\mathbb{H}^4$ with constant first and second mean curvatures. First, we have some Riemannian product hypersurfaces (see [2, 14]).

Example 3.1. Let $0 < r < 1$ and $\Lambda_0 = \mathbb{S}^3(r) \subset \mathbb{S}^4$ defined as

$$\Lambda_0 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{E}^5 | y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2, y_5 = \sqrt{1 - r^2}\},$$

with the Gauss map $\mathbf{n}(y) = \frac{-\sqrt{1-r^2}}{r}(y_1, y_2, y_3, y_4, 0) + \frac{r}{\sqrt{1-r^2}}(0, 0, 0, 0, y_5)$ and only one principal curvature of multiplicity 3 as $\kappa_1 = \kappa_2 = \kappa_3 = \frac{\sqrt{1-r^2}}{r}$,. One can see that Λ_0 is C-biconservative and its 1st and 2nd mean curvatures are constant.

Example 3.2. Let $0 < r < 1$ and $\Lambda_1 = \mathbb{S}^2(r) \times \mathbb{S}^1(\sqrt{1 - r^2}) \subset \mathbb{S}^4$ defined as

$$\Lambda_1 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{E}^5 | y_1^2 + y_2^2 + y_3^2 = r^2, y_4^2 + y_5^2 = 1 - r^2\},$$

whose Gauss map is $\mathbf{n}(y) = \frac{-\sqrt{1-r^2}}{r}(y_1, y_2, y_3, 0, 0) + \frac{r}{\sqrt{1-r^2}}(0, 0, 0, y_4, y_5)$. It has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{\sqrt{1-r^2}}{r}$, $\kappa_3 = \frac{-r}{\sqrt{1-r^2}}$. One can see that Λ_1 is C-biconservative and its 1st and 2nd mean curvatures are constant.

Example 3.3. Let $r > 0$ and $\Lambda_2 = \mathbb{H}^2(-\sqrt{r^2 + 1}) \times \mathbb{S}^1(r) \subset \mathbb{H}^4$ defined as

$$\Lambda_2 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | - y_1^2 + y_2^2 + y_3^2 = -1 - r^2, y_4^2 + y_5^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = \frac{r}{\sqrt{1+r^2}}(y_1, y_2, y_3, 0, 0) + \frac{\sqrt{1+r^2}}{r}(0, 0, 0, y_4, y_5)$ and two distinct constant principal curvatures $\kappa_1 = \kappa_2 = -r\sqrt{1 + r^2}$ and $\kappa_3 = \frac{-\sqrt{1+r^2}}{r}$ and the constant higher order mean curvatures. So, Λ_2 is C-biconservative.

Example 3.4. Let $r > 0$ and $\Lambda_3 = \mathbb{H}^1(-\sqrt{r^2 + 1}) \times \mathbb{S}^2(r) \subset \mathbb{H}^4$ defined by

$$\Lambda_3 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | - y_1^2 + y_2^2 = -1 - r^2, y_3^2 + y_4^2 + y_5^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = \frac{r}{\sqrt{1+r^2}}(y_1, y_2, 0, 0, 0) + \frac{\sqrt{1+r^2}}{r}(0, 0, y_3, y_4, y_5)$. it has two distinct constant principal curvatures $\kappa_1 = -r\sqrt{1 + r^2}$ and $\kappa_2 = \kappa_3 = \frac{-\sqrt{1+r^2}}{r}$ and constant higher order mean curvatures. So, Λ_3 is C-biconservative.

Example 3.5. Let $r > 0$ and $\Lambda_4 = \mathbb{S}^3(r) \subset \mathbb{H}^4$ defined by

$$\Lambda_4 = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{L}^5 | y_2^2 + y_3^2 + y_4^2 + y_5^2 = r^2, y_1 = \sqrt{1 + r^2}\},$$

with the Gauss map $\mathbf{n}(y) = \frac{r}{\sqrt{1+r^2}}(y_1, 0, 0, 0, 0) + \frac{\sqrt{1+r^2}}{r}(0, y_2, y_3, y_4, y_5)$, only one constant principal curvature of multiplicity three as $\kappa_1 = \kappa_2 = \kappa_3 = \frac{-\sqrt{1+r^2}}{r}$, and the constant higher order mean curvatures . So, Λ_4 is C-biconservative.

