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APPLICATIONS OF SOFT $g^{\#}$ SEMI CLOSED SETS IN SOFT TOPOLOGICAL SPACES

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Abstract. In this research work, we introduce and investigate four innovative types of soft spaces, pushing the boundaries of traditional spatial concepts. These new types of soft spaces are named as soft T_b space, soft $T_b^{\#}$ space, soft $T_b^{\#\#}$ space and soft${}_{\alpha}T_b^{\#}$ space. Through rigorous analysis and experimentation, we uncover and propose distinct characteristics that define and differentiate these spaces. In this research work, we have established that every soft $T_{\frac{1}{2}}$ space is a soft ${}_{\alpha}T_b^{\#}$ space, every soft T_b space is a soft ${}_{\alpha}T_b^{\#}$ space, every soft $T_b^{\#}$ space is a soft ${}_{\alpha}T_b^{\#}$ space, every soft T_b space is a soft $T_b^{\#}$ space, every soft $T_b^{\#}$ space is a soft $T_b^{\#\#}$ space, every soft $T_{\frac{1}{2}}$ space is a soft ${}^{\#}T_b$ space and every soft T_b space is a soft ${}^{\#}T_b$ space.

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1. Introduction

In the study of topological spaces, Devi et al. [1] introduced a new space called ${}_{\alpha}T_b$ space in 1998. This new space mainly focuses on the ideas of "generalised-closed maps" and "α-generalised closed maps". These ideas are most likely extensions or variants of standard closed mappings in the setting of a topological space. Closed maps are important in general topology because they help to retain certain aspects of the topological space investigated [2, 3, 4, 5, 6]. The invention of the notion of soft sets by the Russian mathematician Molodtsov in 1999 marked a significant milestone in the field of mathematics [7]. This novel

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notion deviated from standard set theory by proposing a new concept known as soft sets to solve decision-making challenges.

In 2002, the concept of $g^{\#}$ semi-closed sets in the study of topological spaces introduced by Veerakumar [8]. He made an important contribution by inventing the idea of $g^{\#}$-semi-closed sets in the study of topological spaces. In the context of topological spaces, this term is an extension of the classical notions of open and closed sets. He presented some new spaces namely ${}^{\#}T_b$ spaces, ${}_{\alpha}T_b^{\#}$ spaces, $T_b^{\#}$ spaces and $T_b^{\#\#}$ spaces in the study of topological spaces and it has been mentioned that these new spaces are the application of $g^{\#}s$ closed set. This break through intended to broaden our understanding of topological features and functions beyond the traditional open and closed sets.

The soft set theory was thoroughly analysed by Maji et al. in the year 2003. The basic definitions and operations of soft sets, such as soft set union, intersection, and complement have been introduced [9]. Shabir et al [10] initiated an abstract idea of topology associated with soft sets in 2011. The creation of a soft set topology enabled academics to investigate notions such as open and closed soft sets, soft continuity, and other topological qualities that were tailored to the particular traits of soft sets. This advancement laid the groundwork for understanding the behaviour of soft sets in regard to one another, as well as their interactions within a larger mathematical space. Kannan [11] made important contributions to the field of mathematical analysis in 2012 by presenting novel concepts that broadened the scope of soft set theory and soft topology. In particular, pioneered two concepts such as generalised closed soft sets and soft $T_{\frac{1}{2}}$ spaces. These ideas have since been ingrained in many branches of mathematics and have found applications in a wide range of fields. Generalised closed soft sets were introduced as a fresh technique to characterising sets in a more flexible and adaptive manner. Rajendrakumar and V. Kaladevi [12] contributed significantly to the field of soft topological spaces in 2016 by proposing the concept of "soft $g^{\#}s$ semi-closed sets." This breakthrough added a new dimension to the study of soft topological spaces, extending the theoretical framework and introducing unique techniques for analysing various features of soft sets in soft topological spaces.

The focus of this paper revolves around the concepts of soft $g^{\#}s$ closed sets and its applications and it has been introduced the soft ${}^{\#}T_b$ space, soft $T_b^{\#}$ space, soft $T_b^{\#\#}$ and soft ${}_{\alpha}T_b^{\#}$ space are the applications of soft $g^{\#}s$ closed sets and their properties were examined. The soft${}_{\alpha}T_b$ has been introduced here in this paper, and it has been established that the new class say soft${}^{\#}T_b$ spaces appropriately stands in between the classes of soft${}_{\alpha}T_b$ spaces and soft $T_{\frac{1}{2}}$ spaces.

2. Preliminaries

The soft space (X_u, τ_s, E_p) is termed as non-empty soft topological space without separation axioms unless or otherwise mentioned.

The soft set $(A_X, E_P) \tilde{\subseteq} (X_u, \tau_s, E_P)$ is considered as a soft subset, throughout

this article, the closure, interior and complement of the soft set (A_X, E_P) are respectively denoted by cl$((A_X, E_P))$, int$((A_X, E_P))$ and $(A_X, E_P)^c$.
As we will be using various terminology throughout this article, the definitions below will help you recall, what we are talking about.

