JAMI Journal of Applied Mathematics and Informatics

J. Appl. Math. & Informatics Vol. **42***(2024), No. 3, pp. 695 - 708*
https://doi.org/10.14317/jami.2024.695

DEVELOPMENT OF A NON-STANDARD FINITE DIFFERENCE METHOD FOR SOLVING A FRACTIONAL DECAY MODEL†

SAID AL KATHIRI*, EIHAB BASHIER, NUR NADIAH ABD HAMID, NORSHAFIRA RAMLI

Abstract. In this paper we present a non-standard finite difference method for solving a fractional decay model. The proposed NSFDM is constructed by incorporating a non-standard denominator function, resulting in an explicit numerical scheme as easy as the conventional Euler method, but it provides very accurate solutions and has unconditional stability. Two examples from the literature are presented to demonstrate the performance of the proposed numerical scheme, which is compared to three methods from the literature. It is found that the method's estimated errors are extremely minimal, such as within the machine precision.

AMS Mathematics Subject Classification : 26A33, 26D10, 05A30.
Key words and phrases : Fractional differential equation, non-standard finite difference method, decay model.

1. INTRODUCTION

The origins of fractional calculus may be traced to a correspondence between Leibniz and L'Hopital in 1695 on a semi-derivative concept. Many mathematicians have since been interested in fractional calculus, including Euler, Riemann, Fourier, Grunewald, and Liouville [1].

Currently, fractional differential equations (FDEs) are extensively used to model various problems in different fields of science and engineering. These include epidemic diseases, chemical processes [2, 3], anomalous diffusion, and viscoelasticity [4, 5]. One of the simplest FDE models is the decay model, which is of the form

Received January 8, 2024. Revised February 20, 2024. Accepted March 19, 2024.
*Corresponding author.
†This work is supported by Universiti Sains Malaysia's Graduate On Time Incentive with account number 1001/PMATHS/823233.

$$^{c}D^{\alpha}y = -\lambda y, y(0) = y_0. \tag{1}$$

Fractional decay models provide a flexible framework for characterizing phenomena in which conventional exponential decay models fail to accurately depict the decay process. Their wide-ranging applications encompass various disciplines, contributing to a more sophisticated comprehension of decay mechanisms characterized by dynamic and intricate decay rates. [1]. Example to such an application is the work in [12] which aimed to construct fractional Bateman equations capable of simulating memory effects in the transformations of successive isotopes. Another example is the nuclear decay equation that is analyzed in [6] through the application of fractional calculus. Within this particular framework, the first-order time derivative is transformed into a Caputo fractional derivative. As a result, the time fractional nuclear decay equation is obtained.

Some recent numerical techniques for solving fractional order differential equations are as follows: Turkyilmazoglu [8] proposed a modified version of the Adomian Decomposition Method that incorporates a functional term to solve fractional ordinary and partial differential equations. The approach is precise, superior than traditional ADM, and straightforward to implement for fractional ordinary and partial differential equations in mathematical models. In a separate investigation conducted by Alghtani and Saad [9], the spectral collocation methodology was examined for three novel models of the space fractional Fisher equation. The research used several techniques such as exponential decay kernel, Chebyshev polynomial features, finite differences, Newton's method, and power law kernel. The findings indicate a strong correlation between the solutions derived from Chebyshev polynomials and the power law and Mittag-Leffler kernels. The authors Wang et. al presented a novel fourth-order fractional Adams-type implicit-explicit method in their publication [10]. This method is designed to solve nonlinear fractional ordinary differential equations with weakly singular solutions. The authors demonstrated that the method converges and remains stable under the Lipschitz condition.

Kumar et. al [23] analyzes a fractional-order prey–predator system in presence of harvesting and fear effect. Analytically, the condition for existence, uniqueness, nonnegativity, and boundedness are investigated. Existence criteria for feasible equilibrium points are discussed, and the conditions for local and global stability are explored.

In order to obtain a fully discrete method, the standard central finite difference approximation was used to discretize the second-order spatial derivative [24]. By using ADI scheme for the three-dimensional problem, the overall computational cost was reduced significantly. Two new approaches were adopted for theoretical stability analysis. They provided the convergence behaviour of the proposed method and the error bounds were proved.

Fractional Runge-Kutta numerical algorithms for solving the fractional decay model had been devised, but could not achieve more than order 4 accuracy, according to [7].

In general, conventional numerical techniques are meant to handle a class of generic problems without taking into account the structure of any specific problems [11]. As a result, they were unable to capture the qualitative characteristics of the exact solution [2].

An essential aspect of constructing non-standard finite difference methods (NSFDMs) is integrating the qualitative characteristic of the exact model into the non-standard difference scheme [13, 14, 15]. Therefore, while achieving great stability qualities, it is also possible to provide an accurate solution for this fractional model [16].