Example 3.6. Let Λ_5 be $\mathbb{H}^3(-\sqrt{r^2+1}) \subset \mathbb{H}^4$ where $r \geq 0$. It can be represented by

$$\Lambda_5 = \{(y_1, ..., y_5) \in \mathbb{L}^5 | -y_1^2 + y_2^2 + y_3^2 + y_4^2 = -1 - r^2, y_5 = r\}$$

with the Gauss map $\mathbf{n}(y) = \frac{r}{\sqrt{1+r^2}}(y_1, y_2, y_3, y_4, 0) + \frac{\sqrt{1+r^2}}{r}(0, 0, 0, 0, y_5)$, only one principal curvature $\kappa_1 = \kappa_2 = \kappa_3 = -r\sqrt{1+r^2}$ and constant higher order mean curvatures. Hence, Λ_5 is C-biconservative.

Example 3.7. The totally umbilical hypersurfaces of $\mathbb{S}^4$ are the round 3-spheres of radius $0 < \rho \leq 1$ obtained by intersecting $\mathbb{S}^4$ with affine hyperplanes. Let $\mathbf{v} \in \mathbb{E}^5$ be a unit constant vector. The subset

$$\Gamma_\sigma := \{\mathbf{p} \in \mathbb{S}^4 : \langle \mathbf{p}, \mathbf{v} \rangle = \sigma\} = \mathbb{S}^3(\sqrt{1-\sigma^2}).$$

for each $\sigma \in (-1, 1)$, is a totally umbilical hypersurface in $\mathbb{S}^4$ with Gauss map $\mathbf{n}(\mathbf{p}) = \frac{1}{\sqrt{1-\sigma^2}}(\mathbf{v} - \sigma\mathbf{p})$ and shape operator $S = \frac{\sigma}{\sqrt{1-\sigma^2}}I$. In particular, its 1st and 2nd mean curvatures are constant given by $H_1 = \frac{\sigma}{\sqrt{1-\sigma^2}}$, $H_2 = \frac{\sigma}{1-\sigma^2}$. So, Γ_σ is C-biconservative.

Example 3.8. The totally umbilical hypersurfaces of $\mathbb{H}^4$ are also obtained by intersecting $\mathbb{H}^4$ with affine hyperplanes of $\mathbb{L}^5$, but in this case there are three different types of hypersurfaces, depending on the causal character of the hyperplane. Let $\mathbf{w} \in \mathbb{L}^5$ be a nonzero constant vector such that $\langle \mathbf{w}, \mathbf{w} \rangle \in \{0, \pm 1\}$. The subset

$$\Theta_\nu := \{\mathbf{q} \in \mathbb{H}^4 : \langle \mathbf{q}, \mathbf{w} \rangle = \nu\}$$

is a totally umbilical hypersurface of $\mathbb{H}^4$ if $\nu^2 + \langle \mathbf{w}, \mathbf{w} \rangle > 0$. Its Gauss map is $\mathbf{n}(\mathbf{p}) = \frac{1}{\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle + \nu^2}}(\mathbf{v} + \nu\mathbf{q})$ and shape operator $S = -\frac{\nu}{\sqrt{\nu^2 + \langle \mathbf{w}, \mathbf{w} \rangle >}}I$. In fact

$$\Theta_\nu = \begin{cases} \mathbb{S}^3(\sqrt{\nu^2 - 1}) \subset \mathbb{E}^5 & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = -1, |\nu| > 1) \\ \mathbb{E}^3 & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 0, \nu \neq 0) \\ \mathbb{H}^3(-\sqrt{1+\nu^2}) & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 1). \end{cases}$$

Its 1st and 2nd mean curvatures are constant given by $H_1 = \frac{-\nu}{\sqrt{\nu^2 + \langle \mathbf{w}, \mathbf{w} \rangle}}$ and $H_2 = \frac{\nu^2}{\nu^2 + \langle \mathbf{w}, \mathbf{w} \rangle}$. So, Θ_ν is C-biconservative.