2.1. Definition [10]**.** Consider the set $X_U \neq \phi$-the initial universe, the set $E_P \neq \phi$, the parameters set. Then the τ_s = soft open sets of X_U is usually known as a soft topology on X_U if,

- $\emptyset, (X_U, E_P)$ are in τ_s
- The arbitrary $\sqcup\{$soft open sets in $\tau_s\}$ are in τ_s
- The finite $\sqcap\{$soft open set in $\tau_s\}$ are in τ_s

Then the soft space $(X_u, \tau_s,\ E_p)$ is known as a soft toplogical space over X_U. The complement of the members of τ_s are termed as soft closed sets over X_U

2.2. Definition [13]**.** If $(A_X, E_p)\subseteqq Cl\,(Int((A_X, E_p)))$ then the soft subset (A_X, E_p) of $(X_u, \tau_s,\ E_p)$ is said to be soft semiopen and if $Int\,(Cl\ ((A_X, E_p)))$ $\subseteqq(A_X, E_p)$ then (A_X, E_p) is called a soft semiclosed set of $(X_u, \tau_s,\ E_p)$.

2.3. Definition [14]**.** If cl$((A_X\ , E_P))\subseteqq\ (H_X,\ E_P)$ whenever $(A_X\ , E_P)$ $\subseteqq(H_X,\ E_P)$ and $(H_X,\ E_P)\ \subseteqq(X, E_p)$ is soft open, then the soft subset $(A_X, E_p)\subseteqq(X_u, \tau_s,\ E_p)$ is known as a soft generalised closed set.

2.4. Definition [11]**.** If $scl\,((A_X\ , E_P))\subseteqq\ (H_X,\ E_P)$ whenever $(A_X\ , E_P)$ $\subseteqq(H_X,\ E_P)$ and $(H_X,\ E_P)\ \subseteqq(X, E_p)$ is soft open in (X, E_p), then the soft subset $(A_X, E_p)\subseteq(X_u, \tau_s,\ E_p)$ is known as soft generalized semi-closed (soft gs-closed).

2.5. Definition [15]**.** If $(cl\,((A_X\ , E_P))\subseteqq\ (H_X,\ E_P)$ whenever $(A_X\ , E_P)$ $\subseteqq(H_X,\ E_P)$ and $(H_X,\ E_P)\ \subseteqq(X, E_p)$ is soft open then the soft subset $(A_X, E_p)\subseteqq\ (X_u, \tau_s,\ E_p)$ is known as soft α - generalized closed (soft α g-closed).

2.6. Definition [15]**.** If $\alpha cl\,((A_X\ , E_P))\subseteqq\ (H_X,\ E_P)$ whenever $(A_X\ , E_P)$ $\subseteqq(H_X,\ E_P)$ and $(H_X,\ E_P)\ \subseteqq(X, E_p)$ is softα open then the soft subset (A_X, E_p) of $(X_u, \tau_s,\ E_p)$ is known as soft generalized α-closed (soft $g\alpha$-closed).

2.7. Definition [12]**.** If $scl\,((A_X\ , E_P))\subseteqq\ (H_X,\ E_P)$ whenever $(A_X\ , E_P)$ $\subseteqq(H_X,\ E_P)$ and $(H_X,\ E_P)\ \subseteqq(X, E_p)$ is soft α g open then the soft subset (A_X, E_p) of $(X_u, \tau_s,\ E_p)$ is known as soft $g^{\#}s$ closed.

2.8. Definition [16]**.** When all g-closed sets in $(X,\ \tau)$ are closed, the topological space $(X,\ \tau)$ is known as a $T_{1/2}$ space.

2.9. Definition [16]**.** When all gs-closed sets in $(X,\ \tau)$ are closed in $(X,\ \tau)$, the topological space $(X,\ \tau)$ is known as a T_b space.

2.10. Definition [16]**.** When all α g-closed sets in $(X,\ \tau)$ are closed in $(X,\ \tau)$, the topological space $(X,\ \tau)$ is known as an ${}_{\alpha}T_b$ space.

2.11. Definition [11]**.** When all soft-g-closed sets in $(X_u, \tau_s,\ E_p)$ are soft-closed, the soft topological space $(X_u, \tau_s,\ E_p)$ is known as $soft\ T_{1/2}$ space.

3. Applications of soft $g^{\#}s$ closed sets

This section explains, some brand-new classes of soft spaces say, soft ${}_{\alpha}T_b^{\#}$ space, soft $T_b^{\#}$ space, soft $T_b^{\#\#}$ space, and soft ${}^{\#}T_b$ spaces have been introduced and their properties have been analyzed.

3.1. Definition. In $(X_u, \tau_s,\ E_p)$, if every soft gs closed in it is soft closed, then $(X_u, \tau_s,\ E_p)$ is known as a soft T_b space.

Example 3.1: All Soft gs closed sets are soft closed.
Let $X = c_1, c_2, c_3,\ E = \{e_1, e_2\}$, where X is the set of all cars available in a showroom and in a parameter set E, e_1 represents pertrol cars, e_2 represents CNG cars and the soft topology is
$\tau_s = \{\varphi, X_s, (G_1,\ \ E_p), (G_2,\ E_p), (G_3,\ \ E_p), (G_4,\ \ E_p), (G_5,\ E_p)\}$.

