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INVARIANT (α, β)-METRIC OF DOUGLAS AND BERWALD TYPE

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Abstract. In this paper, we find the conditions for a homogeneous Finsler space with an invariant infinite series (α, β)-metric to be of Berwald type. Also, we derive the necessary and sufficient condition for such a metric to be a Douglas metric.

1. Introduction

According to Chern ([6]), Finsler geometry is just the Riemannian geometry without the quadratic restriction. A connected smooth manifold M is called a Finsler space if there exists a function $F\colon TM \longrightarrow [0, \infty)$ such that F is smooth on the slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any $T_p(M),\ p \in M$, is a Minkowski norm. In this case, F is called a Finsler metric. The notion of (α, β)-metric in Finsler geometry was introduced by Matsumoto in 1972 ([17]). An (α, β)-metric is a Finsler metric of the form

$$F = \alpha\phi(s),\ \ s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. It is well known fact that (α, β)-metrics are the generalizations of the Randers metric introduced by Randers in ([20]). (α, β)-metrics have various applications in physics and biology ([1]).

Consider the r^{th} series (α, β)-metric:

$$F(\alpha, \beta) = \beta \sum_{r=0}^{r=\infty} \left(\frac{\alpha}{\beta}\right)^r.$$

If $r = 1$, then it is a Randers metric. If $r = \infty$, then

$$F = \frac{\beta^2}{\beta - \alpha}.$$

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This metric is called an infinite series (α,β)-metric. An interesting fact about this metric is that, it is the difference of a Randers metric and a Matsumoto metric.

Using the properties of geodesics, one can characterize Berwald metrics and Douglas metrics. According to Szabo ([24]), Berwald metrics are "almost Riemannian", i.e., a Finsler metric F on a smooth n-manifold M is a Berwald metric if and only if there exists a Riemannian metric g on M such that (M,g) and (M,F) have same geodesics as parametrized curves. F is a Douglas metric if and only if there exists a Riemannian metric g on M such that (M,g) and (M,F) have same geodesics as point sets ([4], [7]).

In ([10]), Douglas introduced the notion of a new curvature, which later on named as Douglas curvature. A Finsler metric with vanishing Douglas curvature is a Douglas metric. A Randers metric $F=\alpha+\beta$ is a Douglas metric if and only if β is closed and Randers metric have same geodesics as that of Riemannian metric α ([3]).

The authors in ([15]) give a characterization of (α,β)-metrics of Douglas type for $n\geq 3$. Yang ([25]) give a characterization of (α,β)-metrics of Douglas type for $n=2$. Liu and Deng ([16]) study the homogeneous (α,β)-metrics of Douglas type. They prove that a homogeneous (α,β)-metric is a Douglas metric if and only if either F is a Berwald metric or a Douglas metric of Randers type. Also, in ([16]), the authors discuss homogeneous (α,β)-metrics of Berwald type. Some authors ([8], [11, 12, 13], [16], [21, 22, 23] etc.) have studied various properties of homogeneous (α,β)-metrics.

2. Preliminaries

In this section, we discuss Berwald and Douglas metrics which are required to study their counterparts in homogeneous Finsler spaces. Let F be a Finsler metric on a smooth n-manifold and π^*TM be the pulled-back tangent bundle over $TM_0=TM-\{0\}$, by the natural projection $\pi\colon TM_0\to M$. For a standard local coordinate system (x^i,y^i) in TM_0, let $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\right\}$ denote the local natural frame and $\left\{dx^i,dy^i\right\}$ denote the local natural coframe for $T(TM_0)$. Let $G=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$ be a vector field on TM_0, where

$$G^i \;:=\frac{1}{4}g^{il}\left\{\left(F^2\right)_{x^ky^l}y^k-\left(F^2\right)_{x^l}\right\},\quad i=1,2,...,n,\ x\in M,\ y\in T_xM.$$

Here, the vector field G is called a spray which is useful in determining geodesics of F and G^i are called geodesic spray coefficients of Finsler metric F in a standard local coordinate system of TM. The geodesics of F are determined by the following equation

$$\ddot{x}^i+2G^i(x,\dot{x})=0.$$

We can say that the projections of the integral curves of G are the geodesics of F.
If the geodesic coefficients are quadratic in y^i, i.e.,

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k,$$

where $\Gamma^i_{jk}(x)$ are local functions on M, then F is called a Berwald metric.
If the geodesic coefficients can be written as

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $P(x,y)$ is a local positively homogeneous function of degree one on TM, then F is called a Douglas metric ([3], [5]). One can easily see that every Riemannian metric is a Berwald metric and every Berwald metric is a Douglas metric.