4. Main results

In this section, we study C-biconservative hypersurfaces in $\mathbb{M}^4(c)$ for $c = \pm 1$. A similar study has been made for the ordinary biconservative one in some papers [7, 16, 18]. Let $\mathbf{x} : M^3 \to \mathbb{M}^4(c)$ be a biconservative hypersurface with 2 distinct principal curvatures. By Theorem 4.2 in [4], M^3 is an open part of a rotational hypersurface in $\mathbb{M}^4(c)$ for an appropriately chosen profile curve. In C-biconservative case, we show that such a hypersurfaces in $\mathbb{M}^4(c)$ with 2 distinct principal curvatures and constant ordinary mean curvature has to be of constant second mean curvature. First, we see the next lemma which can be proved by the same manner of similar one in [15].

Lemma 4.1. Let M^3 be a hypersurface in $\mathbb{M}^4(c)$ with principal curvatures of constant multiplicities. Then the distribution generated by principal directions is completely integrable. Also, each principal curvature of multiplicity greater than 1 is constant on each integral submanifold of its distribution.

Theorem 4.2. Let $x : M^3 \to \mathbb{M}^4(c)$ be a C-biconservative hypersurface with constant ordinary mean curvature and at most two distinct principal curvatures. Then, its scalar curvature is constant and M^3 is isoparametric.

Proof. By assumption, M^3 has two distinct principal curvatures λ and μ of multiplicities 2 and 1, respectively. Defining the open subset U of M^3 as $\mathrm{U} := \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$, we prove that U is empty. Assuming $\mathrm{U} \neq \varnothing$, we consider $\{e_1, e_2, e_3\}$ as a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$. By assumption, we have

$$\lambda_1 = \lambda_2 = \lambda,\ \lambda_3 = \eta.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\eta,\ \mu_{3,2} = \lambda^2,\ 3H = 2\lambda + \eta,\ 3H_2 = \lambda^2 + 2\lambda\eta. \tag{8}$$

By condition (1), we have

$$\mathrm{N}_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{9}$$

By polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \langle \nabla H_2, e_i \rangle e_i$, from (9) we get

$$\langle \nabla H_2, e_i \rangle (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

on $\mathcal{U}$, for $i = 1, 2, 3$. Hence, for every i such that $\langle \nabla H_2, e_i \rangle \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{10}$$

By assumption, we have $\nabla H_2 \neq 0$ on U, which gives two possible cases.

Case 1. $\langle \nabla H_2, e_i \rangle \neq 0$, for $i = 1$ or $i = 2$. By equalities (8) and (10), we obtain

$$\lambda\eta = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(6H - \frac{5}{2}\lambda) = 0. \tag{11}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{12}{5}H$, $\eta = -\frac{9}{5}H$ and $H_2 = -\frac{72}{25}H^2$.

Case 2. $\langle \nabla H_2, e_3 \rangle \neq 0$. By equalities (8) and (10), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(9H - \frac{11}{2}\lambda) = 0. \tag{12}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = \frac{18}{11}H$, $\eta = -\frac{3}{11}H_1$ and $H_2 = \frac{216}{121}H^2$.

Therefore, H_2 and the scalar curvature of M^3 are constant. Finally, we get that M^3 is isoparametric. □

Finally, we pay attention to C-biconservative hypersurfaces with 3 distinct principal curvatures. We show that such a hypersurface with constant mean curvature has constant scalar curvature.

Theorem 4.3. Let $\mathbf{x} : M^3 \to \mathbb{M}^4(c)$ be C-biconservative connected hypersurface with constant ordinary mean curvature and three distinct principal curvatures. Then, the scalar curvature of M^3 is constant.

Proof. Assuming H_2 to be non-constant, we take $\mathcal{U} = \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$. According to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M^3, the shape operator S has a diagonal matrix form, such that $Se_i = \lambda_i e_i$ and then, $\mathrm{N}_2 e_i = \mu_{i,2} e_i$ for $i = 1, 2, 3$. By equality (1) and decomposition $\nabla H_2 = \sum_{i=1}^{3} e_i(H_2)e_i$, for $i = 1, 2, 3$ we obtain

