ON THE TOPOLOGICAL INDICES OF ZERO DIVISOR GRAPHS OF SOME COMMUTATIVE RINGS†

FARIZ MAULANA, MUHAMMAD ZULFIKAR ADITYA, ERMA SUWASTIKA, INTAN MUCHTADI-ALAMSYAH*, NUR IDAYU ALIMON, NOR HANIZA SARMIN

ABSTRACT. The zero divisor graph is the most basic way of representing an algebraic structure as a graph. For any commutative ring R, each element is a vertex on the zero divisor graph and two vertices are defined as adjacent if and only if the product of those vertices equals zero. In this research, we determine some topological indices such as the Wiener index, the edge-Wiener index, the hyper-Wiener index, the Harary index, the first Zagreb index, the second Zagreb index, and the Gutman index of zero divisor graph of integers modulo prime power and its direct product.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Wiener index, hyper-wiener index, Harary index, edge-Wiener index, first Zagreb index, second Zagreb index, Gutman index, zero divisor graph.

1. Introduction

In chemistry, the graph theory has been widely used to solve molecular problems. The structure of molecules can be represented as a graph where atoms are vertices, and the bonds between atoms are edges. Some can be assumed as a complete molecular graph, while others can be viewed as a skeleton graph [1].

There are many applications of graph theory and group theory in chemistry. One of them is topological indices that represent the chemical structure with numerical values. Additionally, the topological index of a structure is useful for chemical documentation, isomer discrimination, structure-property relationships, and others [2].

For instance, the Wiener index (W) is used to predict the cavity surface area (CSA) of alcohols with the equation \( \ln(\text{CSA}) = 5.229 + 0.144\ln(W) \), predict the...
boiling point (BP) of alcohols with the equation \( \ln(bp) = 4.279 + 0.181 \ln(W) \), and predict the molar refraction (MR) of heterogeneous compounds with the equation \( \ln(mr) = 0.826 + 0.690 \ln(W) \) (Gupta, [3]). Many types of topological indices have been introduced since 1947, which include the Wiener index [4, 5] and the Zagreb index [8, 7]. The present study investigated the structure of these indices in a specific graph, namely the zero divisor graph.

The zero divisor graph is the most basic way of representing an algebraic structure as a graph. Here, for any commutative ring \( R \), each element is a vertex on the zero divisor graph and two vertices are defined as adjacent if and only if the product of those vertices equals 0 [10, 9]. Previous studies refer to the zero divisor graph [11, 12] as a useful way of working with algebraic graphs, for example to manipulate each element in \( R \) or to find the relation between two elements.

The simplest ring (or commutative ring) that can be thought of as a fundamental structure is no other than \( \mathbb{Z}_p \), or the modulo ring over a prime number. This modulo ring can be approached using elementary number theory to discover many modulo ring properties and relations. Any commutative ring can be represented by a modulo ring or the multiplication of some modulo rings. Based on this, the zero divisor graph as a complete representation of any commutative ring is interesting to explore [13, 14].

Rayer and Jeyaraj illustrated the zero divisor graph of commutative rings in 2023. They also examined topological indices for zero divisor graphs, focusing on the eccentricity of the vertices [15]. In the same year, Ghazali et al discovered general zeroth-order Randić index of zero divisor graph for the ring of integers modulo \( p^n \) [6].

In this paper, we present the general formula of the Wiener index, hyper-Wiener index, Harary index, edge-Wiener index, first Zagreb index, second Zagreb index and Gutman index of a zero divisor graph using the modulo ring of a prime power \( \mathbb{Z}_{p^n} \) and its direct product \( \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \) for prime numbers \( p, q \) and natural numbers \( n, m \).

The definitions and basic concepts in graph theory and topological indices that are used to prove the main theorem are presented. The zero divisor graph is defined in the following definition.

**Definition 1.1.** [10] Let \( R \) be a commutative ring. The zero divisor graph of \( R \), denoted by \( \Gamma R \), is a simple graph of the vertex set \( R \) and two distinct vertices \( x \) and \( y \) are joined by an edge whenever \( xy = 0 \).

**Definition 1.2.** [4] Let \( G \) be a connected graph. The Wiener index of \( G \) is the sum of the half of the distances between every unordered pair of vertices of \( G \), written as,

\[
W(G) = \sum_{u,v \in V(G)} d(u,v),
\]
where $d(u, v)$ is the distances of unordered pair of vertices $u$ and $v$.

**Example 1.3.** $W(\Gamma_{\mathbb{Z}_{2^2}}) = d(0, 1) + d(0, 2) + d(0, 3) + d(1, 2) + d(1, 3) + d(2, 3) = 1 + 1 + 1 + 2 + 2 + 2 = 9$.

**Definition 1.4.** [16] Let $G$ be a connected graph. The hyper-Wiener index of $G$, denoted by $WW(G)$, is defined as

$$WW(G) = \frac{1}{2} \left( \sum_{u,v \in V(G)} d(u,v) + \sum_{u,v \in V(G)} d(u,v)^2 \right),$$

where $d(u, v)$ is the distances of unordered pair of vertices $u$ and $v$.

**Example 1.5.** $WW(\Gamma_{\mathbb{Z}_{2^2} \times \mathbb{Z}_3}) = \frac{1}{2} \left[ d(0, 1) + d(0, 2) + d(0, 3) + d(1, 2) + d(1, 3) + d(2, 3) + d(0, 1)^2 + d(0, 2)^2 + d(0, 3)^2 + d(1, 2)^2 + d(1, 3)^2 + d(2, 3)^2 \right]$

$$= \frac{1}{2} (1 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 1 + 4 + 4 + 4) = 12.$$
Definition 1.6. [16] Let $G$ be a connected graph. The Harary index of $G$, denoted by $H(G)$, is defined as

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)},$$

where $d(u, v)$ is the distance of unordered pair of vertices $u$ and $v$.

