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COMPUTATION OF TOTAL CHROMATIC NUMBER FOR CERTAIN CONVEX POLYTOPE GRAPHS

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ABSTRACT. A total coloring of a graph G is an assignment of colors to the elements of a graphs G such that no adjacent vertices and edges receive the same color. The total chromatic number of a graph G , denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. In this paper, we proved the Behzad and Vizing conjecture for certain convex polytope graphs D_n^p, Q_n^p, R_n^p, E_n, S_n, G_n, T_n, U_n, C_n,respectively. This significant result in a graph G contributes to the advancement of graph theory and combinatorics by further confirming the conjecture's applicability to specific classes of graphs. The presented proof of the Behzad and Vizing conjecture for certain convex polytope graphs not only provides theoretical insights into the structural properties of graphs but also has practical implications. Overall, this paper contributes to the advancement of graph theory and combinatorics by confirming the validity of the Behzad and Vizing conjecture in a graph G and establishing its relevance to applied problems in sciences and engineering.

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1. Introduction

Let a graph G be finite and undirected with no loops or multiple edges. If each vertex in $V(G)$ has a degree d, the graph G is called a d -regular graph. In recent years, the study of total coloring in graphs has found important applications in various scientific and engineering domains. The total chromatic number, denoted as $\chi''(G)$, provides a valuable measure for scheduling and resource allocation problems in parallel computing, wireless networks, and telecommunication systems. By assigning distinct colors to vertices and edges such that adjacent elements receive different colors, total coloring ensures the efficient utilization of

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resources and minimizes interference or conflicts. In this paper, we focus on the Behzad [2] and Vizing [15] conjecture, a fundamental problem in graph theory with practical implications. The conjecture proposes a relationship between the total chromatic number of a graph and its maximum degree, stating that$\chi''(G)$ is either equal to the maximum degree or the maximum degree plus one. Validating this conjecture for specific classes of graphs is of great significance, as it not only sheds light on the fundamental properties of graphs but also contributes to the development of efficient resource allocation strategies in real-world applications.

A graph of a convex polytope is formed from its vertices and edges having the same incidence relation. Graphs of convex polytopes were first examined by Baca [9]. He studies graceful and anti-graceful labeling problems for these geometrically important graphs. All the graphs considered here are finite, simple and undirected. let $(V(G), E(G))$ be a graph with set of vertices $V(G)$ and edges $E(G)$ respectively. A total coloring of G is a mapping $f : V(G) \cup E(G) \rightarrow C$, where C is the set of colors, satisfying the following three conditions $(i) - (iii)$.

i) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$,

ii) $f(e) \neq f(e^{'})$ for any adjacent edges $e, e^{'} \in E(G)$,

iii) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and edges $e \in E(G)$ incident to v.

The total chromatic number of a graph G denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. Behzad[2] and Vizing[15] conjectured that for every graph $G, \Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. If a graph G is total colorable with $\Delta(G) + 1$ colors then the graph is called Type-I, and if it is total colorable with $\Delta(G) + 2$ colors but not $\Delta(G) + 1$ colors, then it is Type - II. A graph G is said to be total colorable if the elements of G are colored with atmost $\Delta(G) + 2$ colors. This conjecture was verified by Rosenfeld[13] and Vijayaditya[14] for $\Delta(G) = 3$ and by by Kostochka[6, 7] for $\Delta(G) \leq 5$. For planar graphs, the conjecture was verified by Borodin[4] for $\Delta(G) \geq 9$. In 1992, Yap and Chew [17] proved that any graph G has total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq |V(G)| - 5$, where $|V(G)|$ is the number of vertices in G. Muthuramakrishnan and Jayaraman[12] proved that total chromatic number of twig graph, splitting graph of comb graph and shadow graph of comb graph. In [10] the concept of total chromatic number is applied in complier optimization, register allocation is the process of assigning local automatic variables and expression results to a limited number of processor register. Other applications of the graph coloring concern load balancing problems in multiprocessor machines and results in probability theory (scheduling). Some applications establish the added constraints. For instance, in scheduling problems, workloads, time charts have to be allotted uniformly among the labourers without any chaos. This may be modeled by a graph with elements like vertices and edges representing the task assigned to completed and for every conflicting pair of tasks. Labourers denoted by colors. Coloring of these graphs referred a valid allocation of tasks to the labourers. In this paper, the total coloring conjecture is proved for convex polytope graph with certain pendent edges. Our work contributes to the growing body of research on regular graphs, which have been extensively studied in various areas of mathematics, computer science, and engineering. The properties and structures of regular graphs make them particularly useful in modelling and analyzing real-world systems, and our results showcase the power of regular graphs in solving complex problems in applied mathematics and engineering.