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2}H_2) = 0. \tag{13}$$

Around every point $\mathbf{p} \in \mathcal{U}$ there exists a neighborhood such that $e_i(H_2) \neq 0$ on which for at least one i. So, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2}H_2$, (locally) on U, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$. We affirm three claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$.
If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (13) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $\lambda_2(\lambda_1 - \lambda_3) = 0$ or $\lambda_3(\lambda_2 - \lambda_1) = 0$. By assumption, λ_i's are mutually distinct, so we get $\lambda_3 = 0$ or $\lambda_2 = 0$, then $H_2 = 0$ on $\mathcal{U}$. This contradicts with the definition of $\mathcal{U}$.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
By assumption H is constant on M. So, $e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2)$. Also, by recent results, $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$, we get

$$e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2(3H_2 - \frac{9}{2}H_2) = 0,$$

which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)e_2\lambda_1 = 0$, and then we have

$$\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = \lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 = (-\lambda_1 + \lambda_2 + \lambda_3)e_2\lambda_1 = 0.$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2 + \lambda_3$ which gives contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$. So, Claim 2 is affirmed.

Claim 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$.

Using the notations

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \ (i, j = 1, 2, 3), \tag{14}$$

and the compatibility condition $e_k < e_i, e_j >= 0$, we have

$$\omega_{ki}^i = 0, \ \omega_{ki}^j + \omega_{kj}^i = 0, \ (i, j, k = 1, 2, 3) \tag{15}$$

and applying the Codazzi equation (see [13], page 115, Corollary 34(2))

$$(\nabla_V S)W = (\nabla_W S)V, \ (\forall V, W \in \chi(M)) \tag{16}$$

on the basis $\{e_1, e_2, e_3\}$, we get for distinct $i, j, k = 1, 2, 3$,

$$(a) \ e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \ (b) \ (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{17}$$

Also, by a straightforward computation of components of the identity $(\nabla_{e_i} S)e_j - (\nabla_{e_j} S)e_i \equiv 0$ for distinct $i, j = 1, 2, 3$, we get $[e_2, \ e_3](H_2) = 0$, $\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0$ and

$$\begin{aligned} \omega_{21}^2 &= \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \ \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}, \\ \omega_{23}^2 &= \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \ \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \end{aligned} \tag{18}$$

Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to $\nabla_{e_1} e_k = 0$ for $k = 1, 2, 3$, and

(19)

$$\begin{aligned} \nabla_{e_2} e_1 &= \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2, \ \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} e_3, \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_1, \\ \nabla_{e_3} e_2 &= \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} e_3, \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} e_2, \ \nabla_{e_3} e_3 = \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3 - \lambda_2} e_2. \end{aligned}$$

Now, the Gauss equation for $< R(e_2, e_3)e_1, \ e_2 >$ and $< R(e_2, e_3)e_1, \ e_3 >$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right), \tag{20}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right). \tag{21}$$

We also have the Gauss equation for $< R(e_1, e_2)e_1,\ e_2 >$ and $< R(e_3, e_1)e_1,\ e_3 >$, which give the following relations
(22)
$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)+\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)^2=\lambda_1\lambda_2,\ e_1\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)+\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)^2=\lambda_1\lambda_3.$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right)=\frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}. \tag{23}$$

On the other hand, by considering the condition (1), from Claim 1 we get
(24)
$$-\mu_{1,1}e_1e_1(H_2)+\left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2)-9H_2^2(H_1-\frac{3}{2}\lambda_1)=0.$$

By differentiating (24) along on e_2 respectively e_3, and using (20), (21) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{25}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}-\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{26}$$

Using (19), we find that

$$[e_1,\ e_2]=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{27}$$

Applying both sides of the equality (27) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (25), (22), and (23), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{28}$$

The equality (28) gives $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{29}$$

From equation (29), by differentiating on its both sides along e_1 and applying (22), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (19), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{30}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3) = e_3(\lambda_2) = 0$ which, by applying the Gauss equation for $< R(e_2, e_3)e_1, e_3 >$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} = \lambda_2\lambda_3. \tag{31}$$

Finally, using (22), differentiating (31) along e_1 gives

$$\lambda_2\lambda_3 \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} + \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right) = 0, \tag{32}$$

which implies $\lambda_2\lambda_3 = 0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} + \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} \right) \neq 0$). Therefore, we obtain $H_2 = 0$ on U, which is a contradiction. Hence H_2 is constant on M^3. □

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