Example 1.7. $H(\Gamma_Z^2) = \frac{1}{d(0,1)} + \frac{1}{d(0,2)} + \frac{1}{d(1,2)} + \frac{1}{d(1,3)} + \frac{1}{d(2,3)} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} = \frac{9}{2}$.

Definition 1.8. [17] Let $G$ be a connected graph. The edge-Wiener index of $G$ is the sum of the distances in the line graph between all pairs of edges of $G$, written as,

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e, f),$$

where $E(G)$ is the set of edges in $G$ and $d(e, f)$ are the distances between two edges. The distances between two edges are the distances between the corresponding vertices in the line graph of $G$, denoted by $L(G)$.

![Figure 3. Zero divisor graph of line graph $\mathbb{Z}_2^2$](image)

Example 1.9. $W_e(\Gamma\mathbb{Z}_2^2) = d((0,1), (0,2)) + d((0,1), (0,3)) + d((0,2), (0,3)) = 1 + 1 + 1 = 3$

Definition 1.10. [7] Let $G$ be a connected graph. Then, the first Zagreb index of $G$ is the sum of squares of the degrees of the vertices of $G$, written as,

$$M_1(G) = \sum_{u \in V(G)} \deg(u)^2$$

where $\deg(u)$ is the number of edges connected to vertex $u$.

Example 1.11. $M_1(\Gamma\mathbb{Z}_2^2) = \deg(0)^2 + \deg(1)^2 + \deg(2)^2 + \deg(3)^2 = 3^2 + 1^2 + 1^2 + 1^2 = 12$. 
Definition 1.12. [7] Let $G$ be a connected graph. Then, the second Zagreb index of $G$ is the sum of the product of the degrees of pairs of adjacent vertices of $G$, written as,
\[ M_2(G) = \sum_{u,v\in E(G)} \text{deg}(u)\text{deg}(v) \]
where $u, v$ are the vertices on the edge connecting them.

Example 1.13. $M_2(\Gamma\mathbb{Z}_{2^2}) = \text{deg}(0)\text{deg}(1) + \text{deg}(0)\text{deg}(2) + \text{deg}(0)\text{deg}(3) = 3.1 + 3.1 + 3.1 = 9$.

Definition 1.14. [18] Let $G$ be a connected graph. The Gutman index of $G$, denoted by $\text{Gut}(G)$, written as,
\[ \text{Gut}(G) = \sum_{\{u,v\} \subseteq V(G)} \text{deg}(u)\text{deg}(v)d(u,v) \]

Example 1.15. $\text{Gut}(\Gamma\mathbb{Z}_{2^2}) = \text{deg}(0)\text{deg}(1)d(0,1) + \text{deg}(0)\text{deg}(2)d(0,2) + \text{deg}(0)\text{deg}(3)d(0,3) + \text{deg}(1)\text{deg}(2)d(1,2) + \text{deg}(1)\text{deg}(3)d(1,3) + \text{deg}(2)\text{deg}(3)d(2,3) = 3.1.1 + 3.1.1 + 3.1.1 + 1.1.2 + 1.1.2 + 1.1.2 = 15$.

2. Main results

In this section, the Wiener index, the hyper-Wiener index, the Harary index, the edge-Wiener index, the first Zagreb index, and the second Zagreb index of the zero divisor graph of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ are determined, where their general forms in natural number $n$ and and prime number $p$ are found.

2.1. Topological indices of the graph that has diameter at most 2.

This subsection contains the Wiener index, the hyper-Wiener index, the Harary index, edge-Wiener index and Gutman index of graph that has diameter 2.

In terms of their numbers of vertices and number of edges, these results are used to prove the topological indices of zero divisor graphs of the commutative rings in the following subsections.

Theorem 2.1. Let $G$ be a simple connected graph with $\text{diam}(G) \leq 2$, then the Wiener index of $G$ is $|V(G)|(|V(G)| - 1) - |E(G)|$.

Proof. Since $\text{diam}(G) \leq 2$, the number of unordered pairs of vertices in $G$ that have distance 2 is,
\[ \binom{|V(G)|}{2} - |E(G)|. \]
Hence, the Wiener index of $G$ is,
\[ W(G) = |E(G)| + 2\left( \binom{|V(G)|}{2} - |E(G)| \right) \]
Theorem 2.2. Let $G$ be a simple connected graph with $\text{diam}(G) \leq 2$, then the hyper-Wiener index of $G$ is \[\frac{3}{2}|V(G)||(|V(G)| - 1) - 2|E(G)|.\]

Proof. Since $\text{diam}(G) \leq 2$, the number of unordered pairs of vertices in $G$ that have distance 2 is, \[\binom{|V(G)|}{2} - |E(G)|.\]

Hence, the hyper-Wiener index of $G$ is, \[WW(G) = \frac{1}{2}|E(G)| + 2\left(\binom{|V(G)|}{2} - |E(G)|\right) - |E(G)| = \frac{3}{2}|V(G)||(|V(G)| - 1) - 2|E(G)|.\]

Theorem 2.3. Let $G$ be a simple connected graph with $\text{diam}(G) \leq 2$, then the Harary index of $G$ is \[\frac{1}{4}|V(G)||(|V(G)| - 1) + \frac{1}{2}|E(G)|.\]

Proof. Since $\text{diam}(G) \leq 2$, the number of unordered pairs of vertices in $G$ that have distance 2 is, \[\binom{|V(G)|}{2} - |E(G)|.\]

Hence, the Harary index of $G$ is, \[H(G) = |E(G)| + \frac{1}{2}\left(\binom{|V(G)|}{2} - |E(G)|\right) = \frac{1}{4}|V(G)||(|V(G)| - 1) + \frac{1}{2}|E(G)|.\]

If diameter of the line graph is at most 2, then its edge-Wiener index in terms of number of edges and its first Zagreb index. This following lemma shows that.