Nonstandard finite difference methods have a particular advantage over traditional finite difference schemes because they may provide exact numerical schemes that lead to exact solutions at the nodal points. For the purpose of solving certain classes of first-order differential equations, such as the logistic and exponential growth, systems of linear ODE models, higher order ODEs, some PDE problems, etc., Micken was the first to design and explain the derivation of exact finite difference techniques [2, 20]. Initially, nonstandard finite difference schemes were designed to offer either an accurate finite difference scheme or the optimal numerical scheme, with the goal of achieving all the qualitative elements of the exact solution of the original problem.

The purpose of this paper is to design a numerical scheme employing non-standard finite difference discretization to solve the fractional decay model (1). It compares the performance of the proposed scheme to other methods found in the literature. The main features of the proposed numerical approach are its explicit Euler method-like design, high order of convergence, and unconditional stability. The method's approximation errors, measured using the infinity norm, are very small and fall within or very near the machine precision.

The rest of this paper is organized as follows. Section 2 provides an overview of fractional calculus (FC) and the development of a non-standard scheme using Mickens principles. In Section 3, the proposed methodology is introduced. Section 4 employs numerical examples from existing literature to evaluate the effectiveness of the proposed technique. The obtained findings are then compared with those of other methods mentioned in the literature. Finally, the conclusions and discussions are presented in Section 5.

2. MATERIALS AND METHODS

2.1. Materials. In this section, the basic definitions and properties of FC are presented.

Definition 1: For any $f \in L(D)$, the left -sided Riemann-Liouville fractional integral of order $\alpha > 0$ based at $x = a$, denoted by ${}_aI_x^{\alpha} f$, is defined by [17] as

follows:

$$ {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, $$

and the right -sided Riemann-Liouville fractional integral of order $\alpha > 0$ based at $x = b$, denoted by ${}_bI_x^{\alpha} f$, is defined by [17]

$$ {}_bI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds. $$

Definition 2: For any $f \in L(D)$, and $n-1 < \alpha < n, n \in \mathbb{N}$ its left -sided Riemann-Liouville fractional integral of order α (based at $x = a$), denoted by ${}_a^R D_x^{\alpha} f$, is defined by [17] as follows:

$$ {}_a^R D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^s (x-s)^{n-\alpha-1} f(s) ds, $$

if the integral on the right-hand side exists. Its right-sided Riemann-Liouville fractional derivative of order α (based at $x = b$), denoted by ${}_x^R D_b^{\alpha} f$ is defined by

$$ {}_x^R D_b^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_s^b (s-x)^{n-\alpha-1} f(s) ds. $$

For $n = 1$ and $0 < \alpha < 1$, then the fractional left and right Riemann-Liouville derivatives are

$$ {}_a^R D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-s)^{-\alpha} f(s) ds, $$

and

$$ {}_x^R D_b^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (s-x)^{-\alpha} f(s) ds. $$

Definition 3: For any $f \in L(D)$, and $n-1 < \alpha < n, n \in \mathbb{N}$, its left-sided Djrbashian-Caputo fractional derivative ${}_a^C D_x^{\alpha} f$ of order α based at $x = a$, is defined by [17] as follows:

$$ {}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s) ds, $$

if the integral on the right-hand side exists. Likewise, its right-sided Djrbashian-Caputo fractional derivative ${}_x^C D_b^{\alpha} f$ of order α based at $x = b$, is defined by [17] as follows:

$$ {}_x^C D_b^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (s-x)^{n-\alpha-1} f^{(n)}(s) ds. $$

For $n = 1$ and $0 < \alpha < 1$, then the fractional left and right Caputo derivatives are

$$ {}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-s)^{-\alpha} f^{(')}(s) ds, $$

and

$$ {}_x^C D_b^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b (s-x)^{-\alpha} f^{(')}(s) ds. $$

The relation between the fractional Riemann-Liouville derivative and fractional Caputo derivative are

$$ {}_a^C D_t^\alpha f(x) = {}_a^{RL} D_x^\alpha f(x) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

and

$$ {}_x^C D_b^\alpha f(x) = {}_t^{RL} D_b^\alpha f(x) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}, $$

where $n-1 < \alpha < n, n \in N$.

