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GENERATING FUNCTIONS FOR PHASE-SPACE HERMITE POLYNOMIALS

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ABSTRACT. In this work, we take advantage of Weisners̀ group-theoretic and operational identities technique to establish generating functions for the phase-space Hermite polynomials of two-index and two-variables.

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1. Introduction and Definitions

Dattoli et al. [10] and Dattoli and Torre [11] introduced and discussed the theory of phase-space Hermite polynomials using an operator formalism. These polynomials play a crucial role within the framework of phase-space formalism of classical and quantum mechanics. The phase-space Hermite polynomials $H_{m,n}(x,y)$ are defined by the generating relation(see [11, p. 1637 (1)]; also see [1] and[12]):

$$e^{\underline{h}^T \hat{M}\underline{z}-\frac{1}{2}\underline{h}^T \hat{M}\underline{h}} = \sum_{m,n=0}^{\infty} H_{m,n}(x,y)\frac{t^m u^n}{m!n!}, \tag{1.1}$$

where T denotes the transpose and

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix}, \underline{h} = \begin{pmatrix} t \\ u \end{pmatrix}, \ |t| < \infty, \ |u| < \infty, \ t \neq 0, \ u \neq 0. \tag{1.2}$$

The 2×2 matrix $\hat{M}$ is

$$\hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \ \Delta = ac - b^2 > 0, \ a > 0, \ c > 0. \tag{1.3}$$

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From (1.1) to (1.3), it immediately follows that

$$\exp\left[(ax+by)t+(bx+cy)u-\frac{1}{2}\left(at^2+cu^2+2btu\right)\right]$$
$$=\sum_{m,n=0}^{\infty} H_{m,n}(x,y)\frac{t^m u^n}{m!n!}, \tag{1.4}$$

where

$$H_{m,n}(x,y)=m!n!\sum_{r=0}^{\left[\frac{m-k}{2}\right]}\sum_{s=0}^{\left[\frac{n-k}{2}\right]}$$
$$\times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{r+s}(ax+by)^{m-2r-k}(bx+cy)^{n-2s-k}a^r c^s(-b)^k}{2^{r+s}r!s!k!(m-2r-k)!(n-2s-k)!}. \tag{1.5}$$

The following simultaneous partial differential equations (see [7]-[10]) have the following solutions provided by the Hermite polynomials $f=H_{m,n}(x,y)$:

$$\frac{1}{\Delta}\left[c(ax+by)\frac{\partial f}{\partial x}-b(ax+by)\frac{\partial f}{\partial y}-c\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+\Delta f\right]$$
$$-mf-f=0, \tag{1.6}$$

and

$$\frac{1}{\Delta}\left[-b(bx+cy)\frac{\partial f}{\partial x}+a(bx+cy)\frac{\partial f}{\partial y}-a\frac{\partial^2 f}{\partial y^2}+b\frac{\partial^2 f}{\partial x\partial y}+\Delta f\right]$$
$$-nf-f=0, \tag{1.7}$$

where $\Delta=ac-b^2$, or equivalently

$$\frac{1}{\Delta}\left[c(ax+by)\frac{\partial f}{\partial x}-b(ax+by)\frac{\partial f}{\partial y}-c\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}-m\Delta f\right]=0,$$

and

$$\frac{1}{\Delta}\left[-b(bx+cy)\frac{\partial f}{\partial x}-a(bx+cy)\frac{\partial f}{\partial y}-a\frac{\partial^2 f}{\partial y^2}+b\frac{\partial^2 f}{\partial x\partial y}-n\Delta f\right]=0.$$