The soft open sets are :
$(G_1, E_p) = \{(e_1,\ \ \{c_1,\ \ c_2\}),\ \ (e_2,\ \ \{c_1\})\}$
$(G_2, E_p) = \{(e_1,\ \ \{c_2\}),\ \ (e_2,\ \ \{c_1\})\}$
$(G_3, E_p) = \{(e_1,\ \ \{c_1\}),\ \ (e_2,\ \ \{c_2\})\}$
$(G_4, E_p) = \{(e_1,\ \ \{c_1,\ c_2\}),\ \ (e_2,\ \ \{c_1,\ \ c_2\})\}$
$(G_5, E_p) = \{(e_1,\ \ \{c_1\}),\ \ (e_2,\ \ \{\phi\})\}$

$(F_x, E_p) = \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \ \}$ **is soft gs – closed**.
Soft gs-closed : If $scl\,((F_x, E_p)) \subseteqq (U_x, E_p)$ whenever $(F_x, E_p) \subseteqq (U_x, E_p)$ and (U_x, E_p) is soft open in (X, E_p)

Semi Closure: $scl\,((F_x, E_p)) = \Cap \left\{soft\ semi\ closed\ sets \supseteqq (F_X\ E_P)\right\}$
soft semi closed set: If $int\,(cl\,((F_x, E_p))) \subseteqq\ (F_x, E_p)$ then (F_x, E_p) is soft semi closed.

Now, cl$((F_x, E_p)) = \Cap \left\{soft\ semi\ closed\ sets \supseteqq (F_X\ E_P)\right\} = \{(e_1,\ \{c_3\})\}$,
$(e_2,\ \{c_3\})$=(F_4, E_p)

Now, int$(cl\,((F_x, E_p))) = int\,((F_4, E_p)) = \Cup \left\{soft\ open\ sets \subseteqq (F_4,\ E_P)\right\} = \phi$
Thus $int\,(cl\,((F_x, E_p)))\ \subseteqq (F_x, E_p)$.
Therefore, (F_x, E_p) is soft semi closed.

Now, $scl\,((F_x, E_p)) = \Cap\{soft\ semi\ closed\ sets\ \supseteq (F_X\ , E_P)\}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \ \} = (F_4, E_p)$ Thus $scl\,((F_x, E_p))$ =(F_4, E_p).
Note that $scl\,((F_x, E_p)) \subseteq\ (X,\ E_P)$, Whenever $(F_x, E_p) \subseteq\ (X,\ E_P)$ and $(X,\ E_P)$ is soft open.

Hence (F_x, E_p) is soft gs-closed.

Now, $scl\left((F_x, E_p)\right) = \underline{\cap}\ \left\{soft\ semi\ closed\ sets\ \underline{\supseteq}\ (F_X\ E_P)\right\}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \ \} = (F_4, E_p)$ Thus $scl\left((F_x, E_p)\right) = (F_4, E_p)$.
Note that $scl\left((F_x, E_p)\right) \underline{\subseteq} (X,\ E_P)$, Whenever $(F_x, E_p) \underline{\subseteq} (X,\ E_P)$ and $(X,\ E_P)$ is soft open.
Hence (F_x, E_p) is soft gs-closed.

Now, $cl\left((F_x, E_p)\right) = \underline{\cap}\ \left\{soft\ semi\ closed\ sets\ \underline{\supseteq}\ (F_X\ , E_P)\right\}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \ \} = (F_4, E_p)$ i.e., $(F_x, E_p) = cl\left((F_x, E_p)\right)$. Hence (F_x, E_p) is soft closed.

3.2. Definition. In (X_u, τ_s, E_p), when all soft αg closed sets are soft-closed, the soft space (X_u, τ_s, E_p) is called as a soft ${}_\alpha T_b$ *space*.

Example 3.2: Every soft αg closed sets are soft closed
Let us consider the soft set mentioned in Example 3.1. Now consider the soft set (B, E_p) over X such that $(B, E_p) = \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\}$.
soft α closed : If $cl(int(cl((B, E_p)))) \underline{\subseteq}\ (B, E_p)$ then (B, E_p) is soft α *closed.*

$cl((B, E_p)) = \underline{\cap}\{soft\ closed\ sets\ \underline{\supseteq}\ (B, E_p)\ \} = \{(e_1, \{c_3\}),\ (e_2, \{c_3\})\} = (F_4, E_p)$

Now, $int(cl((B, E_p))) = int((F_4, E_p)) = \underline{\cup}\{soft\ open\ sets\ \underline{\subseteq}\ (F_4, E_p)\} = \phi$

Now, $cl(int(cl((B, E_p)))) = cl(\phi) = \underline{\cap}\{soft\ closed\ sets \underline{\supseteq} \phi\} = \{(e_1, \{c_3\}), (e_2, \{c_3\})\}$
$= (F_4, E_p) = (B, E_p)$
ie., $cl(int(cl((B, E_p)))) \underline{\subseteq}\ (B, E_p)$.
Hence (B, E_p) *is a soft α closed.*

Given that, $(B, E_p) = \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\}$.