Theorem 2.4. Let $G$ be a graph. If line graph of $G$ is connected and has diameter at most 2, then $W_e(G) = |E(G)|^2 - \frac{1}{2}M_1(G)$.

Proof. Two edges $e, f$ in a graph $G$ will be adjacent in $L(G)$ if both $e$ and $f$ share one same vertex in $G$. So every vertex $v$ in $G$ will have $\text{deg}(v)$ edges sharing the vertex $v$ and create \(\binom{\text{deg}(v)}{2}\) different edges in $L(G)$. Hence, we have \[|E(L(G))| = \sum_{v \in V(G)} \binom{\text{deg}(v)}{2}\]
By using Lemma 2.1, we have
\[ W_e(G) = W(L(G)) = |V(L(G))| (|V(L(G))| - 1) - |E(L(G))| = |E(G)| (|E(G)| - 1) - \frac{1}{2} M_1(G) + |E(G)| = |E(G)|^2 - \frac{1}{2} M_1(G). \]

If the graph has diameter at most 2, then there is a relationship between its Gutman index and its Zagreb indices. It shows in the following lemma.

**Theorem 2.5.** Let \( G \) be a simple connected graph with \( \text{diam}(G) \leq 2 \), then the Gutman index of \( G \) is
\[ \text{Gut}(G) = 4|E(G)|^2 - M_1(G) - M_2(G). \]

**Proof.** Note that,
\[ 4|E(G)|^2 = \left( \sum_{u \in V(G)} \text{deg}(u) \right)^2 = \sum_{u \in V(G)} \text{deg}(u)^2 + \sum_{uv \in E(G)} \text{deg}(u)\text{deg}(v) + 2 \sum_{uv \notin E(G)} \text{deg}(u)\text{deg}(v) = M_1(G) + 2M_2(G) + 2 \sum_{uv \notin E(G)} \text{deg}(u)\text{deg}(v). \]

Since \( \text{diam}(G) \leq 2 \), then,
\[ \text{Gut}(G) = \sum_{uv \in E(G)} \text{deg}(u)\text{deg}(v)d(u, v) + \sum_{uv \notin E(G)} \text{deg}(u)\text{deg}(v)d(u, v) = \sum_{uv \in E(G)} \text{deg}(u)\text{deg}(v) + 2 \sum_{uv \notin E(G)} \text{deg}(u)\text{deg}(v) = M_2(G) + 4|E(G)|^2 - M_1(G) - 2M_2(G) = 4|E(G)|^2 - M_1(G) - M_2(G). \]
2.2. Topological Indices of the Zero Divisor Graph of $\mathbb{Z}_p^n$.

This subsection contains the results for the Wiener index, the hyper-Wiener
index, the Harary index, the edge-Wiener index, the first Zagreb index, the
second Zagreb index and Gutman index for $\Gamma_{\mathbb{Z}_p^n}$ for prime number $p$ and $n \in \mathbb{N}$.

Before getting into a deep understanding of how to determine any topological
index result, the first thing to know is the graph’s structure. The neighbors of
each vertex are fundamental for seeing the structure preserved in that graph.

**Lemma 2.6.** If $R$ is a commutative ring, then the diameter of $\Gamma R$ is at most 2.

**Proof.** Let $a, b$ be different vertices in $\Gamma R$ with $ab \neq 0$. Since $a.0 = 0$ and $b.0 = 0,$
$a - 0 - b$ is a path of length 2. □

**Proposition 2.7.** Let $p$ be a prime number, $n \in \mathbb{N}$ and $a \in \mathbb{Z}_{p^n}$ with $\gcd(a, p^n) = p^i$ for $i = 0, 1, 2, ..., n$. Then, the degree of $a$ in $\Gamma_{\mathbb{Z}_p^n}$ is
\[
\deg(a) = \begin{cases} 
p^i, & i \leq \left\lfloor \frac{n-1}{2} \right\rfloor 
p^n - 1, & i > \left\lfloor \frac{n-1}{2} \right\rfloor \end{cases}
\]

**Proof.** Let $a \in \mathbb{Z}_{p^n}$ with $\gcd(a, p^n) = p^i$, and $b \in \mathbb{Z}_{p^n}$ with $\gcd(b, p^n) = p^j$. Then, $a$ is adjacent to $b$ if and only if $j \geq n - i$. So $b \in p^{n-i}\mathbb{Z}_{p^n}$ and $|p^{n-i}\mathbb{Z}_{p^n}| = p^i$.

- If $i > \left\lfloor \frac{n-1}{2} \right\rfloor$, then $a \in p^{n-i}\mathbb{Z}_{p^n}$. So $\deg(a) = p^i - 1$.
- If $i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, then $\deg(a) = p^i$.

□

In the zero divisor graph of any commutative ring, some of the vertices have
the same degree. The following proposition shows the number of vertices that
have the same degree in the respective graphs.

**Proposition 2.8.** Let $V_i = \{a \in \mathbb{Z}_{p^n} : \gcd(a, p^n) = p^i\}$, then $|V_i| = p^{n-i} - p^{n-(i+1)}$ for $0 \leq i \leq n - 1$ and $|V_i| = 1$ for $i = n$.

**Proof.** Let $a \in V_i$. For $0 \leq i \leq n - 1$, we have $a \in p^i\mathbb{Z}_{p^n}$, but $a \notin p^{i+1}\mathbb{Z}_{p^n}$. So $|V_i| = p^{n-i} - p^{n-(i+1)}$. For the case $i = n$, $V_i = \{0\}$ and immediately $|V_i| = 1$. □

Before we determine the Wiener index, Hyper-Wiener index, Harary index,
edge-Wiener index and Gutman index, we need to determine the number of
edges. The number of edges shows in the following lemma.