The polynomials $H_{m,n}(x,y)$ are exploited in many fields of pure and applied mathematics [3,4,5,10,11]. They are very useful in the description of the quantum treatment [14] of coupled harmonic oscillators. In recent years several mathematical physicists showed that Hermite polynomials have applications to the quantum spectrum of the harmonic oscillator and in quantum optics (see [2,6,11,13,17-20]; see also [23, 24]). The study of special functions from the Weisnerś group-theoretic and operational methods point of view has witnessed a significant evolution during recent years. The applications of the theory of representations of group-theoretic and their Lie algebras allow interpretations of many familiar one-variable special functions, for example, Miller [15] and Weisner ([21] and [22]). Recently, a fundamental link between some generalized special functions of mathematical physics and certain Lie groups and Lie algebras has been established, see for example [5]. This paper is set up as follows. In Section 2, using the Weisner-Group-theoretic method approach to identify generating relations, we provide the first-order differential operators, the commutative relations of the operators, and the extended form of the group generated by operators. Acordingly, in Section 3 by employing the results of Section 2, generating functions involving the Hermite polynomials $H_{m,n}(x,y)$ are obtained. Also, we consider some special cases that would yield inevitably to many new and known generating relations for the polynomials $H_{m,n}(x,y)$. In Section 4, we follow the approach of methods, relevant to operational calculus and special function (see [4,6,10,11]) and obtain operational representations for the polynomials, which are further exploited to derive generating relations for $H_{m,n}(x,y)$.

2. Group-theoretic method

Weisner (see [21]) has devised a method for obtaining generating functions for a set of functions that satisfy certain conditions. Among the functions which do are the Hermite functions (see Weisner [22]). From the ordinary differential equation which is satisfied by the set of functions under consideration partial differential equations are constructed group of transformations under which this differential equation is invariant. The method is based on finding non-trivial contentious groups. Starting from equations (1.6) and (1.7) and replacing m and n by $p\frac{\partial}{\partial p}$ and $s\frac{\partial}{\partial s}$ respectively, we obtain

$$\frac{1}{\Delta}\left[c(ax+by)\frac{\partial f}{\partial x}-b(ax+by)\frac{\partial f}{\partial y}-c\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}-\Delta p\frac{\partial}{\partial p}\right]=0, \tag{2.1}$$

and

$$\frac{1}{\Delta}\left[-b(bx+cy)\frac{\partial f}{\partial x}+a(bx+cy)\frac{\partial f}{\partial y}-a\frac{\partial^2 f}{\partial y^2}+b\frac{\partial^2 f}{\partial x\partial y}-\Delta s\frac{\partial}{\partial s}\right]=0. \tag{2.2}$$

Note that, since $f(x,y,p,s)=H_{m,n}(x,y)p^m s^n$ is the solution of (1.6) and (1.7), then it is also a solution of (2.1) and (2.2). Let us define the following first-order differential operators:

$$A_1=\left[p\frac{\partial}{\partial p}\right],\ A_2=\left[s\frac{\partial}{\partial s}\right],\ A_3=\frac{1}{p\Delta}\left[c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right],\ A_4=p\left[(ax+by)-\frac{\partial}{\partial x}\right],$$
$$A_5=\frac{1}{-s\Delta}\left[b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right],\ A_6=s\left[(bx+cy)-\frac{\partial}{\partial y}\right].$$

Consequently, we have

$$A_1\left[H_{m,n}(x,y)p^m s^n\right]=mH_{m,n}(x,y)p^m s^n,$$
$$A_2\left[H_{m,n}(x,y)p^m s^n\right]=nH_{m,n}(x,y)p^m s^n,$$
$$A_3\left[H_{m,n}(x,y)p^m s^n\right]=mH_{m-1,n}(x,y)p^{m-1} s^n,$$

$$A_4\left[H_{m,n}(x,y)p^m s^n\right] = H_{m+1,n}(x,y)p^{m+1}s^n,$$
$$A_5\left[H_{m,n}(x,y)p^m s^n\right] = nH_{m,n-1}(x,y)p^m s^{n-1},$$
$$A_6\left[H_{m,n}(x,y)p^m s^n\right] = H_{m,n+1}(x,y)p^m s^{n+1}.$$

Regarding the Lie bracket $[,]$ defined by $[A,B] = AB - BA$, we lead to

$[A_1, A_2] = 0,\ [A_2, A_3] =, 0\ [A_3, A_4] = 0,\ [A_4, A_5] = 0,\ [A_5, A_6] = 0,$

$[A_1, A_3] = 0,\ [A_2, A_4] = 0,\ [A_3, A_5] = 0,\ [A_4, A_6] = 0,$

$[A_1, A_4] = 0,\ [A_2, A_5] = 0,\ [A_3, A_6] = 0,$

$[A_1, A_5] = 0,\ [A_2, A_6] = 0,$

$[A_1, A_6] = 0, \quad \text{and} \quad [A_i, A_j] = A_iA_j - A_jA_i,\ \ i,j \in \{1,2,3,4,5,6\}\ i \neq j.$