If $\alpha cl((B,\ E_p)) \underline{\subseteq} (H_X,\ E_p)$, whenever $(B,\ E_p) \underline{\subseteq} (H_X,\ E_p)$ *and* $(H_X,\ E_p)$ *is soft open in* $(X,\ E_p)$.
Then $(B,\ E_p)$ *is said to be soft αg closed.*

Now, $\alpha cl((B, E_p)) = \underline{\cap}\{soft\ \alpha\ closed\ sets\ \underline{\supseteq}(B, E_p)\ \} = \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \}$
$= (F_4, E_p)$.

Thus $\alpha cl((B, E_p)) \underline{\subseteq} (X, E_p)$, whenever $(B, E_p) \underline{\subseteq} (X, E_p)$ and (H_X, E_p) *is soft open in* (X, E_p).

Now, $cl((B, E_p)) = \underline{\cap}\{soft\ \alpha\ closed\ sets\ \underline{\supseteq}(B, E_p)\ \}$

$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_3\})\ \} = (F_4, E_p) = (B, E_p)$ i.e., $(B, E_p) = cl((B, E_p))$.
Hence (B, E_p) is soft closed.

3.3. Definition. When every soft $g^{\#}s$ closed set of a soft topological space $(X_U,\ \tau_s,\ E_p)$ is soft closed, the soft space $(X_U,\ \tau_s,\ E_p)$ is said to be a soft $T_b^{\#}$ space.

Remark 3.1. Let us consider the soft set given in example 3.1. Now consider the soft set (A_X, E_p) over X such that $(A_X, E_p) = \{(e_1, \phi), (e_2, \{c_2\})\}$. Then (A_X, E_p) is neither soft closed nor soft α closed.

Given that $(A_X, E_p) = \{(e_1, \phi),\ (e_2, \{c_2\})\}$
Soft closed: If $(A_X, E_p) = cl((A_X, E_p))$ *then* (A_X, E_p) *is soft closed.*
Now, $cl((A_X, E_p)) = \Cap \{soft\ closed\ sets \supseteqq (A_X, E_p)\}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_2, c_3\})\ \} = (F_1, E_p) \neq (A_X, E_p)$
Therefore (A_X, E_p) is not a soft closed.

Soft α closed sets:
If $cl(int(cl((A_X, E_p)))) \subseteqq (A_X, E_p)$ then (A_X, E_p) *is soft* α *closed.*
Given that $(A_X, E_p) = \{(e_1, \phi),\ (e_2, \{c_2\})\}$
$cl((A_X, E_p)) = \Cap \{soft\ closed\ sets \supseteqq (A_X, E_p)\}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_2, c_3\})\ \} = (F_1, E_p)$
Now, $int(cl((A_X, E_p))) = int((F_1, E_p))$
$= \Cup \{soft\ open\ sets \subseteqq (F_1, E_p)\ \} = \phi$
$cl(int(cl((A_X, E_p)))) = cl(\phi)$
$= \Cap \{soft\ closed\ sets \supseteqq \phi\} = (F_4, E_p)$
i.e., $cl(int(cl((A_X, E_p)))) = (F_4, E_p) \nsubseteq (A_X, E_p)$
Therefore, (A_X, E_p) is not a soft α closed.
Hence (A_X, E_p) is neither soft closed nor soft α closed.
Given that $(A_X, E_p) = \{(e_1, \phi),\ (e_2, \{c_2\})\ \}$
Soft semi-closed: If $(A_X, E_p) \supseteqq int(cl((A_X, E_p)))$ then (A_X, E_p) *is called soft semi closed.*
Now, $cl((A_X, E_p)) = \Cap \{soft\ closed\ sets \supseteqq (A_X, E_p)\ \}$
$= \{(e_1,\ \{c_3\}),\ (e_2,\ \{c_2, c_3\})\ \} = (F_1, E_p)$

Now, $int(cl((A_X, E_p))) = int((F_1, E_p)) = \Cup\{soft\ open\ sets \subseteqq (F_1, E_p)\} = \phi$
Thus $(A_X, E_p) \supseteqq int(cl((A_X, E_p)))$, Hence (A_X, E_p) is soft semi-closed.

If $cl(int(cl((B, E_p)))) \subseteqq (B, E_p)$ then (B, E_p) is soft α closed. $cl((B, E_p)) = \Cap \{soft\ closed\ sets \supseteqq (B, E_p)\ \} = \{(e_1,\ \{c_3\}), (e_2, \{c_3\})\} = (F_4, E_p)$

Now, $int(cl((B, E_p))) = int((F_4, E_p))$

$= \Cup \{soft\ open\ sets \subseteqq (F_4, E_p)\} = \phi$
Now, $cl(int(cl((B, E_p)))) = cl(\phi)$
$= \Cap \{soft\ closed\ sets \supseteqq \phi\}$
$= \{(e_1, \{c_3\}), (e_2, \{c_3\})\} = (F_4, E_p) = (B, E_p)$
ie., $cl(int(cl((B, E_p)))) \subseteqq (B, E_p)$.
Hence (B, E_p) is soft α closed.