Definition 4: The one parameter Mittag-Leffler function E_α which was introduced by [1] as follows:

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \ z \in \mathbb{C}, \ Re(\alpha) > 0. $$

For $0 < \alpha < 1$, the Laplace transformation of the Caputo derivative ${}_a^C D_t^\alpha f(t)$ is defined as

$$ \mathcal{L}\{{}_a^C D_t^\alpha f(t)\} = \frac{sF(s) - f(0)}{s^{1-\alpha}} = s^\alpha F(s) - s^{\alpha-1} f(0). $$

The Laplace transformation of the one parameter Mittag-Leffler function $E_\alpha(\lambda z^\alpha)$ is given by [1] as

$$ \mathcal{L}\{E_\alpha(\lambda z^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, $$

which means that

$$ \mathcal{L}^{-1}\{\frac{s^{\alpha-1}}{s^\alpha - \lambda}\} = E_\alpha(\lambda z^\alpha). $$

The properties of Laplace transformation and its inverse, applied to both the Caputo derivative and the Mittag-Leffler function, may be used to determine the analytical solution of the decay model (1). Let's assume that the Laplace transform of the function $y(t)$ is denoted as $Y(s)$. By applying the Laplace transformation to both sides of equation 1, and using the linearity feature of the Laplace transformation [18], we obtain

$$ s^\alpha Y(s) - s^{\alpha-1} y(0) = -\lambda Y(s). $$

Solving for $Y(s)$ and substituting y_0 instead of $y(0)$ we find that

$$ Y(s) = \frac{y_0 s^{\alpha-1}}{s^\alpha + \lambda}. $$

By applying the inverse Laplace transform, we obtain

$$ y(t) = y_0 \mathcal{L}^{-1}\left\{ \frac{s^{\alpha-1}}{s^\alpha + \lambda} \right\} = y_0 E_\alpha(-\lambda t^\alpha). $$

2.2. Methods. In this section, we develop NSFDM for solving the fractional decay model. The exact solution of the fractional decay model 1 is given by

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha),\ t > 0. \tag{2}$$

Let $N > 1$ be a positive integer and $h = T/N$. Let $x_k = kh, k = 0, 1, ..., N$. The basis function of the exact solution is given by $E_\alpha(-\lambda t^\alpha)$. Now, we follow the Micken's rules for constructing nonstandard finite difference schemes. We construct the determinant

$$\begin{vmatrix} y_k & E_\alpha(-\lambda t_k^\alpha) \\ y_{k+1} & E_\alpha(-\lambda t_{k+1}^\alpha) \end{vmatrix} = 0$$

from which we find:

$$y_{k+1}E_\alpha(-\lambda t_k^\alpha) = y_k E_\alpha(-\lambda t_{k+1}^\alpha) \implies y_{k+1} = \frac{E_\alpha(-\lambda t_{k+1}^\alpha)}{E_\alpha(-\lambda t_k^\alpha)} y_k.$$

By subtracting y_k from the two sides, multiplying and dividing the right-hand side by $-\lambda$, we obtain

$$y_{k+1} - y_k = \frac{-\lambda y_k \left(\frac{E_\alpha(-\lambda t_{k+1}^\alpha)}{E_\alpha(-\lambda t_k^\alpha)} - 1 \right)}{-\lambda} = \frac{1 - \frac{E_\alpha(-\lambda t_{k+1}^\alpha)}{E_\alpha(-\lambda t_k^\alpha)}}{\lambda}(-\lambda y_k)$$

from which we find that

$$\frac{y_{k+1} - y_k}{\frac{1 - \frac{E_\alpha(-\lambda t_{k+1}^\alpha)}{E_\alpha(-\lambda t_k^\alpha)}}{\lambda}} = -\lambda y_k.$$

Therefore, a suitable denominator function $\phi_k(\lambda, \alpha, h)$ is defined in $[t_k, t_{k+1}]$ as

$$\phi_k(\lambda, \alpha, h) = \frac{1 - \frac{E_\alpha(-\lambda t_{k+1}^\alpha)}{E_\alpha(-\lambda t_k^\alpha)}}{\lambda} = \frac{1 - \frac{E_\alpha(-(k+1)^\alpha \lambda h^\alpha)}{E_\alpha(-k^\alpha \lambda h^\alpha)}}{\lambda}.$$

Then, the proposed exact finite difference scheme is of the form

$$\frac{y_{k+1} - y_k}{\phi_k(\lambda, \alpha, h)} = -\lambda y_k \tag{3}$$

which can be simplified into

$$y_{k+1} = (1 - \lambda\phi(\lambda, \alpha, h)).y_k,\ k = 0, 1, ..., N-1. \tag{4}$$

Thus, by using the numerical scheme (4), it is possible to produce numerical solutions at the points $x_1, x_2, ..., x_N$.