The authors in [15] give a necessary and sufficient condition for an (α,β)-metric to be of Douglas type which is stated below:

Theorem 2.1. *Suppose that $F = \alpha\phi(s)$ be an (α,β)-metric on an open subset V of $\mathbb{R}^n$ $(n \geq 3)$. Further suppose that the following conditions are satisfied:*

- **(i)** *either b is constant on V or $db \neq 0$ everywhere,*
- **(ii)** *β is not parallel with respect to α,*
- **(iii)** *F is not is of Randers type.*

Then F is a Douglas metric on V if and only if the function $\phi(s)$ satisfies the following differential equation:

$$\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\} \phi^{''}(s) = \left(k_1 + k_2 s^2\right) \left\{\phi(s) - s\phi^{'}(s)\right\} \tag{1}$$

and the covariant derivative $\nabla\beta = b_{i|j}y^i dx^j$ of β with respect to α satisfies the equations

$$b_{i|j} = 2\tau \left\{\left(1 + k_1 b^2\right) a_{ij} + \left(k_2 b^2 + k^3\right) b_i b_j\right\}, \tag{2}$$

where $k_1,\ k_2, k_3$ are constants such that $(k_2, k_3) \neq (0,0)$ and $\tau = \tau(x)$ is a scalar function on V.

Yang ([25]) has discussed Douglas metrics on an open subset V of $\mathbb{R}^2$ written as follows:

Theorem 2.2. *Suppose that $F = \alpha\phi(s)$ be an (α,β)-metric on an open subset $V \subset \mathbb{R}^2$. Further, suppose that F is not of Randers type, and β is not parallel with respect to α. Let F be a Douglas metric on V. Then one of the following two cases hold.*

- **(i)** *The function $\phi(s)$ satisfies equation (1) with $k_2 \neq k_1 k_3$ and β satisfies equation (2).*

(ii) *F can be written as*

$$F = \tilde{\alpha} \pm \frac{\tilde{\beta}^2}{\tilde{\alpha}}$$

with $\tilde{\alpha} = \sqrt{\alpha^2 - k\beta^2}$, $\tilde{\beta} = c\beta$, where k and $c \neq 0$ are constants.

3. Invariant infinite series (α,β)-metric of Berwald type

In this section, we find necessary and sufficient condition for homogeneous infinite series (α,β)-metric to be of Berwald type.

Let (M, F) be a homogeneous Finsler space with G-invariant infinite series (α,β)-metric $F = \dfrac{\beta^2}{\beta - \alpha}$. Then, M can be written as a coset space G/H, where $G = I(M, F)$ is a Lie transformation group of M and H, the compact isotropy subgroup of $I(M, F)$ at some point $x \in M$([8]). Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. If $\mathfrak{g}$ can be written as a direct sum of subspace $\mathfrak{h}$ and subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}\ \ \forall\, h \in H$, then $(G/H, F)$ is called a reductive homogeneous manifold ([19]). Note that a Finsler metric F can be viewed as a G-invariant Finsler metric on M. Thus, we can say that any homogeneous Finsler manifold can be written as a coset space of a connected Lie group with an invariant Finsler metric. Then using lemma 3.3 of ([22]), both α and β are G-invariant. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{m}$ induced by the Riemannian metric α.

We can identify the tangent space $T_{eH}(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{m}$ through the following map:

$$\mathfrak{m} \longrightarrow T_{eH}(G/H)$$
$$v \longrightarrow \frac{d}{dt}(\exp(tv)H)|_{t=0}.$$

Observe that for any $v \in \mathfrak{g}$, the vector field $\tilde{v} = \dfrac{d}{dt}(\exp(tv)H)|_{t=0}$ is called the fundamental Killing vector field generated by v ([14]).

Theorem 3.1. *Let $F = \dfrac{\beta^2}{\beta - \alpha}$ be a G-invariant infinite series metric on a reductive homogeneous Finsler space G/H with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ generated by a Riemannian metric α and a vector $v \in \mathfrak{m}$ such that $\mathrm{Ad}(H)v = v$ and $\langle v, v\rangle < 1$. Then F is a Berwald metric if and only if*

$$\left\langle [u,v]_{\mathfrak{m}}, w \right\rangle + \left\langle [w,v]_{\mathfrak{m}}, u \right\rangle = 0, \tag{3}$$

and

$$\left\langle [u,w]_{\mathfrak{m}}, v \right\rangle = 0, \tag{4}$$

for any $u, w \in \mathfrak{m}$.