**Lemma 2.9.** The number of edges of $\Gamma_{\mathbb{Z}_p^n}$ is
\[
\frac{1}{2} \left( (n+1)p^n - np^{n-1} - p^{\left\lfloor \frac{n+1}{2} \right\rfloor} \right).
\]

**Proof.** The number of edges in any graph will equal half of the sum of the degrees
degrees of all vertices. Using Proposition 2.7 and Proposition 2.8, we have
\[
|E(\Gamma_{\mathbb{Z}_p^n})| = \frac{1}{2} \sum_{a \in V(\Gamma_{\mathbb{Z}_p^n})} \deg(a)
\]
Proof. Using Lemma 2.9 and Theorem 2.1 that state
\( H(S) = 1 \cdot (p^n - 1 + \sum_{i=0}^{n-1} (p^{n-i} - p^{n-(i+1)})p^i + \sum_{i=1+\left\lfloor \frac{n+1}{2} \right\rfloor}^{n-1} (p^{n-i} - p^{n-(i+1)})(p^i - 1)) \)
\[ = \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor. \]

\[ \square \]

**Theorem 2.10.** Let \( \Gamma_{Z_p^n} \) be the zero divisor graph of \( Z_p^n \) with prime number \( p \) and natural number \( n \). Then, the Wiener index of \( \Gamma_{Z_p^n} \) is
\[ W(\Gamma_{Z_p^n}) = p^n(p^n - 1) - \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor. \]

Proof. Using Lemma 2.9 and Theorem 2.1 that state \( |V(\Gamma_{Z_p^n})| = p^n \) and \( |E(\Gamma_{Z_p^n})| = \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor \), the result follows. \( \square \)

**Example 2.11.** \( W(\Gamma_{Z_{22}}) = 2^2(2^2 - 1) - \frac{1}{2} (2 + 1)2^2 - 2.2^{2-1} - 2^\left\lfloor \frac{2+1}{2} \right\rfloor = 9. \)

**Theorem 2.12.** Let \( \Gamma_{Z_p^n} \) be the zero divisor graph of \( Z_p^n \) with prime number \( p \) and natural number \( n \). Then, the hyper-Wiener index of \( \Gamma_{Z_p^n} \) is
\[ WW(\Gamma_{Z_p^n}) = 3 \cdot \frac{3}{2} p^n(p^n - 1) - \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor. \]

Proof. Using Lemma 2.9 and Theorem 2.2 that state \( |V(\Gamma_{Z_p^n})| = p^n \) and \( |E(\Gamma_{Z_p^n})| = \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor \), the result follows. \( \square \)

**Example 2.13.** \( WW(\Gamma_{Z_{22}}) = 3 \cdot \frac{3}{2} 2^2(2^2 - 1) - \left( (2 + 1)2^2 - 2.2^{2-1} - 2^\left\lfloor \frac{2+1}{2} \right\rfloor \right) = 12. \)

**Theorem 2.14.** Let \( \Gamma_{Z_p^n} \) be the zero divisor graph of \( Z_p^n \) with prime number \( p \) and natural number \( n \). Then, the Harary index of \( \Gamma_{Z_p^n} \) is
\[ H(\Gamma_{Z_p^n}) = \frac{1}{4} p^n(p^n - 1) + \frac{1}{4} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor. \]

Proof. Using Lemma 2.9 and Theorem 2.3 that state \( |V(\Gamma_{Z_p^n})| = p^n \) and \( |E(\Gamma_{Z_p^n})| = \frac{1}{2} (n + 1)p^n - np^{n-1} - p^\left\lfloor \frac{n+1}{2} \right\rfloor \), the result follows. \( \square \)

**Example 2.15.** \( H(\Gamma_{Z_{22}}) = \frac{1}{4} 2^2(2^2 - 1) + \frac{1}{4} (2 + 1)2^2 - 2.2^{2-1} - 2^\left\lfloor \frac{2+1}{2} \right\rfloor = \frac{9}{2}. \)

Before we determine the edge-Wiener index of \( \Gamma_{Z_p^n} \), we need to determine the first Zagreb index.

**Theorem 2.16.** Let \( \Gamma_{Z_p^n} \) be the zero divisor graph of \( Z_p^n \) with prime number \( p \) and natural number \( n \). Then, the first Zagreb index of \( \Gamma_{Z_p^n} \) is
\[ M_1(\Gamma_{Z_p^n}) = (p^n - 1)^2 + (p^n - p^{n-1}) \left( \frac{(p^n - 1)^2 - 2\left\lfloor \frac{n-1}{2} \right\rfloor}{p^n - 1} \right) + p^\left\lfloor \frac{n+1}{2} \right\rfloor - 1. \]
Proof. By Proposition 2.7 and Proposition 2.8, we have,
\[ M_1(\Gamma Z_{p^n}) = (p^n - 1)^2 + \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (p^{n-i} - p^{(n+1)-(i+1)})(p^i)^2 + \sum_{i=1+\left\lceil \frac{n-1}{2} \right\rceil}^{n-1} (p^{n-i} - p^{(n+1)-(i+1)})(p^{2i} - 2p^i + 1). \]
= \( (p^n - 1)^2 + \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (p^{n-i} - p^{(n+1)-(i+1)})(p^i)^2 + \sum_{i=1+\left\lceil \frac{n-1}{2} \right\rceil}^{n-1} (p^{n-i} - p^{(n+1)-(i+1)})(p^{2i} - 2p^i + 1). \]
\[ = (p^n - 1)^2 + (p^n - p^{n-1}) \left( \frac{p^{n-1}}{p-1} - 2\left\lfloor \frac{n-1}{2} \right\rfloor \right) + p^{\left\lceil \frac{n+1}{2} \right\rceil} - 1. \]
\[ \square \]