3. CONVERGENCE ANALYSIS OF THE PROPOSED SCHEME

3.1. CONSISTENCY OF THE NUMERICAL SCHEME. The local truncation error LTE_k of (3) is given by the form [21]

$$LTE_k = \frac{y_{k+1} - y_k}{\phi(\lambda, h)} + \lambda y_k. \tag{5}$$

By adding and subtracting the terms $\frac{y_{k+1}-y_k}{\frac{h^\alpha}{\Gamma(1+\alpha)}}$ and $\alpha y(t_k)$ to the right hand side of (3.1) we obtain

$$\begin{aligned} |LTE_k| &\leq \left| \frac{y(t_{k+1})-y(t_k)}{\phi(\alpha,h)} - \frac{y(t_{k+1})-y(t_k)}{\frac{h^\alpha}{\Gamma(1+\alpha)}} + \left| \frac{y(t_{k+1})-y(t_k)}{\frac{h^\alpha}{\Gamma(1+\alpha)})} + \lambda y(t_k) \right| \\ &\leq \left| y(t_{k+1})-y(t_k) \right| \left| \frac{1}{\phi} - \frac{1}{\frac{h^\alpha}{\Gamma(1+\alpha)}} \right| + \frac{\frac{h^\alpha \Gamma(1+\alpha)}{\Gamma(1+2\alpha)}}{2} |D^{2\alpha} y(\zeta|. \end{aligned} \tag{6}$$

The quantity $\frac{1}{\phi} - \frac{1}{\frac{h^\alpha}{\Gamma(1+\alpha)}}$ is of $O(1)$. Also, from Taylor expansion, $|y(t_{k+1}) - y(t_k)| = \frac{h^\alpha}{\Gamma(1+\alpha)} \dot{y}(\eta), \eta \in [t_k, t_{k+1}]$, hence (6) becomes

$$|LTE_k| \leq \frac{h^\alpha}{\Gamma(1+\alpha)} \left[D^\alpha \big(O(1)\big) + \frac{D^{2\alpha} y(\zeta)}{2} \right] \tag{7}$$

where $\eta, \zeta \in (t_k, t_{k+1})$.

From equation (7), we find that as $h \to 0$, $|LTE_k| \to 0$ which proves the consistency of the numerical scheme (3).

3.2. STABILITY OF THE NUMERICAL SCHEME. Let $e_k = y(t_k) - y_k$. We want to ascertain the conditions under which the proposed numerical scheme is stable (*i.e.*, $e_k \to 0$ *as* $k \to \infty$) [21].

We have $y_{k+1} = (1 - \lambda\phi(\lambda, \alpha, h)).y_k$. By substituting y_k instead of y_k in equation (4) and subtracting equation (4) from the resulting equation we obtain:

$$e_{k+1} = (1 - \lambda\phi(\lambda, \alpha, h)).e_k. \tag{8}$$

From equation (8) we have to obtain $e_1 = (1 - \lambda\phi(\lambda, \alpha, h)).e_0, e_2 = (1 - \lambda\phi(\lambda, \alpha, h)).e_1 = (1 - \lambda\phi(\lambda, \alpha, h))^2.e_0$. Generally, $e_k = (1 - \lambda\phi(\lambda, \alpha, h))^k.e_0$. Now, $e_k \to 0$ as $k \to \infty$ if and only if $|1 - \lambda\phi(\lambda, \alpha, h)| < 1$. This latest inequality can be expressed as:

$$e - 1 < 1 - \lambda\phi(\lambda, \alpha, h) < 1 \implies 0 < \phi(\lambda, \alpha, h) < \frac{2}{\lambda}.$$

We then have

$$\phi(\lambda, \alpha, h) = \frac{1 - \frac{E_\alpha(-(k+1)^\alpha \lambda h^\alpha)}{E_\alpha(-k^\alpha \lambda h^\alpha)}}{\lambda}$$

Hence,

$$\phi(\lambda, \alpha, h) < \frac{2}{\lambda} \equiv \frac{1 - \frac{E_\alpha(-(k+1)^\alpha \lambda h^\alpha)}{E_\alpha(-k^\alpha \lambda h^\alpha)}}{\lambda} < \frac{2}{\lambda}$$

$$\implies \frac{-E_\alpha(-(k+1)^\alpha \lambda h^\alpha)}{E_\alpha(-k^\alpha \lambda h^\alpha)} < 1.$$

which is fulfilled always, regardless of the value of the step size h. Therefore, the proposed numerical scheme is unconditionally stable.

Finally, since the proposed scheme is both consistent and stable, it is convergent [22].

4. RESULTS AND DISCUSSION

In this section, we utilize the proposed NSFDM on two examples of initial value problems (IVPs) of Fractional Differential Equations (FDEs) obtained from references [7] and [19]. We then proceed to compare it with the fractional third order Runge-Kutta (RK3) scheme [7] and the fractional strong stability preserving third order Runge-Kutta (SSRK3) [7]. The first example involves solving a linear fractional differential equation (FDE) as an initial value problem (IVP), whereas the second example demonstrates the application of FDE.