Proof. By [18] and Lemma 2.1 of ([22]), we can say that infinite series (α,β)-metric F is of Berwald type if and only if the invariant vector field v is parallel with respect to α. Further, the vector field v is parallel with respect to α if and only if

$$\Gamma^i_{nj} = \frac{1}{2}\left(\left\langle [v_j, v_n]_{\mathfrak{m}}, v_i\right\rangle + \left\langle [v_i, v_n]_{\mathfrak{m}}, v_j\right\rangle + \left\langle [v_i, v_j]_{\mathfrak{m}}, v_n\right\rangle\right) = 0.$$

Therefore for all $u, w \in \mathfrak{m}$, we can write

$$\left\langle [u, v]_{\mathfrak{m}}, w\right\rangle + \left\langle [w, v]_{\mathfrak{m}}, u\right\rangle + \left\langle [u, w]_{\mathfrak{m}}, v\right\rangle = 0. \tag{5}$$

Now, letting $w = u$ in above equation, we get

$$\left\langle [u, v], u\right\rangle = 0, \ \forall\, u \in \mathfrak{m}. \tag{6}$$

We can also write the above equation for $u + w \in \mathfrak{m}$, i.e.,

$$\left\langle [u + w, v], u + w\right\rangle = 0, \ \forall\, u, w \in \mathfrak{m}. \tag{7}$$

From equations (6) and (7), we get

$$\left\langle [u, v], w\right\rangle + \left\langle [w, v], u\right\rangle = 0. \tag{8}$$

From equations (5) and (8), we get

$$\left\langle [u, w], v\right\rangle = 0.$$

Conversely, if equations (3) and (4) hold, then clearly equation (5) is satisfied and so F is a Berwald metric. This completes the proof. □

4. Invariant infinite series (α,β)-metric of Douglas type

In this section, we prove our main results obtained in 4.1 and 4.2.

Theorem 4.1. *Let* $F = \dfrac{\beta^2}{\beta - \alpha}$ *be a* G*-invariant infinite series* (α,β)*-metric on the reductive homogeneous Finsler space* G/H *with a decomposition of the Lie algebra* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$*. Then* F *is a Douglas metric if and only if* F *is a Berwald metric or* F *is a Douglas metric of Randers type.*

Proof. Since Finsler space is homogeneous, it is sufficient to prove the result at the origin eH. We consider two cases:
Case 1: $dim(G/H) \geq 3$. Let F be a Douglas metric and suppose to contrary that neither F is a Berwald metric nor F is of Randers type. We know that in case of a homogeneous Finsler space, Riemannian length b is constant. Therefore using theorem 2.1, we have

$$b_{n|n} = 2\tau(eH)\left\{\left(1 + k_1 b^2\right)\delta_{nn} + \left(k_2 b^2 + k_3\right) b\delta_{nn}\ b\delta_{nn}\right\},$$

i.e.,

$$b_{n|n} = 2\tau(eH)\left\{\left(1 + k_1 b^2\right) + \left(k_2 b^2 + k_3\right) b^2\right\}. \tag{9}$$

Also, we have

$$b_{n|n} = 0. \tag{10}$$

From equations (9) and (10), we get

$$\tau(eH)\left\{\left(1 + k_1 b^2\right) + \left(k_2 b^2 + k_3\right) b^2\right\} = 0.$$

Note that scalar function τ is constant, as α and β are both $G-$invariant. Also, by the assumption that F is not a Berwald metric, we have $\tau \neq 0$.
Therefore

$$\left\{\left(1 + k_1 b^2\right) + \left(k_2 b^2 + k_3\right) b^2\right\} = 0. \tag{11}$$

Now, using Shen's lemma, the condition for infinite series metric $F = \alpha\phi(s) = \alpha\left(\dfrac{s^2}{s-1}\right)$ to be a Finsler metric reduces to

$$\phi(s) - s\phi^{'}(s) + \left(b^2 - s^2\right)\phi^{''}(s) = \frac{s^2}{(s-1)^2} + \left(b^2 - s^2\right)\frac{2}{(s-1)^3} > 0 \tag{12}$$

Now, using equation (1) of theorem 2.1, we have
(13)

$$\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\}\frac{2}{(s-1)^3} = \left(k_1 + k_2 s^2\right)\left\{\frac{s^2}{s-1} - \frac{s\left(s^2 - 2s\right)}{(s-1)^2}\right\},$$

simplifying, we get

$$\frac{2}{(s-1)^3} = \frac{s^2\left(k_1 + k_2 s^2\right)}{(s-1)^2\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\}}. \tag{14}$$