Example 2.17. \( M_1(\Gamma Z_{2^2}) = (2^2 - 1)^2 + (2^2 - 2^{2-1}) \left( \frac{2^{2-1}}{2-1} - 2\left\lfloor \frac{2-1}{2} \right\rfloor \right) + 2^\left\lceil \frac{2+2}{2} \right\rceil - 1 = 12. \)

Theorem 2.18. Let \( \Gamma Z_{p^n} \) be the zero divisor graph of \( \mathbb{Z}_{p^n} \) with prime number \( p \) and natural number \( n \). Then, the edge-Wiener index of \( \Gamma Z_{p^n} \) is 
\[ W_e(\Gamma Z_{p^n}) = \frac{1}{4} \left( (n+1)p^n - np^{n-1} - p^{\left\lceil \frac{n+1}{2} \right\rceil} \right)^2 - \frac{1}{2} \left( p^n - 1 \right)^2 + \frac{1}{2} \left( (p^n - p^{n-1}) \left( \frac{p^{n-1}}{p-1} - 2\left\lfloor \frac{n-1}{2} \right\rfloor \right) + p^{\left\lceil \frac{n+1}{2} \right\rceil} - 1 \right). \]

Proof. For any prime numbers \( p \) and \( n \geq 1 \), let \( a, b, c, d \in \mathbb{Z}_{p^n} \) be all different vertices in \( \Gamma Z_{p^n} \) such that \( ab = cd = 0 \). Then, \( (a, b), (c, d) \in E(\Gamma Z_{p^n}) \) and it follows that \( (a, b), (c, d) \) are different vertices in \( L(\Gamma Z_{p^n}) \).

Let \( 0 \leq i, j, k, l \leq n \) be integers that satisfy \( \gcd(a, p^n) = p^i \), \( \gcd(b, p^n) = p^j \), \( \gcd(c, p^n) = p^k \), and \( \gcd(d, p^n) = p^l \). Note that \( i + j \geq n \) and \( k + l \geq n \), so \( \max\{i, j\} + \max\{k, l\} \geq n \). Choose \( x \in \{a, b\} \) and \( y \in \{c, d\} \) that satisfy \( \gcd(x, p^n) = p^{\max(i, j)} \) and \( \gcd(y, p^n) = p^{\max(k, l)} \). Hence, \( (x, y) \in E(\Gamma Z_{p^n}) \) and \( (a, b) - (x, y) - (c, d) \) is a path of length 2. So \( \text{diam}(L(\Gamma Z_{p^n})) \leq 2 \).

Using Theorem 2.4, we have,
\[ W_e(\Gamma Z_{p^n}) = |E(\Gamma Z_{p^n})|^2 - \frac{1}{2} M_1(\Gamma Z_{p^n}) \]

Using Lemma 2.9 and Theorem 2.16, the result follows. \[ \square \]

Example 2.19. \( W_e(\Gamma Z_{2^2}) = \frac{1}{4} \left( (2+1)2^2 - 2.2^2 - 1 - 2\left\lceil \frac{2+1}{2} \right\rceil \right)^2 - \frac{1}{2} \left( (2^2 - 1)^2 + (2^2 - 2^{2-1}) \left( \frac{2^{2-1}}{2-1} - 2\left\lfloor \frac{2-1}{2} \right\rfloor \right) + 2^{\left\lceil \frac{2+2}{2} \right\rceil} - 1 \right) = 3. \)

Theorem 2.20. Let \( \Gamma Z_{p^n} \) be the zero divisor graph of \( \mathbb{Z}_{p^n} \) with prime number \( p \) and natural number \( n \). Then, the second Zagreb index of \( \Gamma Z_{p^n} \) is
\[ M_2(Z_{p^n}) = (p^n - 1) \left( n(p^n - p^{n-1}) - p^{\left\lceil \frac{n+1}{2} \right\rceil} + 1 \right) + \frac{1}{2} (p^{\left\lceil \frac{n+1}{2} \right\rceil} - 1) (p^{\left\lceil \frac{n+1}{2} \right\rceil} - 2) + \frac{1}{2} (p^n - p^{n-1}) \left( \frac{n(n-1)}{2} + p^n - p^{n-1}) + 2n + 2\left\lceil \frac{n-1}{2} \right\rceil - 1 \right) - \frac{1}{2} (p^n - p^{n-1}) \left( \frac{n(n-1)}{2} + p^n - p^{n-1}) + 2n + 2\left\lceil \frac{n-1}{2} \right\rceil - 1 \right) - np^{\left\lceil \frac{n+1}{2} \right\rceil}. \]
Proof. By the second Zagreb index’s definition, it is obvious that

$$M_2(\Gamma G) = \frac{1}{2} \sum_{a \in V(\Gamma G)} \left( \deg(a) \sum_{b \in N(a)} \deg(b) \right),$$

where $N(a)$ denotes the neighborhood of $a$. Note that, if $a \in V(\Gamma Z_{p^n})$ with $\gcd(a, p^n) = p^i$, then $N(a) = \{ b \in V(\Gamma Z_{p^n}) : \gcd(b, p^n) = p^{n-j}, 0 \leq j \leq i \}$. Hence we have,

$$M_2(\Gamma Z_{p^n}) = \frac{1}{2} \left( \sum_{a \in V(\Gamma Z_{p^n})} \deg(a) \left( \sum_{b \in N(a)} \deg(b) \right) \right)$$

$$= \frac{1}{2} (p^n - 1) \left( (n+1)p^n - np^{n-1} - p^{\left\lceil \frac{n-1}{2} \right\rceil} - (p^n - 1) \right) + \frac{1}{2} \left( \sum_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (p^{n-i} - p^{n-(i+1)})(p^i - 1) \left( \sum_{j=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (p^j - p^{j-1})(p^{n-j} - 1) + p^n - 1 \right) + \frac{1}{2} (p^n - 1) \left( n(p^n - p^{n-1}) - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right)$$