According to the definition of Lie algebra, the above commutation relations show that the set of operators generate some Lie algebra λ. It is clear that the differential operators:

$$Ł_1 = \frac{1}{\Delta}\left[c(ax+by)\frac{\partial}{\partial x} - b(ax+by)\frac{\partial}{\partial y} - c\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y} + \Delta\right] - p\frac{\partial}{\partial p} - 1, \tag{2.3}$$

and

$$L_2 = \frac{1}{\Delta}\left[-b(bx+cy)\frac{\partial}{\partial x} + a(bx+cy)\frac{\partial}{\partial y} - a\frac{\partial^2}{\partial y^2} + b\frac{\partial^2}{\partial x \partial y} + \Delta\right] - s\frac{\partial}{\partial s} - 1, \tag{2.4}$$

can be expressed as

$$L_1 = A_3A_4 - m - 1 \quad \text{and} \quad L_2 = A_5A_6 - n - 1,$$

such that

$$[L_1, A_i] = 0 \quad \text{and} \quad [L_2, A_i] = 0,\ i = \{1,2,3,4,5,6\}. \tag{2.5}$$

The extended forms of the group generated by $\{A_i : i = 1,2,3,4,5,6\}$ are as follows:

$e^{a_1A_1}f(x,y,p,s) = f(x,y,pe^{a_1},s),$

$e^{a_2A_2}f(x,y,p,s) = f(x,y,p,se^{a_2}),$

$$e^{a_3A_3}f(x,y,p,s) = f\left(x+\tfrac{c}{p\Delta}a_3, y-\tfrac{b}{p\Delta}a_3, p, s\right),$$

$$e^{a_4A_4}f(x,y,p,s) = \exp\left(apxa_4+bpya_4\right)f\left(x-pa_4, y, p, s\right),$$

$$e^{a_5A_5}f(x,y,p,s) = f\left(x-\tfrac{b}{s\Delta}a_5, y+\tfrac{a}{s\Delta}a_5, p, s\right),$$

$$e^{a_6A_6}f(x,y,p,s) = \exp\left(bsxa_6+csya_6\right)f\left(x, y-sa_6, p, s\right).$$

Then, we find that

$$\begin{aligned}\exp&\left(a_6A_6+a_5A_5+a_4A_4+a_3A_3+a_2A_2+a_1A_1\right)f(x,y,p,s)\\ &= \exp\left(a\left(x-\frac{b}{s\Delta}a_5\right)a_4+b(y-sa_6)a_4\right)\\ &\times f\left(x-\frac{b}{s\Delta}a_5-a_4+\frac{c}{p\Delta}a_3, y-sa_6+\frac{a}{s\Delta}a_5-\frac{b}{s\Delta}a_3, pe^{a_1}, se^{a_2}\right).\end{aligned} \tag{2.6}$$

3. Generating functions via Weisnerś method

From the discussion in Section 2, we observe that

$$f(x,y,p,s) = H_{m,n}(x,y)p^m s^n,$$

is a solution of the following systems:

$$\begin{cases} L_1 f = 0 \\ (A_3A_4-m)f = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_2 f = 0 \\ (A_5A_6-n)f = 0. \end{cases}$$

From equation (2.5), we easily see that

$$SL_1\left(H_{m,n}(x,y)p^m s^n\right) = L_1 S\left(H_{m,n}(x,y)p^m s^n\right),$$

and

$$SL_2\left(H_{m,n}(x,y)p^m s^n\right) = L_2 S\left(H_{m,n}(x,y)p^m s^n\right),$$

where $S = exp\left(a_6A_6+a_5A_5+a_4A_4+a_3A_3+a_2A_2+a_1A_1\right)$.

Therefore, the transformation $S(H_{m,n}(x,y)p^m s^n)$ is also annulled by L_1 and L_2.