Using equation (14), we can write equation (12) as

$$\begin{aligned}
0 <& \frac{s^2}{(s-1)^2} + \frac{\left(b^2 - s^2\right) s^2\left(k_1 + k_2 s^2\right)}{(s-1)^2\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\}} \\
&= \frac{s^2}{(s-1)^2}\left[1 + \frac{\left(b^2 - s^2\right)\left(k_1 + k_2 s^2\right)}{\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\}}\right] \\
&= \frac{s^2}{(s-1)^2}\left[\frac{1 + k_1 b^2 + \left(k_2 b^2 + k_3\right) s^2}{\left\{1 + \left(k_1 + k_2 s^2 + k_3\right) s^2\right\}}\right]
\end{aligned}$$

letting $s = 0$ in above inequality, we get

$$1 + k_1 b^2 > 0.$$

Also, $1 + k_1 s^2 > min\left\{1, 1 + k_1 b^2\right\}$.
Therefore

$$1 + k_1 s^2 > 0, \ \forall\ |s| \leq b < b_0.$$

Letting $s = b$ in equation (13) and using equation (11), we have

$$k_1 + k_2 b^2 = 0. \tag{15}$$

From equations (11) and (15), we have

$$1 + k_3 b^2 = 0. \tag{16}$$

From equations (15) and (16), we get values of k_2 and k_3 as follows:

$$k_2 = -\frac{k_1}{b^2}, \quad k_3 = -\frac{1}{b^2}.$$

Substituting these values of k_2 and k_3 in equation (1), we get

$$\begin{aligned}
&\left\{1 + \left(k_1 - \frac{k_1 s^2}{b^2} - \frac{1}{b^2}\right) s^2\right\} \phi^{''}(s) = \left(k_1 - \frac{k_1 s^2}{b^2}\right) \left\{\phi(s) - s\phi^{'}(s)\right\} \\
\Longrightarrow\ & \left[b^2 + \left\{k_1 \left(b^2 - s^2\right) - 1\right\} s^2\right] \phi^{''}(s) = k_1 \left(b^2 - s^2\right) \left\{\phi(s) - s\phi^{'}(s)\right\} \\
\Longrightarrow\ & \left\{b^2 \left(1 + k_1 s^2\right) - s^2 \left(1 + k_1 s^2\right)\right\} \phi^{''}(s) = \left(k_1 - \frac{k_1 s^2}{b^2}\right) \left\{\phi(s) - s\phi^{'}(s)\right\} \\
\Longrightarrow\ & \left(b^2 - s^2\right) \left(1 + k_1 s^2\right) \phi^{''}(s) = \left(k_1 - \frac{k_1 s^2}{b^2}\right) \left\{\phi(s) - s\phi^{'}(s)\right\},
\end{aligned}$$

we get the following second order ordinary differential equation

$$\phi^{''}(s) + \frac{k_1 s}{1 + k_1 s^2} \phi^{'}(s) - \frac{k_1}{1 + k_1 s^2} \phi(s) = 0. \tag{17}$$

Next, we solve this equation as follows:
let $P = \dfrac{k_1 s}{1 + k_1 s^2}$ and $Q = -\dfrac{k_1}{1 + k_1 s^2}$, note that $P + sQ = 0$. Therefore, let $\phi(s) = vs$ be a solution of equation (17). Substituting the values $\phi(s) = vs$, $\phi^{'}(s) = v + s\dfrac{dv}{ds}$ and $\phi^{''} = s\dfrac{d^2v}{ds^2} + 2\dfrac{dv}{ds}$ in equation (17) to get the following differential equation

$$\begin{aligned}
& s\frac{d^2v}{ds^2} + 2\frac{dv}{ds} + \frac{k_1 s}{1 + k_1 s^2} \left(v + s\frac{dv}{ds}\right) - \frac{k_1 v s}{1 + k_1 s^2} = 0 \\
\Longrightarrow\ & s\frac{d^2v}{ds^2} + \frac{2 + 3k_1 s^2}{1 + k_1 s^2} \frac{dv}{ds} = 0 \\
\Longrightarrow\ & \frac{d^2v}{ds^2} + \frac{2 + 3k_1 s^2}{s\left(1 + k_1 s^2\right)} \frac{dv}{ds} = 0 \\
\Longrightarrow\ & \frac{d^2v}{ds^2} + \left(\frac{2}{s} + \frac{k_1 s}{1 + k_1 s^2}\right) \frac{dv}{ds} = 0,
\end{aligned}$$