If $\alpha cl((B, E_p)) \subseteqq (H_X, E_p)$, whenever $(B, E_p) \subseteqq (H_X, E_p)$ and (H_X, E_p) is soft open in (X, E_p).
Now, $\alpha cl((B, E_p)) = \Cap \{soft\ \alpha\ closed\ sets \supseteqq (B, E_p)\}$
$= \{(e_1, \{c_3\}), (e_2, \{c_3\})\} = (F_4, E_p)$
Thus $\alpha cl((B, E_p)) \subseteqq (X, E_p)$, whenever $(B, E_p) \subseteqq (X, E_p)$ and (X, E_p) is soft open in (X, E_p).
Hence (B, E_p) is Soft αg-closed.
Therefore its complement $(B, E_p)^c = \{(e_1, \{c_1, c_2\}), (e_2, \{c_1, c_2\})\} = (G_4, E_p)$ is soft αg-open.

Given that $(A_X, E_p) = \{(e_1, \phi), (e_2, \{c_2\})\}$
Now, $scl((A_X, E_p)) \subseteqq (G_4, E_p)$ whenever $(A_X, E_p) \subseteqq (G_4, E_p)$ and (G_4, E_p) is soft αg open in (X, E_p).
Hence (A_X, E_p) soft $g^{\#}s$ *closed.*

Theorem 3.1. *In (X_U, τ_s, E_p), every soft $T_{1/2}$ space is a soft ${}_\alpha T_b^{\#}$ space, but not the converse.*

Proof. Consider a *soft* $T_{1/2}$ *space* (X_U, τ_s, E_p). Let us take (A_X, E_p) be a soft $g^{\#}s$ *closed set* of (X_U, τ_s, E_p). According to Theorem 3.4 of [12], (A_X, E_p) *is a soft gs closed set*. As it has been considered (X_U, τ_s, E_p) as a *soft* $T_{1/2}$ *space*, every soft gs closed set must be a soft closed. Thus we have the soft gs closed set (A_X, E_p) is a soft closed. From the known fact we can easily see that every soft closed sets are soft semi closed sets, according to Remark 3.2 [17]. As a result (X_U, τ_s, E_p) is a soft${}_\alpha T_b^{\#}$ space.
Conversely, assume that (X_U, τ_s, E_p) is a soft ${}_\alpha T_b^{\#}$ space and (A_X, E_p) is a soft $g^{\#}s$ closed in (X, E_p). According to Remark 3.1, we see that (A_X, E_p) is neither soft-closed nor soft α closed. But it is a soft $g^{\#}s$ closed. Therefore (X_U, τ_s, E_p) will not be a soft $T_{1/2}$ space.

□

Theorem 3.2. *All the soft T_b spaces are a soft ${}_\alpha T_b^{\#}$ spaces, but the converse is not.*

Proof. Let $(A_X, E_p) \subseteqq (X_U, \tau_s, E_p)$ be a soft $g^{\#}s$ closed subset of a soft T_b *space* (X_U, τ_s, E_p). Then (A_X, E_p) must be a soft gs closed set, according to theorem 3.4 [12]. As we have considered (X_U, τ_s, E_p) is a soft T_b space,

(A_X, E_p) must be a soft closed. Already it has been proved that every soft closed set is soft semi closed, by Remark 3.2 [17]. Thus we can easily see that (A_X, E_p) is soft semi closed. Hence the soft T_b space $(X_U,\ \tau_s,\ E_p)$ is a soft ${}_{\alpha}T_b^{\#}$ space.

Conversely, Assume that $(X_U,\ \tau_s,\ E_p)$ is a soft ${}_{\alpha}T_b^{\#}$ space and (A_X, E_p) is soft $g^{\#}s$ closed set of $(X_U,\ \tau_s,\ E_p)$. The soft set (A_X, E_p) as defined in Example 3.1, $(A_X, E_p) = \{(e_1, \{\phi\}), (e_2,\ \{b\})\}$ is neither soft closed nor soft α closed and hence not soft semi-closed. Therefore (A_X, E_p) cannot be a soft T_b space. Hence the soft ${}_{\alpha}T_b^{\#}$ space does not fit in a soft T_b space. □

Theorem 3.3. *Every soft $T_b^{\#}$ space is a soft ${}_{\alpha}T_b^{\#}$ space but does not hold the converse.*

Proof. Assume that $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#}$ space and (A_X, E_p) is a soft $g^{\#}s$ closed in $(X_U,\ \tau_s,\ E_p)$. According to the definition 3.03 of soft $T_b^{\#}$ space, (A_X, E_p) is a soft closed. Already it has been shown that all soft closed sets are soft semi closed, by Remark 3.2 [17]. Therefore, every soft $g^{\#}s$ closed set (A_X, E_p) is a soft semi closed. As a result $(X_U,\ \tau_s,\ E_p)$ is a soft ${}_{\alpha}T_b^{\#}$ space.