Example 1 Consider the following linear IVP application of FDE [7]:

$$ {}^{C}D^{\alpha}_{0.1^{+}} y = -y,\ 0.1 < x < 1,\ y(o.1) = E_{\alpha}((-0.1)^{\alpha}). $$

The analytic solution of the above example is

$$ y(x) = E_{\alpha}(-x^{\alpha}). $$

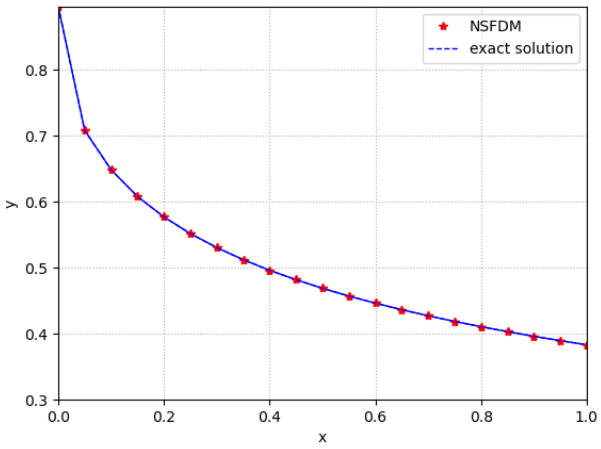


FIGURE 1. Solution from NSFDM for Example 1 with step size $h = 1/10$ and $\alpha = 1/2$.

The proposed scheme with step size $h = 1/10$ and $\alpha = 1/2$ was used to calculate the corresponding infinity norm error. The last column of Table 1 provides an illustration of these errors.

TABLE 1. Errors and orders of convergences obtained by two fractional Runge-Kutta methods from the literature and the proposed NSFDM.

	IEM [7]		SSRK3 [7]		NSFDM
N	Error	Order	Error	Order	Error
10	4.972e-03	-	8.461e-06	-	5.551e-17
20	1.285e-03	1.9523	1.125e-06	2.7265	1.110e-16
40	4.972e-03	1.9859	1.432e-07	2.9098	2.220e-16
80	8.130e-05	1.9963	1.822e-08	2.9750	2.220e-16
160	4.067e-05	0.9991	2.288e-09	2.9935	3.330e-16
320	2.034e-05	0.9998	2.863e-10	2.9984	5.551e-16
640	1.017e-05	0.9999	3.580e-11	2.9993	5.551e-16
1280	2.543e-06	2.0000	4.451e-12	3.0077	1.776e-15
2560	6.357e-07	2.0000	5.747e-13	2.9533	9.992e-16
5120	1.589e-07	2.0000	7.156e-14	3.00559	2.554e-15

The first 5 columns of Table 1 demonstrate the errors and convergence rates achieved by two fractional Runge-Kutta techniques, namely the Implicit Euler Method (IEM) and the Symmetric Splitting Runge-Kutta (SSRK) method [7]. The solutions generated by the proposed NSFDM were calculated and visually compared to the actual solution, as shown in Figure 1.

Table 1 indicates that the suggested NSFDM yields almost exact solutions for Example 1. However, when the value of N rises, so does the inaccuracy. This is due to the accumulation of round-off mistakes caused by machine accuracy. It is observed that the suggested NSFDM's error for $N = 10, 20, 40, \ , and\ 80$ falls within the precision of the machine, which is $\epsilon \approx 2.2202260e^{-16}$ [20]. As N rises, we see the influence of the round-off error for $N = 160, 320, 640, 1280, 2560$, and 5120. When we compare the performance of the suggested technique to the IEM and SSRK3 found in [7], which have orders of convergence of 2 and 3, we can observe how significantly superior the proposed NSDFM is.

Example 2 Consider the following time fractional radioactive decay equation [7]:

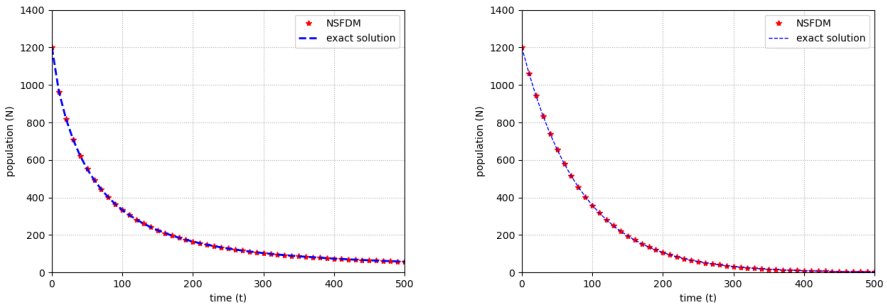
$$ {}^{C}D^{\alpha}_{0^+} N(t) = -\lambda N(t),\ t > t_0,\ N(t_0) = N_0 $$

whose exact solution is

$$ N(t) = N_0 E_{\alpha}(-\lambda t^{\alpha}),\ t > t_0 $$

In [19], this example was developed and explored in relation to the radioactivity of aluminum and silver, where experimental results indicated that an exponential decay model fits the data well. In this model, the radioactive element's mean lifespan is denoted by τ, and $\lambda = 1/\tau$. When $\lambda = 0.0121$, the traditional

exponential decay model with $\alpha = 1$ provided the best fit for λ. According to [19], $\alpha = 0.8252$ and $\lambda = 0.0314$ are the best fits for α and λ.