Proposition 3.1. The following generating equation holds:

$$\begin{aligned}\exp(bxh+cyh)&H_{m,n}\left(x-\frac{b}{s\Delta}g, y-sh+\frac{a}{s\Delta}g\right)p^m s^n\\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{g^l h^k}{l!k!}H_{m,n+k-l}(x,y)p^m s^{n+k-l}.\end{aligned} \tag{3.1}$$

Proof. By putting $a_i = 0, \{i = 1, 2, 3, 4\}, a_5 = g, a_6 = h$ and writing $f(x, y, p, s) = H_{m,n}(x, y)p^m s^n$ in (2.6), we get

$$e^{hA_6}e^{gA_5}(H_{m,n}(x,y)p^m s^n)$$
$$= \exp(bxh + cyh)H_{m,n}\left(x - \frac{b}{s\Delta}g, y - sh + \frac{a}{s\Delta}g\right)p^m s^n. \tag{3.2}$$

Also, we have

$$e^{hA_6}e^{gA_5}(H_{m,n}(x,y)p^m s^n)$$
$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{g^l h^k}{l!k!}\left(H_{m,n+k-l}(x,y)p^m s^{n+k-l}\right). \tag{3.3}$$

Now, combining (3.2) and (3.3), we obtain the generating relation (3.1).□

Now, some special cases of equation (3.1) are of interest. By putting $g = 0, p = s = 1$ in (3.1), we find that

$$\exp(bxh + cyh)H_{m,n}(x, y - h) = \sum_{k=0}^{\infty}\frac{h^k}{k!}H_{m,n+k}(x,y). \tag{3.4}$$

Again, by putting $h = 0, p = s = 1$ in (3.1), we obtain

$$H_{m,n}\left(x - \frac{b}{s\Delta}g, y + \frac{a}{s\Delta}g\right) = \sum_{l=0}^{\infty}\frac{g^l}{l!}H_{m,n-l}(x,y). \tag{3.5}$$

Next, we derive another generating relation.

Proposition 3.2. The following generating equation holds:

$$\exp(axt + byt)H_{m,n}\left(x + \frac{c}{p\Delta}r - tp, y - \frac{b}{p\Delta}r\right)p^m s^n$$
$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{r^l t^k}{l!k!}\left(H_{m+k-l,n}(x,y)p^{m+k-l}s^n\right). \tag{3.6}$$

Proof. By putting $a_i = 0,\ (i = 1, 2, 5, 6),\ a_3 = r,\ a_4 = t$ and writing $f(x, y, s, p) = H_{m,n}(x, y)p^m s^n$ in (2.6), we get

$$e^{tA_4}e^{rA_3}(H_{m,n}(x,y)p^m s^n)$$
$$= \exp(axt + byt)H_{m,n}\left(x + \frac{c}{p\triangle}r - tp, y - \frac{b}{p\triangle}r\right)p^m s^n.$$

Also, we have

$$e^{tA_4}e^{rA_3}(H_{m,n}(x,y)p^m s^n = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{r^l t^k}{l!k!}\left(H_{m+k-l,n}(x,y)p^{m+k-l}s^n\right).$$

Combining the above two equations, we get the desired generating relation (3.6).□

By putting $r=0,\ p=s=1$ in (3.6), we get

$$\exp(axt+byt)H_{m,n}\left(x-t,y\right)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(H_{m+k,n}(x,y)\right). \tag{3.7}$$

On setting $t=0,\ p=s=1$ in (3.6), we obtain

$$H_{m,n}\left(x+\frac{c}{\Delta}r,y-\frac{b}{\Delta}r\right)=\sum_{l=0}^{\infty}\frac{r^l}{l!}\left(H_{m-l,n}(x,y)\right). \tag{3.8}$$

Noteworthy is the following special case. Indeed, by letting $a=2,\ b=-t,\ y=1,$ and $c=m=n=0$, in (3.7), we get the known result due to Miller [15]:

$$\exp\left(2xt-t^2\right)=\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(x). \tag{3.9}$$

4. Generating functions via operational identities

In this section, we will utilize operational identities to obtain generating functions for the Hermite polynomials $H_{m,n}(x,y)$. First of all, if we let $z=ax+by$ and $u=bx+cy$, the series representation (1.5) can then be adjusted as follows:

$$H_{m,n}(z,u)=m!n!\sum_{r=0}^{\left[\frac{m-k}{2}\right]}\sum_{s=0}^{\left[\frac{n-k}{2}\right]}$$

$$\times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{r+s}z^{m-2r-k}u^{n-2s-k}a^rc^s(-b)^k}{2^{r+s}r!s!k!(m-2r-k)!(n-2s-k)!}. \tag{4.1}$$

