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THE RESULTS ON UNIQUENESS OF LINEAR DIFFERENCE DIFFERENTIAL POLYNOMIALS WITH WEAKLY WEIGHTED AND RELAXED WEIGHTED SHARING

HARINA P. WAGHAMORE* AND M. ROOPA

Abstract. In this paper, we investigate the uniqueness of linear difference differential polynomials sharing a small function. By using the concepts of weakly weighted and relaxed weighted sharing of transcendental entire functions with finite order, we obtained the corresponding results, which improve and extend some results of Chao Meng [14].

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1. Introduction

In this article, we assume that the reader is familiar with basic notations of Nevanlinna value distribution theory like $T\left(r,f\right), m\left(r.f\right), N\left(r,f\right), E\left(r,f\right), ...$ (see [7],[16],[14]). A meromorphic function f means meromorphic in the complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \mathbb{C}\cup\{\infty\}$. Set $E=(a,f)=\{z : f(z)-a=0\}$, where zero point with multiplicity k is counted k times in the set. If these zeros are counted only once, then we denote the set by $\overline{E}\left(a,f\right)$.

Let f and g be two nonconstant meromorphic functions. If $E(a,f)=E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}\big(r,\frac{1}{f-a}\big)$ the counting function for zeros of $f-a$ with multiplicity not more than k, and by $\overline{N}_{k)}\big(r,\frac{1}{f-a}\big)$

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*Corresponding author..

the corresponding one for which multiplicity is not counted.
Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ be the corresponding one for which multiplicity is not counted. We set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Let $N_E\left(r, a; f, g\right)\left(\overline{N}_E(r, a; f, g)\right)$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_E\left(r, a; f, g\right) = S\left(r, f\right) + S\left(r, g\right),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0\left(r, a; f, g\right) = S\left(r, f\right) + S\left(r, g\right),$$

then we say that f and g share a "IM". Throughout the paper, we denote by $\rho(f)$ the order of f (see [5],[14],[19]).

We define shift and difference operators of $f(z)$ by $f(z+c)$ and $\triangle_c f(z) = f(z+c) - f(z)$, respectively. Note that $\triangle_c^n f(z) = \triangle_c^{n-1}\left(\triangle_c f(z)\right)$, where c is a nonzero complex number and $n \geq 2$ is a positive integer.

For further generalization of $\triangle_c f$, now we define the difference operator of an entire(meromorphic) function f as $\mathscr{L}_c\left(f\right) = f(z+c) + c_0 f(z)$, where c_0 is a non-zero complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $\mathscr{L}_c\left(f\right) = \triangle_c f(z)$.

In 1967, W.K. Hayman[6] and Clunie[3] proposed the following result.

Theorem 1.1. [6, 3] *Let f be a transcendental entire function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, corresponding to the famous conjecture of Hayman[6], Yang and Ua[16] studied the unicity of differential monomials and obtained the following theorem.

Theorem 1.2. [16] *Let f and g be two nonconstant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where $c_1, c_2 c$ are three constants satisfying $\left(c_1 c_2\right)^{n+1} c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

In 2001, Fang and Hong [4] studied the unicity of differential polynomials of the form $f^n\left(f-1\right) f'$ and proved the following uniqueness theorem.

Theorem 1.3. [4] *Let f and g be two transcendental entire functions, $n \geq 11$ an integer. If $f^n\left(f-1\right) f'$ and $g^n\left(g-1\right) g'$ share the value 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi [12] extended the above theorem as to the fixed point. They proved the following result.

Theorem 1.4. [12] *Let f and g be two transcendental entire functions, $n \geq 7$ an integer. If $f^n\,(f-1)\,f'$ and $g^n\,(g-1)\,g'$ share the value z CM, then $f \equiv g$.*

In 2010, Zhang [19] got an analogue result for translates.

Theorem 1.5. [19] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant and $n \geq 7$ is an integer. If $f^n\,(f-1)\,f(z+c)$ and $g^n\,(g-1)\,g(z+c)$ share the value $\alpha(z)$ CM, then $f \equiv g$.*

C. Meng [14] demonstrated the subsequent findings in the same study in 2014.

Theorem 1.6. [14] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant and $n \geq 7$ is an integer. If $f^n\,(f-1)\,f(z+c)$ and $g^n\,(g-1)\,g(z+c)$ share the value $(\alpha(z), 2)$, then $f \equiv g$.*

Theorem 1.7. [14] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant and $n \geq 10$ is an integer. If $f^n\,(f-1)\,f(z+c)$ and $g^n\,(g-1)\,g(z+c)$ share $(\alpha(z), 2)^*$, then $f \equiv g$.*

Theorem 1.8. [14] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant and $n \geq 16$ is an integer. If $\overline{E}_{2)}\,(\alpha(z), f^n(f-1)f(z+c)) = \overline{E}_{2)}\,(\alpha(z), g^n(g-1)f(g+c))$, then $f \equiv g$.*

Motivation: In this paper, we consider Theorems 1.6, 1.7 and 1.8 motivate us to think that, for difference implies that if $f^n(z)f(z+c)$ is replaced by $\left(f^n\,(\alpha f^m+\beta)^s\,\mathscr{L}_c\,(f)\right)^{(k)}$ in Theorems 1.6, 1.7 and 1.8 herein, where α, β are complex constants with $|\alpha+\beta| \neq 0$ and c is a non-zero complex constant, $n \geq 1, k \geq 0, s \geq 1, m \geq 1$ are positive integers . In this way, we prove the results which improve and extend Theorems 1.6, 1.7 and 1.8.