2. Preliminaries

Definition 2.1. [10] The plane graph D_n^p (p from pendent) is obtained from a graph of convex polytope D_n^* by attaching a pendent edges at each vertex of outer cycle of D_n^*. So the graph D_n^p has the vertex set and the edge set given by $V(D_n^p) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$ and $E(D_n^p) = \bigcup_{i=1}^{n}\{a_ia_{i+1}; b_ic_i; b_{i+1}c_i; d_id_{i+1}; a_ib_i; c_id_i; d_ie_i\}$ with $5n$ vertices and $7n$ edges respectively.

Definition 2.2. [10] The plane graph Q_n^p(p from pendent) is obtained from a graph of convex polytope Q_n by attaching a pendent edges at each vertex of outer cycle of graph of convex polytope graph Q_n. So the graph Q_n^p has the vertex set and edge set given by $V(Q_n^p) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$ and $E(Q_n^p) = \bigcup_{i=1}^{n}\{a_ia_{i+1}; b_ic_i; b_{i+1}c_i; d_id_{i+1}; a_ib_i; c_id_i; d_ie_i\}$ with $4n$ vertices and $8n$ edges respectively.

Definition 2.3. [10] In the graph R_n^p(p from pendent) is obtained as a combination of the graph, prism and the graph of an antiprism by attaching a pendant edge at each vertex of outer cycle. We make the convention that $a_{n+1} = a_1, b_{n+1} = b_n, c_{n+1} = c_1$ to simply the notation, we have $V(R_n^p) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i\}$ and $E(R_n^p) = \bigcup_{i=1}^{n}\{a_ia_{i+1}; b_ib_{i+1}; c_ic_{i+1}; c_ib_i; a_ib_i; b_ia_{i+1}; c_id_i\}$ with $4n$ vertices, $7n$ edges and the subscripts being taken modulo n.

Definition 2.4. [1] The E_n is the combination of convex polytope denoted as T_n and A_n by adding new edges $a_{i+1}b_i$ and having the same vertex $V(E_n)$ and $E(E_n)$. The E_n consisting of $3-$sided faces,$5-$sided faces and $n-$sided faces.

Definition 2.5. [12] The convex polytope S_n consists of $2n, 3-$ sided faces, $2n, 4-$ sided faces and a pair of $n-$ sided faces, and is obtained by the combination of the graph of convex polytope R_n and the graph of a prism D_n. We have $V(S_n) = \{a_i; b_i; c_i; d_i : 1 \leq i \leq n\}$ and $E(S_n) = \{a_ia_{i+1}; b_ib_{i+1}; c_ic_{i+1}; d_id_{i+1} : 1 \leq i \leq n\} \cup \{a_{i+1}b_i; a_ib_i; b_ic_i; c_id_i : 1 \leq i \leq n\}$.

Definition 2.6. [18] The convex polytope G_n consists of $2n, 3-$ sided faces, $2n, 4-$ sided faces and a pair of $n-$ sided faces, and is obtained by the combination of the graph of convex polytope R_n and the graph of a prism D_n. We have $V(G_n) = \{a_i; b_i; c_i; d_i : 1 \leq i \leq n\}$ and $E(G_n) = \{a_ia_{i+1}; b_ib_{i+1}; c_ic_{i+1}; d_id_{i+1} : 1 \leq i \leq n\} \cup \{a_{i+1}b_i; a_ib_i; b_ic_i; c_id_i : 1 \leq i \leq n\}$.

Definition 2.7. [12] The convex polytope T_n consists of $4n, 3-$ sided faces, $2n, 4-$ sided faces and a pair of $n-$ sided faces, and is obtained by the combination of the graph of convex polytope R_n and the graph of a prism A_n. We have

$V(T_n) = \{a_i; b_i; c_i; d_i : 1 \leq i \leq n\}$ and $E(T_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1} : 1 \leq i \leq n\} \cup \{a_{i+1} b_i; a_i b_i; b_i c_i; c_i d_i; c_{i+1} d_i : 1 \leq i \leq n\}$.

Definition 2.8. [12] The convex polytope U_n consists of $n, 4-$ sided faces, $2n, 5-$ sided faces and a pair of $n-$ sided faces, and is obtained by the combination of the graph of convex polytope D_n and the graph of a prism D_n. We have $V(U_n) = \{a_i; b_i; c_i; d_i; e_i : 1 \leq i \leq n\}$ and $E(U_n) = \{a_i a_{i+1}; b_i b_{i+1}; e_i e_{i+1} : 1 \leq i \leq n\} \cup \{a_i b_i; b_i c_i; c_i d_i; d_i e_i; c_{i+1} d_i : 1 \leq i \leq n\}$.