$$+ \frac{1}{2} \left( \sum_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (p^{n-i} - p^{n-(i+1)})p^i + \frac{1}{2} \sum_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (p^{n-i} - p^{n-(i+1)})(p^i - 1) \right) (p^n - 1) + \frac{1}{2} \left( \sum_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (p^{n-i} - p^{n-(i+1)})(p^i - 1) \right) (p^n - 1)$$

$$+ \frac{1}{2} \left( \sum_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (p^{n-i} - p^{n-(i+1)}) (p^n - 1) + 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} - p^i \right)$$

$$= (p^n - 1) \left( n(p^n - p^{n-1}) - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) + \frac{1}{2} \left( p^n - (p^n - 1) \left( n - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) \left( 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} \right) \right)$$

$$+ \frac{1}{2} \left( p^n - p^{n-1} \right) \left( p^n - p^{n-1} \right) \left( n - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) \left( 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} \right)$$

$$= (p^n - 1) \left( n(p^n - p^{n-1}) - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) + \frac{1}{2} \left( p^n - p^{n-1} \right) \left( n - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) \left( 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} \right)$$

$$+ \frac{1}{2} \left( p^n - p^{n-1} \right) \left( n - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) \left( 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} \right)$$

$$= (p^n - 1) \left( n(p^n - p^{n-1}) - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) + \frac{1}{2} \left( p^n - p^{n-1} \right) \left( n - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 1 \right) \left( 1 - p^{\left\lceil \frac{n-1}{2} \right\rceil} \right).$$

\(\square\)
Example 2.21. Let \( Z_p^n \) be the zero divisor graph of \( Z_p^n \) with prime number \( p \) and natural number \( n \). Then, the Gutman index of \( Z_p^n \) is
\[
\text{Gut}(Z_p^n) = (n+1)p^n - np^{n-1} - p^{\frac{n-1}{2}} - \left( p^n - 1 \right) + \frac{1}{2} \left( (n^n - p^n - 1) - p^{\frac{n-1}{2}} \right) + \frac{1}{2} \left( (n^n - p^n - 1) - p^{\frac{n-1}{2}} \right) = 9.
\]

Theorem 2.22. Let \( \Gamma Z_{p^n} \) be the zero divisor graph of \( Z_{p^n} \) with prime number \( p \) and natural number \( n \). Then, the degree of \( a, b \) in the zero divisor graph of \( Z_{p^n} \times Z_{q^m} \) is
\[
\text{deg}(a, b) = \left\{ \begin{array}{ll}
p^i q^j, & i = 0, 1, 2, ..., n, j \leq \left\lfloor \frac{n-1}{2} \right\rfloor \
p^i q^j - 1, & i > \left\lfloor \frac{n-1}{2} \right\rfloor, j \geq \left\lfloor \frac{n-1}{2} \right\rfloor\end{array} \right.
\]

Proposition 2.23. Let \( p, q \) be prime numbers, \( n, m \in \mathbb{N} \) and \( (a, b) \in Z_{p^n} \times Z_{q^m} \). Then, the degree of \( a, b \) in the zero divisor graph of \( Z_{p^n} \times Z_{q^m} \) is
\[
\text{deg}(a, b) = \left\{ \begin{array}{ll}
p^i q^j, & i = 0, 1, 2, ..., n, j \leq \left\lfloor \frac{n-1}{2} \right\rfloor \
p^i q^j - 1, & i > \left\lfloor \frac{n-1}{2} \right\rfloor, j \geq \left\lfloor \frac{n-1}{2} \right\rfloor\end{array} \right.
\]

Proposition 2.24. Let \( V'_{ij} = \{(a, b) \in Z_{p^n} \times Z_{q^m} : \text{gcd}(a, p^n) = p^i, \text{gcd}(b, q^m) = q^j \} \), then
\[
|V'_{ij}| = \left\{ \begin{array}{ll}
p^{n-i} - p^{n-i-1}(q^{m-j} - q^{m-j-1}), & i = 0, 1, 2, ..., n-1, j = 0, 1, 2, ..., m-1 \\
p^{n-i} - p^{n-i-1}, & i = 0, 1, 2, ..., n-1, j = m \\
q^{m-j} - q^{m-j-1}, & j = 0, 1, 2, ..., m-1 \\
1, & i = n, j = m\end{array} \right.
\]

Proof. Since \( |\{(a, b) \in Z_{p^n} \times Z_{q^m} : \text{gcd}(a, p^n) = p^i, \text{gcd}(b, q^m) = q^j \}| = |\{a \in Z_{p^n} : \text{gcd}(a, p^n) = p^i\}| \times |\{b \in Z_{q^m} : \text{gcd}(b, q^m) = q^j\}| \), the proof follows from Proposition 2.8.
Lemma 2.25. The number of edges of $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})$ is $\frac{1}{2} \left( nm(p^n - p^{n-1})(q^m - q^{m-1}) + m(q^m - q^{m-1})p^n + n(p^n - p^{n-1})q^m + p^n q^m - p^{\lceil \frac{n-1}{2} \rceil} q^{\lceil \frac{m-1}{2} \rceil} \right)$.