(**a**) $\lambda = 0.0314$ and $\alpha = 0.8252$ (**b**) $\lambda = 0.021$ and $\alpha = 1.0$

FIGURE 2. Solution from NSFDM for Example 2 with step size $h = 10$ sec and $\alpha = 1/2$.

Consequently, [7] examined two sets of parameters, λ and α, in order to solve the model using fractional IEM and RK3 discretization techniques. The first set consists of two elements: $\{\lambda = 0.0314, \alpha = 0.8252\}$. In contrast, the second set comprises two elements: $\{\lambda = 0.0121, \alpha = 1.0\}$.Table 2 presents the analytical solution of Example 2, as well as the solutions and errors found by the IEM, RK3, and the suggested NSFD approach for the initial set of parameters $\lambda = 0.0314$ and $\alpha = 0.8252$.

TABLE 2. Numerical solutions and errors obtained by the IEM, RK3 and NSFDM, for $\lambda = 0.0314, \alpha = 0.8252$ with step size $h = 10$ sec

	Analytical Solution[7]	IEM [7]		RK3[7]		NSFDM	
t(sec)	N_{Frac}	N_{IEM}	Error	N_{RK3}	Error	N_{NSFDM}	Error
0	1200.00	1200.00	0.000e+00	1200.00	0.000e+00	1200.00	0.000e+00
50	551.999	552.321	3.216e-01	552.160	1.608e-01	551.999	0.000e+00
100	326.316	326.690	3.734e-01	326.503	1.867e-01	333.282	5.684e-14
150	225.163	225.506	3.428e-01	225.334	1.714e-01	225.522	5.684e-14
200	164.801	165.105	3.043e-01	164.953	1.521e-01	164.974	8.526e-14
250	127.698	127.963	2.649e-01	127.831	1.324e-01	127.812	4.263e-14
300	103.296	103.527	2.311e-01	103.412	1.156e-01	103.410	5.684e-14
350	86.3853	86.5884	2.032e-01	86.4868	1.016e-01	86.4947	5.684e-14
400	74.1339	74.3144	1.804e-01	74.2241	9.019e-02	74.2345	2.842e-14
450	64.9211	65.0839	1.619e-01	65.0029	8.092e-02	65.0142	1.421e-14
500	57.7777	57.9244	1.467e-01	57.8510	7.329e-02	57.8629	1.421e-14

According to Table 2, the errors produced by the NSFDM approach are consistently close to zero at all time points. In comparison, the errors obtained

by the fractional IEM and RK3 methods are significantly larger. The NSFDM method demonstrates superior performance, as evidenced by the fact that the infinity norm of the error for the IEM method and RK3 method is approximately $O(10^{-1})$, whereas for the NSFDM method it is approximately $O(10^{-14})$. The second set of parameters, $\lambda = 0.0121$ and $\alpha = 1.0$, are used in Example 2 to compare the analytical solution with the solutions and errors produced by the IEM, SSRK3, and the suggested NSFDM techniques. The results are shown in Table 3.

TABLE 3. Numerical solutions and errors obtained by the IEM, RK3 and NSFDM , for $\lambda = 0.0121, \alpha = 1.0$ with step size $h = 10$ sec

	Analytical Solution[7]	IEM [7]		SSRK3[7]		NSFDM	
t(sec)	N_{Frac}	N_{IEM}	Error	N_{SSRK3}	Error	N_{NSFDM}	Error
0	1200.00	1200.00	0.000e+00	1200.00	0.000e+00	1200.00	0.000e+00
50	655.289	552.321	3.216e-01	552.0156	1.608e-02	655.289	0.000e+00
100	357.837	326.690	3.734e-01	326.335	1.867e-02	357.837	5.684e-14
150	195.406	225.506	3.428e-01	225.180	1.714e-02	195.406	5.684e-14
200	106.706	165.105	3.043e-01	165.816	1.521e-02	106.706	0.000e+00
250	58.2694	127.963	2.649e-01	127.712	1.324e-02	58.2694	0.000e+00
300	31.8194	103.527	2.311e-01	103.307	1.155e-02	31.8194	3.553e-15
350	17.3758	86.5884	2.032e-01	86.3954	1.015e-02	17.3758	0.000e+00
400	9.48846	74.31438	1.804e-01	74.14301	9.015e-03	9.48847	0.000e+00
450	5.18141	65.0839	1.619e-01	64.9301	8.089e-03	5.18141	8.8821e-16
500	2.82943	57.9244	1.467e-01	57.7851	7.329e-02	2.82943	4.441e-16

Table 3 displays similar results to Table 2. It is evident that the errors obtained by the NSFDM are extremely close to zero at all time points, in contrast to the errors obtained by the fractional IEM and SSRK3 methods. The NSFDM exhibits significantly superior performance, with the infinity norm of the IEM error being on the order of $O(10^{-1})$ and the SSRK3 method being on the order of $O(10^{-2})$, while the NSFDM is on the order of $O(10^{-14})$. The graphs depicting the answers for both sets are shown in Figure 2.