Definition 2.9. [1] The graph of convex polytope C_n consists of $3n, 3-$ sided faces,$n, 4-$ sided faces, $n, 5-$ sided faces and a pair of $n-$ sided faces. There sets of vertices $V(C_n)$ and sets of edges $E(C_n)$ are given us $V(C_n) = \{a_i; b_i; c_i; d_i; e_i : 1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i+1}; b_i b_{i+1}; d_i d_{i+1}; e_i e_{i+1} : 1 \leq i \leq n-1\} \cup \{a_i b_i; b_i c_i; b_{i+1} c_i; c_i d_i; d_i e_i; d_{i+1} e_i : 1 \leq i \leq n-1\}$.

3. Main results

Theorem 3.1. *Let D_n^p be the plane graph with n pendent edges, then $\chi''(D_n^p) = 5$.*

Proof. Let $V(D_n^p) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle produced by $\{a_i : 1 \leq i \leq n\}$ be the $a-$ cycle; the cycle is induced by $\{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\}$ be the $b-$ cycle; cycle produced by $\{d_i : 1 \leq i \leq n\}$ be the outer cycle and the set of pendent vertices $\{e_i : 1 \leq i \leq n\}$. Let $E(D_n^p) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)}; p_i^{(4)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\}$. The outer cycle vertices $\{u_1, u_2, \cdots u_n\}$ are adjacent to each other and form a cycle. Thus, they must have distinct colors from the total coloring concept.
The inner cycle vertices $\{v_1, v_2, \cdots v_n\}$ also form a cycle and are adjacent to each other. They must have distinct colors from the outer cycle vertices and, therefore, also have distinct colors among themselves. But the vertices $\{v_2, v_4, u_2, u_3\}$ dominated by v_3, by the definition of independent dominating set choosing v_3 is not consideration, focussing on the minimum of an independent dominating set. The edges $E(D_n^p)$ are classified as:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i c_i$
$p_i^{(4)}$	$1 \leq i \leq n$	$c_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$c_i d_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$d_i d_{i+1(modn)}$
$q_i^{(3)}$	$1 \leq i \leq n$	$d_i e_i$

Based on the total coloring conjecture, since $\chi''(D_n^p) \geq \Delta(D_n^p) + 1 = 4 + 1 \geq 5$ then the lower bound of D_n^p is $\chi''(D_n^p) \geq 5$. We now need to prove upper bound of total coloring conjecture of D_n^p is $\chi''(D_n^p) \leq\ 5$. Define total coloring f, such that $f : V(D_n^p) \cup E(D_n^p) \to \{1, 2, 3, 4, 5\}$ as follows:

Case (i): when $n \equiv 0\ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = f(c_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$; $f(d_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;
$f(b_i) = 3$; $f(e_i) = 5$
The edge coloring is formulated as follows:
(2) $f(p_i^{(1)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;
$f(q_i^{(2)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(3)}) = 4$; $f(p_i^{(4)}) = 5$; $f(q_i^{(1)}) = 3$;
$f(q_i^{(3)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$

Case (ii): When $n \equiv 1\ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)}), f(p_i^{(2)}), f(p_i^{(3)}), f(p_i^{(4)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}, f(a_n) = f(d_n) = 5, f(c_n) = 2, f(e_n) = 1, f(p_n^{(1)}) = 5, f(p_1^{(2)}) = 2, f(p_n^{(2)}) = 2, f(q_n^{(2)}) = 4, f(q_n^{(3)}) = 2$. For $i = n-1, f(p_{n-1}^{(2)}) = 1, f(q_{n-1}^{(2)}) = 1, f(q_{n-1}^{(3)}) = 4, f(r_{n-1}^{(2)}) = 3$. It is evident that $\chi''(D_n^p) \leq\ 5$. We can conclude that $\chi''(D_n^p) = 5$. □

Theorem 3.2. *Let Q_n^p be the plane graph with n pendent edges, then* $\chi''(Q_n^p) = 6$.