putting $\dfrac{dv}{ds} = z$, we get

$$\frac{dz}{ds} + \left(\frac{2}{s} + \frac{k_1 s}{1 + k_1 s^2}\right) z = 0,$$

i.e.,

$$\frac{dz}{z} + \left(\frac{2}{s} + \frac{k_1 s}{1 + k_1 s^2}\right) ds = 0,$$

integrating the above equation, we get

$$\log z + \log s^2 + \log \sqrt{1 + k_1 s^2} = \log c_1, \text{ for some constant } c_1,$$

i.e.,

$$zs^2 \sqrt{1 + k_1 s^2} = c_1,$$

which further implies

$$dv = \frac{c_1}{s^2 \sqrt{1 + k_1 s^2}} ds,$$

putting $s = \dfrac{1}{t}$ in above equation and integrating, we get

$$v = -\int \frac{c_1 t}{\sqrt{t^2 + k_1}} dt + c_2, \text{ for some constant } c_2,$$

putting $t^2 + k_1 = u^2$, and solving, we get

$$v = -c_1 u + c_2.$$

Finally, we get

$$\begin{aligned} \phi(s) = vs &= (-c_1 u + c_2) \\ &= -sc_1 \sqrt{t^2 + k_1} + c_2 s \\ &= -c_1 \sqrt{1 + k_1 s^2} + c_2 s, \end{aligned}$$

which shows that ϕ is of Randers type, we get a contradiction.
Hence the result is proved.
Case (ii): If $dim\,(G/H) = 2$, then using (i) of theorem (2.2), the result is proved. □

Theorem 4.2. *Let* $F = \dfrac{\beta^2}{\beta - \alpha}$ *be a* G*-invariant infinite series* (α, β)*-metric on the reductive homogeneous Finsler space* G/H *with a decomposition of the Lie algebra* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$*. Further, suppose that* $\mathfrak{g}$ *be perfect. Then* F *is a Douglas metric if and only if it is a Riemannian metric.*

Proof. In theorem 4.1, we have proved that homogeneous infinite series (α, β)-metric is a Douglas metric if and only if it is a Berwald metric or Douglas metric of Randers type.
Firstly, if F is a Douglas metric of Randers type, then it can be written as $F = \tilde{\alpha} + \tilde{\beta}$, where $\tilde{\alpha}$ is a G-invariant Riemannian metric and $\tilde{\beta}$ be a G-invariant closed 1-form. Then there is a vector $v \in \mathfrak{m}$ such that $\mathrm{Ad}(h)(v) = v \ \forall\ h \in H$, which corresponds to $\tilde{\beta}$ with respect to inner product $\langle , \rangle$ corresponding to $\tilde{\alpha}$

and this vector v satisfying the equation (4) (see [9]).
Therefore, we have

$$[v,u]=0 \ \forall \ u\in\mathfrak{h}. \tag{18}$$

Also, since $\tilde{\alpha}$ is G-invariant, therefore $\langle,\rangle$ is Ad-invariant scalar product on $\mathfrak{m}$ ([2]) and it satisfies the following condition:

$$\left\langle [u,w]_{\mathfrak{m}}, z\right\rangle+\left\langle [u,z]_{\mathfrak{m}}, w\right\rangle=0 \ \forall \ u\in\mathfrak{h},\ w,z\in\mathfrak{m}.$$

In particular, we can write

$$\left\langle [u,w]_{\mathfrak{m}}, v\right\rangle+\left\langle [u,v]_{\mathfrak{m}}, w\right\rangle=0 \ \forall \ u\in\mathfrak{h},\ w\in\mathfrak{m}. \tag{19}$$

From equation (18) and (19), we get

$$\left\langle [u,w]_{\mathfrak{m}}, v\right\rangle=0 \ \forall \ u\in\mathfrak{h},\ w\in\mathfrak{m}. \tag{20}$$

Next, since Lie algebra $\mathfrak{g}$ of G is perfect, i.e., $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$.
Therefore, for vector v, there exist $u,w\in\mathfrak{g}$ such that $v=[u,w]$.
Using reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, we can write $u=u_1+u_2$ and $w=w_1+w_2$, where $u_1,w_1\in\mathfrak{m}$ & $u_2,w_2\in\mathfrak{h}$. Finally, we have

$$\begin{aligned}\left\langle v,v\right\rangle &=\left\langle v,[u,w]_{\mathfrak{m}}\right\rangle\\ &=\left\langle v,[u_1+u_2,w_1+w_2]_{\mathfrak{m}}\right\rangle\\ &=0,\ \text{using equations (4) and (20).}\end{aligned}$$

Therefore, $v=0$. Hence F is a Riemannian metric.
If F is a Berwald metric, then theorem (3.1) implies that equation (4) holds. Then, by following the similar steps as above, we show that F is a Riemannian metric. □

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