Conversely, suppose that $(X_U,\ \tau_s,\ E_p)$ is a soft ${}_{\alpha}T_b^{\#}$ space and (A_X, E_p) is a soft $g^{\#}s$ closed in $(X_U,\ \tau_s,\ E_p)$. Then (A_X, E_p) is a soft semi closed. But by Remark 3.1, it has been discovered that (A_X, E_p) is not a soft closed. Hence $(X_U,\ \tau_s,\ E_p)$ is not a soft $T_b^{\#}$ space. Thus all the soft ${}_{\alpha}T_b^{\#}$ spaces entirely contains the class of soft $T_b^{\#}$ spaces. □

Theorem 3.4. *All the soft T_b spaces are soft $T_b^{\#}$ spaces, but not the converse.*

Proof. Assume that $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space and (A_X, E_p) is a soft $g^{\#}s$ closed in $(X_U,\ \tau_s,\ E_p)$. According to Theorem 3.01 [12], (A_X, E_p) is a soft-gs-closed set of $(X_U,\ \tau_s,\ E_p)$. Since we have taken $(X_U,\ \tau_s,\ E_p)$ as a soft T_b space, (A_X, E_p) is soft-closed. Hence $(X_U,\ \tau_s,\ E_p)$ is soft $T_b^{\#}$ space.

Conversely, suppose that $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#}$ space. Now, the soft space $(X_U,\ \tau_s,\ E_p)$ in Example 3.4 [12] is a soft $T_b^{\#}$ space but not a soft T_b space. Hence we conclude that the converse need not be true. □

Remark 3.2. Soft $T_b^{\#}$ space is independent from soft ${}_{\alpha}T_b$ space and soft $T_{1/2}$ space.

Proof. Let us take $(A_X,\ E_p)$ is a soft $g^{\#}s$ closed set of the soft $T_b^{\#}$ space $(X_U,\ \tau_s,\ E_p)$. Then $(A_X,\ E_p)$ is soft-closed. As per example 3.6 [12], $(A_X,\ E_p)$ is not soft g-closed and hence $(A_X,\ E_p)$ is neither soft gα closed nor soft αg closed in $(X_U,\ \tau_s,\ E_p)$. Therefore, $(X_U,\ \tau_s,\ E_p)$ is neither soft ${}_{\alpha}T_b$ space nor soft $T_{1/2}$ space. □

3.4. Definition. If every soft $g^{\#}s$ closed set $(A_X,\ E_p)$ in $(X_U,\ \tau_s,\ E_p)$ is a soft α closed then $(X_U,\ \tau_s,\ E_p)$ is called a soft $T_b^{\#\#}$ space.

Theorem 3.5. *Every soft $T_b^{\#}$ space (resp. soft T_b space) is a soft $T_b^{\#\#}$ space, but not the converse.*

Proof. Let $(A_X,\ E_p)$ be a soft $g^{\#}s$ closed set of a soft $T_b^{\#}$ space $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#}$ space, then $(A_X,\ E_p)$ is soft-closed. Since by Remark 13 [18] , $(A_X,\ E_p)$ would be a soft α closed. Therefore $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#\#}$ space.
Let $(X_U,\ \tau_s,\ E_p)$ be a soft T_b space and $(A_X,\ E_p)$ be a soft $g^{\#}s$ closed set of $(X_U,\ \tau_s,\ E_p)$. Note that every soft $g^{\#}s$ closed set is a soft gs-closed by theorem 3.4 [12]. So $(A_X,\ E_p)$ is a soft gs-closed. Since $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space, $(A_X,\ E_p)$ is soft closed. By Remark 13 [18] , every soft closed is soft α closed. Hence $(A_X,\ E_p)$ is a soft $T_b^{\#\#}$ space. Therefore every soft T_b space is a soft $T_b^{\#\#}$ space.
Conversely suppose that $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#\#}$ space and $(A_X,\ E_p)$ is a soft $g^{\#}s$ closed set of $(X_U,\ \tau_s,\ E_p)$. Since $(A_X,\ E_p)$ is soft $g^{\#}s$ closed, it is soft α closed. According to Example 14 [18] , the soft α closed set is not a soft closed. Hence $(A_X,\ E_p)$ is not a soft closed. Therefore $(X_U,\ \tau_s,\ E_p)$ cannot be a soft $T_b^{\#}$ space. It is noted that from remark 3.1, the soft space $(A_X,\ E_p)$ is soft $g^{\#}s$ closed set of $(X_U,\ \tau_s,\ E_p)$ but it is neither soft closed nor soft gs closed. Therefore the soft space mentioned in Example 3.2 [12] is a soft $T_b^{\#\#}$ space, but not a soft $T_b^{\#}$ space and a soft T_b space. Thus the class of soft $T_b^{\#}$ spaces and soft T_b spaces are perfectly contained in the class of soft $T_b^{\#\#}$ spaces. □

Theorem 3.6. *When $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#\#}$ space, every singleton of $(X,\ E_p)$ becomes soft αg closed or soft αg open. But the converse does not hold.*

Proof. Let $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#\#}$ space. Assume that the soft singleton $\{e_1\ , \{x_1\}\}$ in $(X,\ E_p)$ is not a soft αg closed. So its complement is not soft αg open. Therefore, $(X,\ E_p)$ is the only soft αg open set containing $(X,\ E_p) - \{e_1\ , \{x_1\}\}$. Hence $(X,\ E_p) - \{e_1\ , \{x_1\}\}$ is soft $g^{\#}s$ closed of $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#\#}$ space, $(X,\ E_p) - \{e_1\ , \{x_1\}\}$ is soft αg closed or equivalently $\{e_1\ , \{x_1\}\}$ is soft αg open. Example 3.03 [12] , supply the evidence to prove that the converse part does not hold.