Proof. Let $V(Q_n^p) = \bigcup_{i=1}^n \{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the $a-cycle$; the cycle is induced by $\{b_i : 1 \leq i \leq n\}$ be the $b-cycle$; set of vertices $\{c_i : 1 \leq i \leq n\}$ be the inner vertices; the cycle induced by $\{d_i : 1 \leq i \leq n\}$ be the $d-cycle$ and the set of pendent vertices $\{e_i : 1 \leq i \leq n\}$. Let $E(Q_n^p) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)} : 1 \leq i \leq n\}$
The edges $E(Q_n^p)$ are classified as:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \le i \le n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \le i \le n$	$a_i b_i$
$p_i^{(3)}$	$1 \le i \le n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \le i \le n$	$b_i c_i$
$q_i^{(2)}$	$1 \le i \le n$	$c_i b_{i+1(modn)}$
$q_i^{(3)}$	$1 \le i \le n$	$c_i d_i$
$r_i^{(1)}$	$1 \le i \le n$	$d_i d_{i+1(modn)}$
$r_i^{(2)}$	$1 \le i \le n$	$d_i e_i$

Based on the total coloring conjecture, since $\chi''(Q_n^p) \ge \Delta(Q_n^p) + 1 = 5 + 1 \ge 6$ then the lower bound of Q_n^p is $\chi''(Q_n^p) \ge 6$. We now need to prove upper bound of total coloring conjecture of Q_n^p is $\chi''(Q_n^p) \le 6$. Define total coloring f, such that $f : V(Q_n^p) \cup E(Q_n^p) \to \{1, 2, 3, 4, 5, 6\}$ as follows:
when $n \equiv 0 \ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \le i \le n$

(1) $f(a_i) = f(d_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$; $f(c_i) = 3$; $f(e_i) = 6$

The coloring of edges is formulated as follows:

(2) $f(p_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = 3$; $f(q_i^{(2)}) = 4$; $f(q_i^{(3)}) = 5$;

$f(p_i^{(3)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(1)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;

$f(r_i^{(1)}) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(2)}) = 4$

When $n \equiv 1 \ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)}), f(p_i^{(2)}), f(p_i^{(3)}), f(p_i^{(4)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(r_i^{(1)}), f(r_i^{(2)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}$. $f(a_n) = 5$, $f(b_n) = f(d_n) = 4$, $f(p_n^{(1)}) = 2$, $f(p_n^{(3)}) = 1$, $f(q_1^{(1)}) = 2$, $f(q_1^{(1)}) = 6$, $f(q_n^{(2)}) = 4$, $f(q_n^{(3)}) = 5$, $f(r_n^{(1)}) = 3$, $f(r_n^{(2)}) = 2$. For $i = n-1$, $f(q_{n-1}^{(2)}) = 5$, $f(q_{n-1}^{(3)}) = 4$, $f(r_{n-1}^{(1)}) = 1$, $f(r_{n-1}^{(2)}) = 3$.For $i = n-2$, $f(q_{n-2}^{(1)}) = 1$, $f(r_{n-2}^{(2)}) = 4$. It is evident that $\chi''(Q_n^p) \le 6$. We can conclude that $\chi''(Q_n^p) = 6$. □

Theorem 3.3. *Let R_n^p be the plane graph with n pendent edges, then* $\chi''(R_n^p) = 6$.

Proof. Let $V(R_n^p) = \bigcup_{i=1}^{n} \{a_i; b_i; c_i; d_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \le i \le n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \le i \le n\}$ be

the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$ be the outer cycle and the set of vertices $\{d_i : 1 \leq i \leq n\}$ be the pendant vertices.
Let $E(R_n^p) = \{p_i^1; p_i^2 : 1 \leq i \leq n\} \cup \{q_i^1; q_i^2 : 1 \leq i \leq n\} \cup \{r_i^1; r_i^2; r_i^3 : 1 \leq i \leq n\}$
The edges classification as shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$a_i b_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$b_i a_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$b_i c_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$c_i c_{i+1(modn)}$
$r_i^{(3)}$	$1 \leq i \leq n$	$c_i d_i$

Based on the total coloring conjecture, since $\chi''(R_n^p) \geq \Delta(R_n^p) + 1 = 5 + 1 \geq 6$, then the lower bound of R_n^p is $\chi''(R_n^p) \geq 6$. We now need to prove upper bound of total coloring conjecture of R_n^p is $\chi''(R_n^p) \leq 6$. Define total coloring f, such that $f : V(R_n^p) \cup E(R_n^p) \to \{1, 2, 3, 4, 5, 6\}$ as follows:
Case(i): When $n \equiv 0\ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$;
$f(c_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(d_i) = 6$
The coloring of edges is formulated as follows:
(2) $f(p_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;
$f(q_i^{(1)}) = 6$; $f(q_i^{(2)}) = 3$; $f(r_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;
$f(r_i^2) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(3)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$
Case(ii): When $n \equiv 1\ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(p_i^{(1)}), f(p_i^{(2)}), f(q_i^{(1)}), f(q_i^{(2)}), f(r_i^{(1)}), f(r_i^{(2)}), f(r_i^{(3)})$ is given in equation (1) and (2),
for $i \in \{1, 2, \ldots, n-1\}$. $f(a_n) = f(c_n) = 4$, $f(b_n) = 2$, $f(p_n^{(1)}) = 2$, $f(p_1^{(2)}) = 2$, $f(p_n^{(2)}) = 1$, $f(r_n^{(1)}) = 5$, $f(r_n^{(2)}) = 3$, $f(r_n^{(3)}) = 1$. For $i = n-1$, $f(p_{n-1}^{(1)}) = 1$, $f(p_{n-1}^{(2)}) = 4$, $f(r_{n-1}^{(1)}) = 1$, $f(r_{n-1}^{(2)}) = 6$, $f(r_{n-1}^{(3)}) = 3$. For $i = n-2$, then