Conclusion

A nonstandard finite difference method for the fractional decay model's solution was introduced in this work. Equidistant points divide the temporal space, and the one-parameter Mittag-Leffler function—which serves as the basis function for the exact solution—is used to construct the denominator function. The NSFD scheme that is produced is explicit. To demonstrate the effectiveness of the suggested NSFDM through comparing it with the techniques mentioned in [7], two examples from the literature were taken into consideration.

The findings of the NSFDM approach were compared with those of the IEM, RK3, and SSRK3 methods in Table 1. The RK3 and SSRK3 had orders of convergence of three, whereas the IEM had orders of convergence of two. The

NSFDM solves Example 1; for small numbers of grid points, the outputs of the solutions for different grid point values are within machine precision; nevertheless, round-off errors accumulate and reduce accuracy for large numbers of grid points.

Tables 2 and 3 compare the performance of the NSFDM with the IEM, RK3, and SSRK3 techniques while solving Example 2. The IEM and RK3 showed errors of order $O(10^{-1})$ at the infinity norm, the SSRK3 showed errors of order $O(10^{-2})$, and the NSFDM showed an error of order $O(10^{-14})$.

Consequently, we may deduce from Tables 1, 2, and 3 that the recommended NSFDM technique produces results that are almost exact, with small absolute errors that lie close to or within the machine precision.

The paper's principal contributions are as follows: The objective is to create an explicit nonstandard finite difference scheme that possesses the same level of complexity as the explicit Euler method. The solutions produced by the suggested approach exhibit a high degree of precision, approaching or lying within the machine precision. The method is unconditionally stable. It is rare to come across explicit finite difference schemes that exhibit this characteristic.

We will benefit from and extend the nonstandard finite difference scheme presented in this paper to more difficult classes of fractional differential equations in the future, such as systems of *alpha*-order FDEs, fractional differential equations with nonlinear terms, and singularly perturbed fractional differential equations.

Conflicts of interest : We disclose that we don't have any financial and personal relationships with other people or organizations that could inappropriately influence (bias) our work.

Data availability : The data is available on the request to authors.

Acknowledgments : Financial support from Universiti Sains Malaysia's Graduate On Time Incentive with account number 1001/PMATHS/823233 is gratefully acknowledged.

References

1. C. Milici, G. Drăgănescu and J.T. Machado, *Introduction to Fractional Differential Equations*, Springer International Publishing, Cham., 2019.
2. R.E. Mickens, *Advances in the Applications of Nonstandard Finite Difference Schemes*, World Scientific Publishing Danvers, 2005.
3. M.M. Hikal and W.K. Zahra, *On Fractional Model of an HIV/AIDS with Treatment and Time Delay*, 02- Progress in Fractional Differentiation and Applications **2** (2016), 55–66. DOI 10.18576/pfda/020106
4. K. Dolfsson, M. Enelund and S. Larsson, *Space-time Discretization of an Integro-differential Equation Modeling Quasi-static Fractional-order Viscoelasticity*, Journal of Vibration and Control **14** (2008), 1631–1649. DOI 10.1177/1077546307087399