Proof. The number of edges in any graph will be equal to half of the sum of the degrees of all vertices in the graph. By using Proposition 2.23 and Proposition 2.24, we have,

\[ \sum_{a \in V(\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}))} \deg(a) = \sum_{i=0}^{n-1} \left( \sum_{j=0}^{m-1} (p^n - p^{n-1})(q^m - q^{m-1}) + \frac{m}{2} (q^m - q^{m-1})p^n + \frac{n}{2} (p^n - p^{n-1})q^m \right) + \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (p^n - p^{n-1})(q^m - q^{m-1}) + m(q^m - q^{m-1})p^n + n(p^n - p^{n-1})q^m + p^n q^m - p^{\lceil \frac{n-1}{2} \rceil} q^{\lceil \frac{m-1}{2} \rceil} + p^n q^m. \]

Theorem 2.26. Let $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})$ be the zero divisor graph of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ with prime numbers $p, q$ and natural numbers $n, m$. Then, the Wiener index of $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})$ is $W(\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})) = p^n q^m(p^n - p^{n-1})(q^m - q^{m-1}) + m(q^m - q^{m-1})p^n + n(p^n - p^{n-1})q^m + p^n q^m - p^{\lceil \frac{n-1}{2} \rceil} q^{\lceil \frac{m-1}{2} \rceil} + p^n q^m$.

Proof. This is clear by using Lemma 2.1 and Theorem 2.25. □

Example 2.27. $W(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)) = 2^2 \cdot 3^1(2^2 - 3^1 - 1) + 1(3^1 - 3^1 - 1)2^2 + 2(2^2 - 2^1 - 1)3^1 + 2^2 \cdot 3^1 - 2^2 \cdot 3^1 \cdot 3^1 = 113.$
Theorem 2.28. Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ be the zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$ with prime numbers $p, q$ and natural numbers $n, m$. Then, the hyper-Wiener index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $WW(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \frac{3}{2}p^n q^m (p^n q^m - 1) - (nm(p^n - p^{n-1}))(q^m - q^{m-1}) + m(q^m - q^{m-1})p^n + n(p^n - p^{n-1})q^m + p^n q^m - p^\left(\frac{n+1}{2}\right)q^\left(\frac{m+1}{2}\right)$.

Proof. This is clear by using Theorem 2.2 and Lemma 2.25. \square

Example 2.29. $WW(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)) = \frac{3}{2}2^2 3^1(2^2 3^1 - 1) - \left[2.1(2^2 - 2^{2-1})(3^1 - 3^{1-1}) + 1(3^1 - 3^{1-1})2^2 + 2(2^2 - 2^{2-1})3^1 + 2^2 3^1 - 2^2 2^{2-1}3^1\right] = 160$.

Theorem 2.30. Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ be the zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$ with prime numbers $p, q$ and natural numbers $n, m$. Then, the Harary index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $H(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \frac{1}{2}p^n q^m (p^n q^m - 1) + \frac{1}{2}(nm(p^n - p^{n-1}))(q^m - q^{m-1}) + m(q^m - q^{m-1})p^n + n(p^n - p^{n-1})q^m + p^n q^m - p^\left(\frac{n+1}{2}\right)q^\left(\frac{m+1}{2}\right)$.

Proof. This is clear by using Theorem 2.3 and Lemma 2.25. \square

Example 2.31. $H(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)) = \frac{1}{2}2^2 3^1(2^2 3^1 - 1) + \frac{1}{2}\left[2.1(2^2 - 2^{2-1})(3^1 - 3^{1-1}) + 1(3^1 - 3^{1-1})2^2 + 2(2^2 - 2^{2-1})3^1 + 2^2 3^1 - 2^2 2^{2-1}3^1\right] = \frac{85}{2}$.

Theorem 2.32. Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ be the zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$ with prime numbers $p, q$ and natural numbers $n, m$. Then, the first Zagreb index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $M_1(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = (p^{2n-1} - p^{n-1})(q^{2m-1} - q^{m-1}) + p^{2n}(q^{2m-1} - q^{m-1}) + q^{2m}(p^{2n-1} - p^{n-1}) - 2[p^{n-1}/2][q^{m-1}/2](p^n - p^{n-1})(q^m - q^{m-1}) + (p^{n+1}/2 - 1)(q^{m+1}/2 - 1) - 2[p^{n-1}/2](p^n - p^{n-1})q^m + (p^{n+1}/2 - 1)(q^{m+1}/2 - 1) - (p^n q^m - p^{n-1})q^m + q^{m-1}p^n + q^{m-1}p^n$.

Proof. Using Proposition 2.23 and Proposition 2.24, we have, $M_1(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \sum_{a \in V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))} \deg(a)^2$.
Example 2.33. \( M_1(\Gamma(Z_{2^2} \times Z_3)) = (2^{2^2-1} - 2^{2-1}) (3^{2^2-1} - 3^{2-1}) + 2^{2^2} (3^{2^2-1} - 3^{2-1}) + 3^{2^2} (2^{2^2-1} - 2^{2-1}) - 2 [\frac{2^2}{2^2-1}] (2^{2^2} - 2^{2-1}) (3^{2^2} - 3^{2-1}) + (2^{2^2} - 1) (3^{2^2} - 1) - 2 [\frac{2^2}{2^2-1}] (2^{2^2} - 2^{2-1}) 3^{1} + (2^{2^2} - 1) - 2 [\frac{2^2}{2^2-1}] (3^{2^2} - 3^{2-1}) 2^{2} + (3^{2^2} - 1) + (2^2, 3^2 - 1)^2 = 6.2 + 16.2 + 9.6 - 0 + 0 - 2 - 1.2 + 3 + 1 - 0 + 0 + 121 = 208.\)

Theorem 2.34. Let \( \Gamma(Z_{p^n} \times Z_{q^m}) \) be the zero divisor graph of \( Z_{p^n} \times Z_{q^m} \) with prime numbers \( p, q \) and natural numbers \( n, m \). Then, the edge-Wiener index of \( \Gamma(Z_{p^n} \times Z_{q^m}) \) is \( W_e(\Gamma(Z_{p^n} \times Z_{q^m})) = |E(\Gamma(Z_{p^n} \times Z_{q^m}))|^2 - \frac{1}{2} M_1(\Gamma(Z_{p^n} \times Z_{q^m})) \).