$f(r^{(2)}_{n-2}) = 4$. It is evident that $\chi''(R^p_n) \leq 6$. We can conclude that $\chi''(R^p_n) = 6$. □

Theorem 3.4. *Let E_n denotes the graph of convex polytope, then $\chi''(E_n) = 7$.*

Proof. Let $V(E_n) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$, the cycle produced by $\{d_i : 1 \leq i \leq n\}$ be the cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$ be the outer cycle. Let $E(E_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)}; p_i^{(4)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)}; q_i^{(4)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)}; r_i^{(3)} : 1 \leq i \leq n\}$
The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i a_{i+1(modn)}$
$p_i^{(4)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$c_i b_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$c_i b_{i+1(modn)}$
$q_i^{(3)}$	$1 \leq i \leq n$	$c_i d_i$
$q_i^{(4)}$	$1 \leq i \leq n$	$d_i d_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$d_i e_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$e_i d_{i+1(modn)}$
$r_i^{(3)}$	$1 \leq i \leq n$	$e_i e_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(E_n) \geq \Delta(E_n) + 1 = 6 + 1 \geq 7$ then the lower bound of E_n is $\chi''(E_n) \geq 7$. We now need to prove upper bound of total coloring conjecture of E_n is $\chi''(E_n) \leq 7$. Define total coloring f, such that $f : V(E_n) \cup E(E_n) \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows:
Case(i): When $n \equiv 0\ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

$$(1) \quad f(a_i) = f(d_i)\begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}; f(b_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases};$$

$$f(e_i) = \begin{cases} 7, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}; f(c_i) = 3.$$

The coloring of edges is formulated as follows:

$$(2) \quad f(p_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}; f(p_i^{(2)}) = 6; f(p_i^{(3)}) = 3;$$

$f(p_i^{(4)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(1)}) = 7$; $f(q_i^{(2)}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$;
$f(q_i^{(3)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(4)}) = \begin{cases} 7, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(1)}) = 5$;
$f(r_i^{(2)}) = 4$; $f(r_i^{(3)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$

Case(ii): When $n \equiv 1\ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)})$, $f(p_i^{(2)}), f(p_i^{(3)}), f(p_i^{(4)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(q_i^{(4)}), f(r_i^{(1)}), f(r_i^{(2)}), f(r_i^{(3)})$ is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}$. $f(a_n) = f(d_n) = 4$, $f(b_n) = 2$, $f(e_n) = 6$, $f(p_n^{(1)}) = 2$, $f(p_n^{(4)}) = 1$, $f(q_n^{(2)}) = 5$, $f(q_n^{(3)}) = 2$, $f(q_n^{(4)}) = 3$, $f(r_n^{(2)}) = 4$, $f(r_n^{(3)}) = 3$. For $i = n-1, f(e_{n-1}) = 3, f(p_{n-1}^{(1)}) = 1, f(p_{n-1}^{(4)}) = 4, f(r_{n-1}^{(2)}) = 7, f(r_{n-1}^{(3)}) = 2$. For $i = n-2$, then $f(r_{n-2}^{(3)}) = 1$. For $i = n-3$, then $f(p_{n-3}^{(1)}) = 4$. It is evident that $\chi''(E_n) \leq 7$. We can conclude that $\chi''(E_n) = 7$. □

Theorem 3.5. *Let S_n denotes the graph of convex polytope, then $\chi''(S_n) = 6$.*

Proof. Let $V(S_n) = \bigcup_{i=1}^{n} \{a_i; b_i; c_i; d_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$ be the inner cycle and the set of vertices $\{d_i : 1 \leq i \leq n\}$ be the outer cycle. Let $E(S_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)} : 1 \leq i \leq n\}$.
The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$c_i b_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$c_i b_{i+1(modn)}$
$q_i^{(3)}$	$1 \leq i \leq n$	$c_i c_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$c_i d_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$d_i d_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(S_n) \geq \Delta(S_n) + 1 = 5 + 1 \geq 6$ then the lower bound of S_n is $\chi''(S_n) \geq 6$. We now need to prove upper bound of total coloring conjecture of S_n is $\chi''(S_n) \leq 6$. Define total coloring f, such that $f : V(S_n) \cup E(S_n) \to \{1, 2, 3, 4, 5, 6\}$ as follows:
Case(i): When $n \equiv 0\ (mod\ 2)$
The coloring of vertices is formulated as follows:

For $1 \leq i \leq n$

(1) $f(a_i) = f(c_i) \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = f(d_i) \begin{cases} 4, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$;

The coloring of edges is formulated as follows:

(2) $f(p_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;

$f(p_i^{(3)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(1)}) = 5$; $f(q_i^{(2)}) = 6$;

$f(q_i^{(3)}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(1)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$;

$f(r_i^{(2)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$

Case(ii): When $n \equiv 1 \ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(p_i^{(1)}), f(p_i^{(2)}), f(p_i^{(3)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(r_i^{(1)}), f(r_i^{(2)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}, f(a_n) = 5, f(b_1) = 2, f(b_n) = 1, f(c_n) = 4, f(d_n) = 5, f(p_n^{(1)}) = 2, f(p_n^{(2)}) = 3, f(p_n^{(3)}) = 4, f(q_n^{(3)}) = 2, f(r_1^{(1)}) = 3, f(r_n^{(1)}) = 1, f(r_n^{(2)}) = 2$. For $i = n-1$, $f(b_{n-1}) = 3$. It is evident that $\chi''(S_n) \leq 6$. We can conclude that $\chi''(S_n) = 6$. □

Theorem 3.6. *Let T_n denotes the graph of convex polytope, then $\chi''(T_n) = 7$.*

Proof. Let $V(T_n) = \bigcup_{i=1}^{n} \{a_i; b_i; c_i; d_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$ be the inner cycle and the set of vertices $\{d_i : 1 \leq i \leq n\}$ be the outer cycle. Let $E(T_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)} : 1 \leq i \leq n\}$

The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$a_{i+1(modn)} b_i$
$q_i^{(1)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(2)}$	$1 \leq i \leq n$	$c_i b_i$
$q_i^{(3)}$	$1 \leq i \leq n$	$c_i b_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$c_i d_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$d_i d_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(T_n) \geq \Delta(T_n) + 1 = 6 + 1 \geq 7$ then the lower bound of T_n is $\chi''(T_n) \geq 7$. We now need to prove upper bound of total coloring conjecture of T_n is $\chi''(T_n) \leq 7$. Define total coloring f, such that $f : V(T_n) \cup E(T_n) \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows:
Case(i): When $n \equiv 0\ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 7, & \text{if } i \text{ is even} \end{cases}$;

$f(c_i) = 3$; $f(d_i) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$

The coloring of edges is formulated as follows:

(2) $f(p_i^{(1)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = 6$; $f(p_i^{(3)}) = 5$;

$f(q_i^{(1)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(2)}) = 1$; $f(q_i^{(3)}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 7, & \text{if } i \text{ is even} \end{cases}$;

$f(r_i^{(1)}) = 2$; $f(r_i^{(2)}) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 7, & \text{if } i \text{ is even} \end{cases}$

Case(ii): When $n \equiv 1\ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(p_i^{(1)}), f(p_i^{(2)}),$ $f(p_i^{(3)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(r_i^{(1)}), f(r_i^{(2)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}, f(a_n) = 5, f(b_n) = 7, f(c_n) = 3, f(d_n) = 4, f(p_n^{(1)}) = 2, f(q_n^{(1)}) = 3, f(q_n^{(2)}) = 2, f(q_n^{(3)}) = 7, f(r_n^{(1)}) = 1, f(r_n^{(2)}) = 3$. For $i = n - 1, f(b_{n-1}) = 3, f(c_{n-1}) = 1, f(d_{n-1}) = 3, f(q_{n-1}^{(1)}) = 1, f(q_{n-1}^{(2)}) = 7, f(q_{n-1}^{(3)}) = 4, f(r_{n-1}^{(1)}) = 2$. It is evident that $\chi''(T_n) \leq 7$. We can conclude that $\chi''(T_n) = 7$. □

Theorem 3.7. *Let G_n denotes the graph of convex polytope, then $\chi''(G_n) = 5$.*

Proof. Let $V(G_n) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$, the cycle produced by $\{d_i : 1 \leq i \leq n\}$ be the cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$ be the outer cycle. Let $E(G_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)} : 1 \leq i \leq n\}$
The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i a_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$b_i c_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$d_i c_i$
$q_i^{(3)}$	$1 \leq i \leq n$	$d_i c_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$d_i e_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$e_i e_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(G_n) \geq \Delta(G_n) + 1 = 4 + 1 \geq 5$ then the lower bound of T_n is $\chi''(G_n) \geq 5$. We now need to prove upper bound of total coloring conjecture of G_n is $\chi''(G_n) \leq 5$. Define total coloring f, such that $f : V(G_n) \cup E(G_n) \to \{1, 2, 3, 4, 5\}$ as follows:
Case(i): When $n \equiv 0 \ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = 3$; $f(c_i) = 1$;