5. M. Dalir, and M. Bashour, *Applications of Fractional Calculus*, Applied Mathematical Sciences **4** (2010), 1021-1032. DOI 10.4236/jamp.2018.66105
6. A.E. Çalik, H. Ertik, B. Oder, H. Şirin, *Fractional calculus approach to investigate the alpha decay processes*, Int. J. Mod. Phys. E. **22** (2013), 1350049.
7. S.S. Alzaid, P.K. Shaw and S. Kumar, *A numerical study of fractional population growth and nuclear decay model*, AIMS Math. **7** (2022), 11417–11442. DOI 10.3934/math.2022637
8. M. Turkyilmazoglu, *An Efficient Computational Method for Differential Equations of Fractional Type*, CMES-Computer Modeling in Engineering & Sciences **133** (2002), 47–65.
9. M. Alqhtani, K.M. Saad, *Numerical solutions of space-fractional diffusion equations via the exponential decay kernel[J].* AIMS Mathematics **7** (2022), 6535-6549. doi: 10.3934/math.2022364
10. Y.M. Wang, B. Xie, *A fourth-order fractional Adams-type implicit–explicit method for nonlinear fractional ordinary differential equations with weakly singular solutions*, Mathematics and Computers in Simulation **212** (2023), 21-48.
11. E.B.M. Bashier, *Fitted Numerical Methods for Delay Differential Equations Arising in Biology*, Ph.D. thesis, University of the Western Cape, South Africa, 2009.
12. C.A. Cruz-López, G. Espinosa-Paredes, *Fractional radioactive decay law and Bateman equations*, Nuclear Engineering and Technology **54** (2002), 275-282. https://doi.org/10.1016/j.net.2021.07.026
13. L. Banjai, M.L. Fernández, *Efficient high order algorithms for fractional integrals and fractional differential equations*, Numer. Math. **141** (2019), 289–317. https://doi.org/10.1007/s00211-018-1004-0
14. D. Baffet, *A Gauss–Jacobi Kernel Compression Scheme for Fractional Differential Equations*, J. Sci. Comput. **79** (2019), 227–248. https://doi.org/10.1007/s10915-018-0848-x
15. D. Baffet and J.S. Hesthaven, *A Kernel Compression Scheme for Fractional Differential Equations*, SIAM J. Numer. Anal. **55** (2017), 496-520. DOI 10.1137/15m1043960
16. M.Y. Ongun, D. Arslan and R. Garrappa, *Nonstandard finite difference schemes for a fractional-order Brusselator system*, Adv. Differ. Equations **102** (2013), 1–13. DOI 10.1186/1687-1847-2013-102
17. B. Jin, *Fractional Differential Equations*, Springer International Publishing, Cham, 2021. DOI 10.1007/978-3-030-76043-4.
18. D.G. Zill and M.R. Cullen, *Differential Equations with Boundary-Value Problems*, Brooks/Cole, Belmont, 2001.
19. M. Awadalla and Y. Yameni, *Modeling Exponential Growth and Exponential Decay Real Phenomena by ψ-Caputo Fractional Derivative*, J. Adv. Math. Comput. Sci. **28** (2018), 1–13. DOI 109734/jamcs/2018/43054
20. E.B.M. Bashier, *Practical Numerical and Scientific Computing with MATLAB® and Python*, CRC Press, Boca Raton, 2020. DOI:10.1201/9780429021985
21. R.L. Burden and J.D. Faires, *Numerical Analysis*, 9th Edition, Brookscole, Boston, 2011.
22. H. Shintani, *Convergence, consistency and stability of step-by-step methods for ordinary differential equations*, Hiroshima mathematical journal **15** (1985), 89-107.
23. S. Kumar, R. Singh, R.P. Chauhan, *Investigation of an Interacting Fractional-Order Predator–Prey System in Presence of Fear and Harvesting*, Iran J. Sci. (2023). https://doi.org/10.1007/s40995-023-01540-5
24. H. Zhang, Y. Liu and X. Yang, *An efficient ADI difference scheme for the nonlocal evolution problem in three-dimensional space*, J. Appl. Math. Comput. **69** (2023), 651–674. https://doi.org/10.1007/s12190-022-01760-9

Said Al Kathiri received M.Sc. from Near East University, Turkey and presently pursuing Ph.D. at University Sains Malaysia. His research interests are Fractional Calculus, Numerical Techniques.

School of Mathematical Sciences, University Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia.
e-mail: said2021@student.usm.my, dsslil.2018@gmail.com

Eihab B.M. Bashier obtained his PhD in 2009 from the University of the Western Cape in South Africa. He is an associate professor of applied mathematics at the faculty of mathematical sciences and information technology, University of Khartoum, since 2013. Currently, he is working at the Faculty of Education and Arts, Sohar University, Oman. The research interests of Dr. Bashier are mainly in numerical methods for differential equations with applications to biology and in information and computer security with focus in cryptography. In 2011, Dr. Bashier won the African Union and the Third World Academy of Science (AU-TWAS) young scientists national award in basic sciences, technology and Innovation. Dr. Bashier is a reviewer for some international journals and an IEEE member.

Faculty of Education and Arts, Sohar University, Sohar, Sultanate of Oman.
e-mail: ebashier@su.edu.om, eihabbash@aims.ac.za

Nur Nadiah Abd Hamid received her Master of Science (M.Sc.) and Doctor of Philosophy (Ph.D.) in Applied Mathematics from USM in 2010 and 2016, respectively. Her research interests are mainly in the fields of numerical analysis and computer-aided geometric design. One of her niche areas is in manipulating splines developed in the field of CAGD to solve numerous differential equations.

Academic Services, Butterworth, Pulau Pinang, Malaysia.
e-mail: abdhamid.nn@nn-as.com

Norshafira Ramli is a Senior Lecturer at the School of Mathematical Sciences, Universiti Sains Malaysia, Malaysia. She graduated from Universiti Sains Malaysia with a Ph.D. degree in Mathematical Modelling. Her research interests are applied mathematics: fluid mechanics, boundary layer flow and heat transfer, and mathematical modelling.

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia.
e-mail: norshafiraramli@usm.my