Proof. Let \((a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2) \in Z_{p^n} \times Z_{q^m}\) be all different vertices in \( \Gamma(Z_{p^n} \times Z_{q^m}) \) such that \((a_1, a_2)(b_1, b_2) = (c_1, c_2)(d_1, d_2) = (0, 0)\).

Then \((a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2) \in E(\Gamma(Z_{p^n} \times Z_{q^m}))\) and follows that \((a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2) \) are different vertices in \( L(\Gamma(Z_{p^n}))\).

Let \( 0 \leq i, j, k, l \leq n \) and \( 0 \leq i', j', k', l' \leq m \) be integers that satisfy \( \gcd(a_1, p^n) = p^i, \gcd(a_2, q^m) = q^{j'}, \gcd(b_1, p^n) = p^{i'}, \gcd(b_2, q^m) = q^{j}, \gcd(c_1, p^n) = p^k, \gcd(c_2, q^m) = q^{k'}, \gcd(d_1, p^n) = p^{l}, \gcd(d_2, q^m) = q^{l'}.\)

Note that \( i + j \geq n, \ i' + j' \geq m \) and \( k + l \geq n, \ k' + l' \geq m \), so \( \max\{i, j\} + \max\{k, l\} \geq n \) and \( \max\{i', j'\} + \max\{k', l'\} \geq m.\)

Choose \( x_1 \in \{a_1, b_1\}, x_2 \in \{a_2, b_2\} \) and \( y_1 \in \{c_1, d_1\}, y_2 \in \{c_2, d_2\} \) that satisfy \( \gcd(x_1, p^n) = p^{\max\{i, j\}}, \gcd(x_2, q^m) = q^{\max\{i', j'\}} \) and \( \gcd(y_1, p^n) = p^{\max\{k, l\}}, \gcd(y_2, q^m) = q^{\max\{k', l'\}}\).

Hence \((x_1, x_2, (y_1, y_2)) \in \Gamma(Z_{p^n} \times Z_{q^m})\) and \((a_1, a_2, (b_1, b_2)) - (x_1, x_2, y_1, y_2) - ((c_1, c_2), (d_1, d_2)) \) is a path with length 2. So \( \text{diam}(L(\Gamma(Z_{p^n} \times Z_{q^m}))) \leq 2.\)

By Theorem 2.4, we have,

\[
W_e(\Gamma(Z_{p^n} \times Z_{q^m})) = |E(\Gamma(Z_{p^n} \times Z_{q^m}))|^2 - \frac{1}{2} M_1(\Gamma(Z_{p^n} \times Z_{q^m})).
\]
In this paper, we have determined the Wiener index, the edge Wiener index, the Harary index, the first Zagreb index, the second Zagreb index and the Gutman index of the zero divisor graph of $Z_p^n$ and the zero divisor graph of $Z_p^n \times Z_q^m$. In future research, we will determine the topological indices of the zero divisor graph of $Z_p^{k_1} \times Z_p^{k_2} \times ... \times Z_p^{k_m}$ for $m > 2$.

3. Conclusion

In this paper, we have determined the Wiener index, the edge Wiener index, the hyper Wiener index, the Harary index, the first Zagreb index, the second Zagreb index and the Gutman index of the zero divisor graph of $Z_p^n$ and the zero divisor graph of $Z_p^n \times Z_q^m$. In future research, we will determine the topological indices of the zero divisor graph of $Z_p^{k_1} \times Z_p^{k_2} \times ... \times Z_p^{k_m}$ for $m > 2$.

Conflicts of interest: The authors declare no conflict of interest.

Data availability: Not applicable
REFERENCES


Fariz Maulana finished a Master degree in Institut Teknologi Bandung, Indonesia. He is currently a freelance teacher. His research interest are Algebra, and Graph Theory.

Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl Ganesha No 10, Bandung, 40132, Indonesia.
e-mail: faruzholmes@gmail.com

Mochammad Zulfikar Aditya finished a Master degree in Institut Teknologi Bandung, Indonesia. He is currently a freelance teacher. His research interest are Algebra, Hadamard Matrix, and Graph Theory.

Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl Ganesha No 10, Bandung, 40132, Indonesia.
Erma Suwastika finished a doctoral degree in Institut Teknologi Bandung, Indonesia. She is currently a lecturer in Mathematics at Institut Teknologi Bandung, Indonesia. Her research interests are Algebra and Graph Theory.

Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl Ganesha No 10, Bandung, 40132, Indonesia.

E-mail: ermasuwastika@math.itb.ac.id

Intan Muchtadi Alamsyah received a Ph.D. from Universite de Picardie, Amiens, France. She is currently a Professor in Mathematics at Institut Teknologi Bandung, Bandung, Indonesia. Her research interests are Algebra, Representation Theory, Coding Theory and Cryptography.

Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl Ganesha No 10, Bandung, 40132, Indonesia.

E-mail: ntan@math.itb.ac.id

Nur Idayu Alimon received Ph.D. from Universiti Teknologi Malaysia (UTM) Johor Bahru. She is currently a senior lecturer at Universiti Teknologi MARA (UiTM) Johor Branch Pasir Gudang Campus, Malaysia. Her research interests are algebra, group theory, graph theory, topological index and their applications.

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Johor Branch, Pasir Gudang Campus, 81750, Masai, Johor, Malaysia.

E-mail: idayualimon@uitm.edu.my

Nor Haniza Sarmin received her BSc (Hons), Master, and Ph.D. in Mathematics from State University of New York (now known as Binghamton University), Binghamton, New York, USA. She is currently a Professor in Mathematics at the Faculty of Science, Universiti Teknologi Malaysia. Her specialization of research is in Group Theory, Graph Theory, Formal Language Theory, Splicing Systems, and Their Applications.

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310, UTM Johor Bahru, Johor, Malaysia.

E-mail: nhs@utm.my