2. Main results

Theorem 2.1. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, $n, k(\geq 0), s, m(\geq k+1)$ are positive integers and α, β are two complex constants with $|\alpha|+|\beta| \neq 0$, such that $n \geq 2k+ms+6$ when $ms \leq k+1$ and $n \geq 4k-ms+10$ when $ms > k+1$. If $\left(f^n\,(\alpha f^m+\beta)^s\,\mathscr{L}_c\,(f)\right)^{(k)}$ and $\left(g^n\,(\alpha g^m+\beta)^s\,\mathscr{L}_c\,(g)\right)^{(k)}$ share " $(\alpha(z), 2)$ " and $f(z), f(z+c)$ share 0 CM, then either $f \equiv g$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by*

$$R(w_1, w_2) = w_1^n(w_1-1)^m w_1(z+\eta) - w_2^n(w_2-1)^m w_2(z+\eta) \tag{1}$$

Theorem 2.2. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, $n, k(\geq 0), s, m(\geq k+1)$ are positive integers and α, β are two complex constants with $|\alpha| + |\beta| \neq 0$, such that $n \geq 3k + 2ms + 8$ when $ms \leq k+1$ and $n \geq 6k - ms + 13$ when $ms > k+1$. If $\left(f^n\left(\alpha f^m + \beta\right)^s \mathscr{L}_c(f)\right)^{(k)}$ and $\left(g^n\left(\alpha g^m + \beta\right)^s \mathscr{L}_c(g)\right)^{(k)}$ share " $(\alpha(z), 2)^*$ " and $f(z), f(z+c)$ share 0 CM, then either $f \equiv g$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g)$ is given in (1). Then conclusion of Theorem 2.1 holds.*

Theorem 2.3. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant, $n \geq 1, k \geq 0, s \geq 1, m \geq 1$ are positive integers and α, β are two complex constants with $|\alpha| + |\beta| \neq 0$, satisfying $n \geq 5k + 4ms + 1$ when $ms \leq k+1$ and $n \geq 10k - ms + 19$ when $ms > k+1$. If $\overline{E}_{2)}\left(\alpha(z), \left(f^n\left(\alpha f^m + \beta\right)^s \mathscr{L}_c(f)\right)^{(k)}\right) = \overline{E}_{2)}\left(\alpha(z), \left(g^n\left(\alpha g^m + \beta\right)^s \mathscr{L}_c(g)\right)^{(k)}\right)$, then $f \equiv g$. Then conclusion of Theorem 2.1 holds.*

Remark 2.1. Clearly, for the $c_0 = 0$, $\mathscr{L}_c(f)$ becomes $f(z+c)$, and therefore Theorem 2.1 coincides with Theorem 1.6 and Theorem 2.2 with Theorem 1.7.

Remark 2.2. The value is decreased in Theorem 1.7, if one replaces

$$\overline{E}_{2)}\left(\alpha(z), f(z)\right) = \overline{E}_{2)}\left(\alpha(z), g(z)\right)$$

for any two meromorphic functions f and g. We have proven the Theorem 2.3 by noting the lower bound of n exists.

Since for particular case $c_0 = -1$, $\mathscr{L}_c(f) = \triangle_c(f)$, we observe the following corollaries.

Corollary 2.4. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, $n, k(\geq 0), s, m(\geq k+1)$ are positive integers and α, β are two complex constants with $|\alpha| + |\beta| \neq 0$, such that $n \geq 2k + ms + 6$ when $ms \leq k+1$ and $n \geq 4k - ms + 10$ when $ms > k+1$. If $\left(f^n\left(\alpha f^m + \beta\right)^s \triangle_c(f)\right)^{(k)}$ and $\left(g^n\left(\alpha g^m + \beta\right)^s \triangle_c(g)\right)^{(k)}$, share " $(\alpha(z), 2)$ ", then conclusion of theorem 2.1, holds.*