According to the derivative operator (see [16]):

$$\left(\frac{\partial}{\partial x}\right)^k(ax+by)^m=\frac{a^k}{(m-k)!}(ax+by)^{m-k},$$

and the abbreviations z and u, we can show that

$$\left(\frac{\partial z}{\partial x}\right)^{2r+k}z^m=\frac{a^{2r+k}m!}{(m-2r-k)!}z^{m-2r-k}, \tag{4.2}$$

$$\left(\frac{\partial u}{\partial y}\right)^{2s+k}u^n=\frac{c^{2s+k}n!}{(n-2s-k)!}u^{n-2s-k}, \tag{4.3}$$

$$\left(\frac{\partial u}{\partial x}\right)^{2s+k}u^n=\frac{b^{2s+k}n!}{(n-2r-k)!}u^{n-2s-k}, \tag{4.4}$$

and

$$\left(\frac{\partial z}{\partial y}\right)^{2r+k}z^m=\frac{b^{2r+k}m!}{(m-2r-k)!}z^{m-2r-k}. \tag{4.5}$$

By using the identities (4.2) and (4.3), the series representation (4.1), can be rewritten in the form

$$H_{m,n}(z,u) = \sum_{r=0}^{\left[\frac{m-k}{2}\right]} \sum_{s=0}^{\left[\frac{n-k}{2}\right]}$$

$$\sum_{k=0}^{\min(m,n)} \frac{(-1)^{r+s+k} \left(\frac{\partial z}{\partial x}\right)^{2r+k} z^m \left(\frac{\partial u}{\partial y}\right)^{2s+k} u^n a^r c^s b^k}{2^{r+s} a^{2r+k} c^{2s+k} r! s! k!}, \tag{4.6}$$

which further yields the following operational identity.

Proposition 4.1. The following operational identity holds:

$$H_{m,n}(z,u) = \exp\left[-\frac{\left(\frac{\partial z}{\partial x}\right)^2}{2a} - \frac{\left(\frac{\partial u}{\partial y}\right)^2}{2c} - \frac{b\frac{\partial z}{\partial x}\frac{\partial u}{\partial y}}{ac}\right]\{z^m u^n\}. \tag{4.7}$$

Similarly, using the identities (4.4) and (4.5), we can easily establish the following operational identity.

Proposition 4.2. The following operational identity holds:

$$H_{m,n}(z,u) = \exp\left[-\frac{a\left(\frac{\partial z}{\partial y}\right)^2}{2b^2} - \frac{c\left(\frac{\partial u}{\partial x}\right)^2}{2b^2} - \frac{\frac{\partial z}{\partial y}\frac{\partial u}{\partial x}}{b}\right]\{z^m u^n\}. \tag{4.8}$$

The result of applying the exponential operator

$$\exp\left[\frac{a\left(\frac{\partial z}{\partial y}\right)^2}{2b^2} + \frac{c\left(\frac{\partial u}{\partial x}\right)^2}{2b^2} + \frac{\frac{\partial z}{\partial y}\frac{\partial u}{\partial x}}{b}\right],$$

to both sides of the equation (4.7) is

$$z^m u^n = \exp\left[\frac{1}{2a}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{2c}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{b}{ac}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right] H_{m,n}(z,u)$$

which can be further exploited to derive the following generating relation

$$z^m u^n = (-1)^{\frac{1}{2}(m+n)} \sum_{r=0}^{\left[\frac{m-k}{2}\right]} \sum_{s=0}^{\left[\frac{n-k}{2}\right]} \sum_{k=0}^{\min(m,n)} \frac{(2r+k)!(2s+k)!}{2^{r+s} r! s! k!} \binom{m}{2r+k} \binom{n}{2s+k}$$

$$H_{m-2r-k,n-2s-k}\left(\frac{z}{\sqrt{(-1)}}, \frac{u}{\sqrt{(-1)}}\right) a^r c^s b^k. \tag{4.9}$$

Next, in equation (4.7), multiply throughout by the factor $\frac{t^m h^n}{m!n!}$ and then take the double summation of both the sides, we obtain:

$$\sum_{m,n=0}^{\infty} H_{m,n}(z,u)\frac{t^m h^n}{m!n!} = \exp\left[-\frac{\left(\frac{\partial z}{\partial x}\right)^2}{2a} - \frac{\left(\frac{\partial u}{\partial y}\right)^2}{2c} - \frac{\frac{\partial z}{\partial x}\frac{\partial u}{\partial y}}{ac}\right] \times e^{zt+uh}. \quad (4.10)$$