□

Definition 3.7. A soft space $(X_U,\ \tau_s,\ E_p)$ is known as a soft ${}^{\#}T_b$ space , when all the soft-gs-closed sets are soft $g^{\#}s$ closed in $(X_U,\ \tau_s,\ E_p)$

Theorem 3.8. *All the soft $T_{1/2}$ spaces are a soft ${}^{\#}T_b$ space, but not the converse.*

Proof. Let $(A_X,\ E_p)$ be a soft gs closed subset of the soft $T_{1/2}$ space $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is soft $T_{1/2}$ space, (A_X, E_p) is soft-semi-closed in $(X_U,\ \tau_s,\ E_p)$. Then by theorem 3.1 [12], (A_X, E_p) is soft $g^{\#}s$ closed. Therefore $(X_U,\ \tau_s,\ E_p)$ is a soft ${}^{\#}T_b$ space. Let $(X_U,\ \tau_s,\ E_p)$ be a soft ${}^{\#}T_b$ space and (A_X, E_p) be a soft-gs-closed set of $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is soft ${}^{\#}T_b$ space,

$(A_X,\ E_p)$ is soft $g^{\#}s$ closed. But the soft $g^{\#}s$ closed sets are not soft gs closed, Remark 3.1 [12]. Therefore $(X_U,\ \tau_s,\ E_p)$ cannot be a soft $T_{1/2}$ space. Hence the converse is not true. □

Theorem 3.9. *All the soft T_b spaces are soft $^{\#}T_b$ spaces, but converse does not hold.*

Proof. Let $(X_U,\ \tau_s,\ E_p)$ be a soft T_b space and (A_X, E_p) be a soft-gs-closed subset of $(X_U,\ \tau_s,\ E_p)$. Now, what we have to prove is (A_X, E_p) is a soft $g^{\#}s$ closed. Since $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space, (A_X, E_p) is a soft closed. Then by Remark 3.2 [18] , (A_X, E_p) is a soft-semi closed. According to theorem 3.1 [12], each soft semi closed sets are soft $g^{\#}s$ closed. Therefore (A_X, E_p) is soft $g^{\#}s$ closed. Hence $(X_U,\ \tau_s,\ E_p)$ is a soft $^{\#}T_b$ space. Let $(X_U,\ \tau_s,\ E_p)$ be a $soft$ $^{\#}T_b$ *space* and (A_X, E_p) be a soft-gs-closed subset of $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is $soft$ $^{\#}T_b$ *space*, (A_X, E_p) is $soft$ $g^{\#}s$ *closed* but not soft-closed by Remark 3.1. Hence $soft$ $^{\#}T_b$ *space* is not a $soft$ T_b *space.* □

Theorem 3.10. *A soft space $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space if and only if $(X_U,\ \tau_s,\ E_p)$ is soft $T_b^{\#}$ space and soft $^{\#}T_b$ space.*

Proof. Assume that $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space. Then by theorem 3.04, $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#}$ space and also by theorem 3.08, $(X_U,\ \tau_s,\ E_p)$ is a soft $^{\#}T_b$ space.
Conversely assume that $(X_U,\ \tau_s,\ E_p)$ is both soft $T_b^{\#}$ space and soft $^{\#}T_b$ space. Let (A_X, E_p) be a soft-gs-closed set of $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is soft $^{\#}T_b$ space, then (A_X, E_p) is a soft $g^{\#}s$ closed in $(X_U,\ \tau_s,\ E_p)$. Also it has been assumed that $(X_U,\ \tau_s,\ E_p)$ is a soft $T_b^{\#}$ space, then (A_X, E_p) would be a soft-closed set in $(X_U,\ \tau_s,\ E_p)$. Hence $(X_U,\ \tau_s,\ E_p)$ is a soft T_b space. □

Theorem 3.11. *A soft space $(X_U,\ \tau_s,\ E_p)$ is a soft $T_{1/2}$ space if and only if it is soft $^{\#}T_b$ space and soft $_{\alpha}T_b^{\#}$ space.*

Proof. By theorems 3.07 and 3.01 a soft $T_{1/2}$ space is soft $^{\#}T_b$ space and soft $_{\alpha}T_b^{\#}$ space.
Conversely suppose that $(X_U,\ \tau_s,\ E_p)$ is both soft $^{\#}T_b$ space and soft $_{\alpha}T_b^{\#}$ space. Let (A_X, E_p) be a soft-gs-closed set of $(X_U,\ \tau_s,\ E_p)$. Since $(X_U,\ \tau_s,\ E_p)$ is soft $^{\#}T_b$ space, then (A_X, E_p) is a soft $g^{\#}s$ closed set of $(X_U,\ \tau_s,\ E_p)$. Also it has been assumed that $(X_U,\ \tau_s,\ E_p)$ is a soft $_{\alpha}T_b^{\#}$ space, so (A_X, E_p) is soft semi closed in $(X_U,\ \tau_s,\ E_p)$. Hence $(X_U,\ \tau_s,\ E_p)$ is a soft $T_{1/2}$ space.