$f(d_i) = 2$; $f(e_i) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;

The coloring of edges is formulated as follows:

(2) $f(p_i^{(1)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = 5$; $f(p_i^{(3)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;

$f(q_i^{(1)}) = 4$; $f(q_i^{(2)}) = 3$; $f(q_i^{(3)}) = 5$; $f(r_i^{(1)}) = 1$; $f(r_i^{(2)}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$

Case(ii): When $n \equiv 1 \ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)}), f(p_i^{(2)}), f(p_i^{(3)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(r_i^{(1)}), f(r_i^{(2)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}$, $f(a_n) = 3, f(b_n) = 2, f(c_n) = 1, f(d_n) = 4, f(e_n) = 5, f(p_n^{(1)}) = 5, f(p_n^{(2)}) = 1, f(p_n^{(3)}) = 4, f(q_n^{(1)}) = 3, f(q_n^{(2)}) = 3, f(r_n^{(2)}) = 4$. For $i = n-1$, $f(b_{n-1}) = 1, f(c_{n-1}) = 3, f(d_{n-1}) = 4$, $f(e_{n-1}) = 2$, $f(p_{n-1}^{(2)}) = 5$, $f(q_{n-1}^{(1)}) = 4$, $f(q_{n-1}^{(2)}) = 2$, $f(r_{n-1}^{(2)}) = 3$. For $i = n-2$, $f(d_{n-2}) = 2, f(q_{n-2}^{(2)}) = 3, f(r_{n-2}^{(2)}) = 4$. It is evident that $\chi''(G_n) \leq 5$. We can conclude that $\chi''(G_n) = 5$. □

Theorem 3.8. *Let U_n denotes the graph of convex polytope, then $\chi''(U_n) = 5$.*

Proof. Let $V(U_n) = \bigcup_{i=1}^{n} \{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$, the cycle produced by

$\{d_i : 1 \leq i \leq n\}$ be the cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$ be the outer cycle.
Let $E(U_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)} : 1 \leq i \leq n\}$
The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$b_i c_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$c_i d_i$
$q_i^{(3)}$	$1 \leq i \leq n$	$d_i c_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$d_i e_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$e_i e_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(U_n) \geq \Delta(U_n) + 1 = 4 + 1 \geq 5$ then the lower bound of U_n is $\chi''(U_n) \geq 5$. We now need to prove upper bound of total coloring conjecture of U_n is $\chi''(U_n) \leq 5$. Define total coloring f, such that $f : V(U_n) \cup E(U_n) \to \{1, 2, 3, 4, 5\}$ as follows:
Case(i): When $n \equiv 0 \ (mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$;

$f(c_i) = 3$; $f(d_i) = 1$; $f(e_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$

(2) $f(p_i^{(1)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = 5$; $f(p_i^{(3)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;

$f(q_i^{(1)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(2)}) = 5$; $f(q_i^{(3)}) = 4$; $f(r_i^{(1)}) = 3$;

$f(r_i^{(2)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$.

Case(ii): When $n \equiv 1 \ (mod\ 2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)}), f(p_i^{(2)}), f(p_i^{(3)}), f(q_i^{(1)}), f(q_i^{(2)}), f(q_i^{(3)}), f(r_i^{(1)}), f(r_i^{(2)})$, is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}$, $f(a_n) = 3, f(b_1) = 3, f(c_1) = 2, f(d_n) = 1, f(e_n) = 5, f(p_n^{(1)}) = 2, f(p_1^{(3)}) = 2, f(q_2^{(1)}) = 4, f(q_1^{(3)}) = 2, f(q_n^{(3)}) = 4, f(r_n^{(2)}) = 4$. For $i = n-1, f(c_{n-1}) = 3, f(q_{n-1}^{(1)}) = 2, f(r_{n-1}^{(2)}) = 2$. For $i = n-2, f(r_{n-2}^{(2)}) = 1$.