Yet another generating function from the assertion (4.7) would occur when we multiply both sides of (4.7) by $(\alpha)_m(\beta)_n \frac{t^m h^n}{m!n!}$. If in this case, we take the double sum and then use the binomial theorem, we shall obtain the following generating function:

$$\sum_{m,n=0}^{\infty} (\alpha)_m(\beta)_n H_{m,n}(z,u)\frac{t^m h^n}{m!n!}$$
$$= \exp\left[-\frac{\left(\frac{\partial z}{\partial x}\right)^2}{2a} - \frac{\left(\frac{\partial u}{\partial y}\right)^2}{2c} - \frac{\frac{\partial z}{\partial x}\frac{\partial u}{\partial y}}{ac}\right] \times (1-zt)^{-\alpha}(1-uh)^{-\beta}. \quad (4.11)$$

Similarly, as in the proof of (4.11), we can show that

$$\sum_{m,n=0}^{\infty} (\alpha)_{m+n} H_{m,n}(z,u)\frac{t^m h^n}{m!n!}$$
$$= \exp\left[-\frac{\left(\frac{\partial z}{\partial x}\right)^2}{2a} - \frac{\left(\frac{\partial u}{\partial y}\right)^2}{2c} - \frac{\frac{\partial z}{\partial x}\frac{\partial u}{\partial y}}{ac}\right] \times (1-z-u)^{-\alpha}. \quad (4.12)$$

Additionally, using the operational identity (4.8), we can use an analogous procedure as in the proofs of the formulas (4.10), (4.11), and (412) to obtain three more generating functions as follows:

$$\sum_{m,n=0}^{\infty} H_{m,n}(z,u)\frac{t^m h^n}{m!n!} = \exp\left[-\frac{\left(\frac{\partial z}{\partial y}\right)^2}{2ab^2} - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{2cb^2} - \frac{\frac{\partial u}{\partial x}\frac{\partial z}{\partial y}}{b}\right] \times e^{zt+uh}, \quad (4.13)$$

$$\sum_{m,n=0}^{\infty} (\alpha)_m(\beta)_n H_{m,n}(z,u)\frac{t^m h^n}{m!n!}$$
$$= \exp\left[-\frac{\left(\frac{\partial z}{\partial y}\right)^2}{2ab^2} - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{2cb^2} - \frac{\frac{\partial u}{\partial x}\frac{\partial z}{\partial y}}{b}\right] \times (1-zt)^{-\alpha}(1-uh)^{-\beta}, \quad (4.14)$$

and

$$\sum_{m,n=0}^{\infty} (\alpha)_{m+n} H_{m,n}(z,u)\frac{t^m h^n}{m!n!}$$

$$= \exp\left[-\frac{\left(\frac{\partial z}{\partial y}\right)^2}{2ab^2} - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{2cb^2} - \frac{\frac{\partial u}{\partial x}\frac{\partial z}{\partial y}}{b}\right] \times (1 - zt - uh)^{-\alpha}. \tag{4.15}$$

We can also rewrite the series representation (4.1) as follows:

$$H_{m,n}(z,u) = m!n! \sum_{k=0}^{\min(m,n)} \frac{(-1)^k b^k}{k!(m-k)!(n-k)!}$$

$$\times (m-k)! \sum_{r=0}^{\left[\frac{m-k}{2}\right]} \frac{(-1)^r a^r z^{m-2r-k}}{2^r r!(m-2r-k)!} \times (n-k)! \sum_{s=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^s c^s z^{n-2s-k}}{2^s s!(n-2s-k)!}. \tag{4.16}$$

Now, by the result:

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, \quad n \geq k \geq 0,$$

and the definition of the ordinary Hermite polynomials (see [2]):

$$He_n(x) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!}, \tag{4.17}$$

the assertion (4.16) can be written as

$$H_{m,n}(z,u) = \sum_{k=0}^{\min(m,n)} \frac{(-zub)^k (-m)_k)(-n)_k}{k!(\sqrt{a})^{m-k}(\sqrt{c})^{n-k}} \times He_{m-k}\left(\frac{z}{\sqrt{a}}\right)$$

$$\times He_{n-k}\left(\frac{u}{\sqrt{c}}\right). \tag{4.18}$$

The next generating function is now obtained by starting with (4.18), multiplying both sides by $t^m h^n$, and then taking the double sums:

$$\sum_{m,n=0}^{\infty} H_{m,n}(z,u) t^m h^n = \sum_{k=0}^{\min(m,n)} \sum_{m,n=0}^{\infty} \frac{(-zub)^k (-m)_k (-n)_k (\sqrt{a})^k (\sqrt{c})^k}{k!}$$

$$\times He_{m-k}\left(\frac{z}{\sqrt{a}}\right) \times He_{n-k}\left(\frac{u}{\sqrt{c}}\right) (t\sqrt{a})^m (h\sqrt{c})^n. \tag{4.19}$$

Moreover, according to the fact that

$$(\sqrt{a})^{m-k}\left(1 - \frac{\partial z}{\partial x}\frac{a}{2z}\right)^{m-k} z^{m-k} = He_{m-k}\left(\frac{z}{\sqrt{a}}\right),$$

the generating function (4.19), further yields the interesting generating case:

$$\sum_{m,n=0}^{\infty} H_{m,n}(z,u) t^m h^n$$

$$= \sum_{m,n=0}^{\infty} {}_2F_0 \left[-m, -n; -; \left(1 - \frac{\partial z}{\partial x}\frac{a}{2z}\right)^{-1} \left(1 - \frac{\partial u}{\partial y}\frac{c}{2u}\right)^{-1} \right]$$

$$\times \left(\left(1 - \frac{\partial z}{\partial x}\frac{a}{2z}\right) t \right)^{m} \left(\left(1 - \frac{\partial u}{\partial y}\frac{c}{2u}\right) h \right)^{n}, \tag{4.20}$$

where ${}_2F_0(.)$ is a special case of the generalized hypergeometric function ${}_pF_q$ (see[16]). The previously outlined procedure offers a useful tool for the derivation of other families of generating functions for the polynomials $H_{m,n}(x, y)$. For instance, let

$$z = ax + by, u = bx + cy, \overline{z} = a\tau + b\nu, \overline{u} = b\tau + c\nu$$

and let us consider the generating relation

$$f(z, u, \overline{z}, \overline{u}|t, h) = \sum_{m,n=0}^{\infty} H_{m,n}(z, u) \times H_{m,n}(\overline{z}, \overline{u}) \frac{t^m h^n}{m!n!}, \tag{4.21}$$

which because of the assertion (4.7) yields the following bilinear-generating function

$$\sum_{m,n=0}^{\infty} H_{m,n}(z, u) \times H_{m,n}(\overline{z}, \overline{u}) \frac{t^m h^n}{m!n!} = \exp \left[-\frac{\left(\frac{\partial z}{\partial x}\right)^2}{2a} - \frac{\left(\frac{\partial u}{\partial y}\right)^2}{2c} - \frac{b \frac{\partial z}{\partial x}\frac{\partial u}{\partial y}}{ac} \right]$$

$$\exp \left[-\frac{\left(\frac{\partial \overline{z}}{\partial \tau}\right)^2}{2a} - \frac{\left(\frac{\partial \overline{u}}{\partial \nu}\right)^2}{2c} - \frac{b \frac{\partial \overline{z}}{\partial \tau}\frac{\partial \overline{u}}{\partial \nu}}{ac} \right] \times e^{z\overline{z}t + u\overline{u}h}. \tag{4.22}$$

5. Concluding Remarks

The present paper concerns with the two-variables and two index polynomials $H_{m,n}(x, y)$ introduced by Hermite (see [1] and [12]), whose application within the context of the phase space approach to physical problems is suggested. This paper is devote to discuss with some details the generating relations of the Hermite polynomials $H_{m,n}(x, y)$ by using the Weisnerś group-theoretic and operational representations technique. Accordingly, in Section 3, generating functions involving the Hermite polynomials $H_{m,n}(x, y)$ are obtained by utilizing the outcomes of Section 2. The operational representations for the polynomials$H_{m,n}(x, y)$ are obtained in Section 4 by employing operational calculus methods. These representations are then used to derive generating relations for $H_{m,n}(x, y)$.

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