□

4. Conclusion

This research contributes to the field of soft spaces by offering a comprehensive characterization of four novel types, shedding light on their distinctive traits, and suggesting properties that encapsulate their essential qualities. These findings have the potential to guide future developments and applications of soft spaces across various domains, fostering innovation and adaptability in dynamic environments. From this investigation we conclude the following:

- The **soft $\mathbf{T_{1/2}}$ spaces** are entirely contained in the **soft$_\alpha \mathbf{T_b^{\#}}$ space** classes.
- The **soft $\mathbf{T_b}$ spaces** are always a **soft $_\alpha\mathbf{T_b^{\#}}$ spaces**.
- The **soft $\mathbf{T_b^{\#}}$ space** is properly contained in the **soft $_\alpha\mathbf{T_b^{\#}}$ space** classes.
- The **soft $\mathbf{T_b^{\#}}$ spaces** are appropriately containing the **soft $\mathbf{T_b}$ space** classes.
- The **soft $\mathbf{T_b^{\#}}$ spaces** and soft T_b spaces perfectly contained in the **soft $\mathbf{T_b^{\#\#}}$ space** classes.
- The **soft $^{\#}\mathbf{T_b}$ spaces** exactly contain the **soft $\mathbf{T_{1/2}}$ space** classes.
- The **soft $\mathbf{T_b}$ spaces** and the **soft $_\alpha\mathbf{T_b}$ spaces** are properly contained in the **soft $^{\#}\mathbf{T_b}$ space** classes.

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References

1. R. Devi, K. Balachandran and H. Maki, *Generalized α-closed maps and α -generalized closed maps*, Indian J. Pure. Appl. Math. **29** (1998), 37-49.
2. Bekir Tanay, M. Burc Kandemir, *Topological structure of fuzzy soft sets*, Computers and Mathematics with Applications **61** (2011), 2952-2957.
3. Naim Cagman, Serkan Karatas and Serdar Enginoglu, *Soft topology*, Computers and Mathematics with Applications **62** (2011), 351-358.
4. Won Keun Min, *A note on soft topological spaces*, Computers and Mathematics with Applications **62** (2011), 3524-3528.
5. Sabir Hussain and Bashir Ahmad, *Some properties of soft topological spaces*, Computers and Mathematics with Applications **62** (2011), 4058-4067.
6. Zhaowen Li, Dingwei Zheng and Jing Hao, *L-fuzzy soft sets based on complete Boolean lattices*, Computers and Mathematics with Applications **64** (2012), 2558-2574.
7. D. Molodtsov, *Soft Set Theory-First Results*, Computers and Mathematics with Applications **37** (1999), 19-31.
8. S.R.K.M. Veera Kumar, *semi-closed sets in topological spaces*, Indian J. Math. **44** (2002), 73-87.
9. P.K. Maji, R. Biswas and A.R. Roy, *Soft Set Theory*, Computers and Mathematics with Applications **45** (2003), 555-562.
10. Muhammad Shabir and Munazza Naz, *On soft topological spaces*, Computers and Mathematics with Applications **61** (2011), 1786-1799.

11. K. Kannan, *Soft Generalized closed sets in soft topological spaces*, J. Theo. and App. Inf. Tech. **37** (2012), 17-21.
12. T. Rajendrakumar and V. Kaladevi, *Soft $g^{\#}s$-closed sets in Soft Topological Spaces*, Aryabhatta J. Mat. and Inf. **8** (2016), 1-7.
13. A. Aygunoglu and H. Aygun, *Some Notes on soft topological spaces*, Neural Comput. and Applic. **21** (2012), 113-119.
14. I. Arockiarani and A. Arockia Lancy, *Generalized soft αg closed sets and soft αgs closed sets in soft topological spaces*, Int. J. Mat. Arch. **4** (2013), 17-23.
15. S. Pious Missier and S. Jackson, *A New Notion of Generalized Closed Sets in Soft Topological Spaces*, IOSR J. Mat. **10** (2014), 122-128.
16. R. Devi, K. Balachandran and H. Maki, *Semi-generalized closed maps and generalized semi-closed maps*, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math. **14** (1993), 41-54.
17. Bin Chen, *Soft semi-open sets and related properties in Soft topological spaces*, An Int. J. Appl. Math. Inf. Sci. **7** (2013), 287-294.
18. Metin Akdag and Alkan Ozkan, *Soft α-Open Sets and Soft α-Continuous Function*, Abstr. Appl. Anal. **7** (2014), 287-294. https://doi.org/10.1155/2014/891341

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