For $i = n-3$, $f(r_{n-3}^{(2)}) = 5$. It is evident that $\chi''(U_n) \leq 5$. We can conclude that $\chi''(U_n) = 5$. □

Theorem 3.9. *Let C_n denotes the graph of convex polytope, then $\chi''(C_n) = 6$.*

Proof. Let $V(C_n) = \bigcup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$. For instance, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ be the inner cycle; the cycle induced by $\{b_i : 1 \leq i \leq n\}$ be the interior cycle; the cycle induced by $\{c_i : 1 \leq i \leq n\}$, the cycle produced by $\{d_i : 1 \leq i \leq n\}$ be the cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$ be the outer cycle. Let $E(C_n) = \{p_i^{(1)}; p_i^{(2)}; p_i^{(3)} : 1 \leq i \leq n\} \cup \{q_i^{(1)}; q_i^{(2)}; q_i^{(3)}; q_i^{(4)} : 1 \leq i \leq n\} \cup \{r_i^{(1)}; r_i^{(2)}; r_i^{(3)} : 1 \leq i \leq n\}$
The edges classification is shown in the following table:

Edges	Range of n	Links between the vertices
$p_i^{(1)}$	$1 \leq i \leq n$	$a_i a_{i+1(modn)}$
$p_i^{(2)}$	$1 \leq i \leq n$	$a_i b_i$
$p_i^{(3)}$	$1 \leq i \leq n$	$b_i b_{i+1(modn)}$
$q_i^{(1)}$	$1 \leq i \leq n$	$b_i c_i$
$q_i^{(2)}$	$1 \leq i \leq n$	$c_i b_{i+1(modn)}$
$q_i^{(3)}$	$1 \leq i \leq n$	$c_i d_i$
$q_i^{(4)}$	$1 \leq i \leq n$	$d_i d_{i+1(modn)}$
$r_i^{(1)}$	$1 \leq i \leq n$	$d_i e_i$
$r_i^{(2)}$	$1 \leq i \leq n$	$e_i d_{i+1(modn)}$
$r_i^{(3)}$	$1 \leq i \leq n$	$e_i e_{i+1(modn)}$

Based on the total coloring conjecture, since $\chi''(C_n) \geq \Delta(C_n) + 1 = 5 + 1 \geq 6$ then the lower bound of C_n is $\chi''(C_n) \geq 6$. We now need to prove upper bound of total coloring conjecture of C_n is $\chi''(C_n) \leq 6$. Define total coloring f, such that $f : V(C_n) \cup E(C_n) \to \{1, 2, 3, 4, 5, 6\}$ as follows:
Case(i): When $n \equiv 0$ $(mod\ 2)$
The coloring of vertices is formulated as follows:
For $1 \leq i \leq n$

(1) $f(a_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(b_i) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;

$f(c_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$; $f(d_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(e_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$

(2) $f(p_i^{(1)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(p_i^{(2)}) = 6$; $f(p_i^{(3)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$;

$f(q_i^{(1)}) = 3$; $f(q_i^{(2)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$; $f(q_i^{(3)}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$;

$f(q_i^{(4)}) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(1)}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$;

$f(r_i^{(2)}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$; $f(r_i^{(3)}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$.

Case(ii): When $n \equiv 1\,(mod\,2)$ then $f(a_i), f(b_i), f(c_i), f(d_i), f(e_i), f(p_i^{(1)}), f(p_i^{(2)})$, $f(p_i^{(3)}), f(q_i^{(1)})$,$f(q_i^{(2)})$, $f(q_i^{(3)})$, $f(q_i^{(4)})$, $f(r_i^{(1)})$, $f(r_i^{(2)})$, $f(r_i^{(3)})$,is given in equation (1) and (2), for $i \in \{1, 2, \ldots, n-1\}$ $f(a_n) = 3$, $f(b_n) = 1$, $f(c_n) = 4$, $f(d_n) = 2$, $f(e_n) = 5$, $f(p_n^{(1)}) = 2$, $f(p_n^{(3)}) = 4$, $f(q_n^{(2)}) = 2$, $f(q_n^{(3)}) = 5$, $f(q_n^{(4)}) = 3$, $f(r_n^{(1)}) = 6$, $f(r_n^{(2)}) = 2$, $f(r_n^{(3)}) = 4$. For $i = n-1$, $f(c_{n-1}) = 2$, $f(d_{n-1}) = 3$, $f(e_{n-1}) = 4$, $f(q_{n-1}^{(2)}) = 5$, $f(q_{n-1}^{(3)}) = 4$, $f(q_{n-1}^{(4)}) = 1$, $f(r_{n-1}^{(1)}) = 5$, $f(r_{n-1}^{(2)}) = 4$. For $i = n-2$, $f(r_{n-2}^{(1)}) = 5$, $f(r_{n-2}^{(2)}) = 2$. For $i = n-3$, $f(r_{n-3}^{(1)}) = 4$. It is evident that $\chi''(C_n) \leq 6$. We can conclude that $\chi''(C_n) = 6$. □

Conflicts of interest : The authors declare no conflicts of interest.

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