Corollary 2.5. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose c be a non-zero complex constant, $n, k(\geq 0), s, m(\geq k+1)$ are positive integers and α, β are two complex constants with $|\alpha| + |\beta| \neq 0$, such that $n \geq 3k + 2ms + 8$ when $ms \leq k+1$ and $n \geq 6k - ms + 13$ when $ms > k+1$. If $\left(f^n\left(\alpha f^m + \beta\right)^s \triangle_c(f)\right)^{(k)}$ and $\left(g^n\left(\alpha g^m + \beta\right)^s \triangle_c(g)\right)^{(k)}$, share " $(\alpha(z), 2)^*$ ", then conclusion of Theorem 2.1, holds.*

Corollary 2.6. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is an non-zero complex constant, $n \geq 1, k \geq 0, s \geq 1, m \geq 1$ are positive integers and α, β are two complex constants with $|\alpha| + |\beta| \neq 0$, satisfying $n \geq 5k + 4ms + 1$ when $ms \leq k + 1$ and $n \geq 10k - ms + 19$ when $ms > k + 1$. If $\overline{E}_{2)}\left(\alpha(z), (f^n(\alpha f^m + \beta)^s \triangle_c(f))^{(k)}\right) = \overline{E}_{2)}\left(\alpha(z), (g^n(\alpha g^m + \beta)^s \triangle_c(g))^{(k)}\right)$, then the conclusion of Theorem 2.1 holds.*

Example 2.7. Let $f(z) = e^z$ and $g(z) = te^z$ where $t^{n+m+1} = 1, k = 0, c_0 = 0, \alpha = 1, \beta = -1, s = 1$. Then it is easy to verify that $(f^n(\alpha f^m + \beta)^s \mathscr{L}_c(f))^{(k)}$ and $(g^n(\alpha g^m + \beta)^s \mathscr{L}_c(g))^{(k)}$ share $\alpha(z)$ CM. Here f and g satisfy the conclusion of Theorem 2.1.

Example 2.8. Let $f(z) = sinz$ and $g(z) = cosz, k = 0, c_0 = -1, c = 2\pi, \alpha = 1, \beta = -1, s = 1, m = n = 1$. Then one can easily verify that $(f^n(\alpha f^m + \beta)^s \mathscr{L}_c(f))^{(k)}$ and $(g^n(\alpha g^m + \beta)^s \mathscr{L}_c(g))^{(k)}$ share $\alpha(z)$. Here $R(f, g) = 0$, then f and g satisfy the algebraic equations of the conclusion of Theorem 2.1.

3. Auxiliary Definitions

Definition 3.1. [11] Let f and g share the value a "IM" and k be a positive integer or infinity. Then $\overline{N}^E_{k)}(r, a; f, g)$ denotes the reduced counting function of those a points of f whose multiplicities are equal to the corresponding a points of g, and both of their multiplicities are not less than k, $\overline{N}^0_{(k}(r, a; f, g)$ denotes the reduced counting function those a-points of f which are a points of g, and both of their multiplicities are not less than k.

Definition 3.2. [11] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If

$$\overline{N}(r, a; f| \leq k) - \overline{N}^E_{k)}(r, a; f, g) = S(r, f),$$
$$\overline{N}(r, a; g| \leq k) - \overline{N}^E_{k)}(r, a; f, g) = S(r, g),$$
$$\overline{N}(r, a; f| \geq k + 1) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, f),$$
$$\overline{N}(r, a; g| \geq k + 1) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, g), if\, k = 0$$
$$\overline{N}\left(r, \frac{1}{f - a}\right) - \overline{N}_0(r, a; f, g) = S(r, f),$$
$$\overline{N}\left(r, \frac{1}{g - a}\right) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say that f and g share the value a weakly with weight k and we write f and g share "(a, k)".

Definition 3.3. [1] Let k be a positive integer and for $a \in \mathbb{C} - \{0\}$, $E_{k)}(a; f) = E_{k)}(a; g)$. Let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$

of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ $\big(\overline{N}_L(r,a;g)\big)$ the reduced counting function of those $a-$ points of f and g for which $p>q\geq k+1$ $(q>p\geq k+1)$, by $\overline{N}_E^{(k+1}(r,a;f)$ the reducing counting function of those $a-$ points of f and g for which $p=q\geq k+1$, by $\overline{N}_{f\geq k+1}(r,a;f|g\neq a)$ the reduced counting functions of those $a-$ points of f and g for which p$\geq k+1$ and $q=0$.

Definition 3.4. [1] Let k be a positive integer and for $a\in\mathbb{C}-\{0\}$, let f and g share a "IM". Let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_{f\geq k+1}(r,a;f|g=m)$ the reduced counting functions of those $a-$ points of f and g for which p$\geq k+1$ and $q=m$. We can define $\overline{N}_L(r,a;f)$ $\big(\overline{N}_L(r,a;g)\big)$ and $\overline{N}_E^{(k+1}(r,a;f)$ in a similar manner as defined in the previous definition.

The term "relaxed weighted sharing", which is weaker than "weakly weighted sharing" was introduced by A. Banerjee et.al [1] scaling between IM and CM.

Definition 3.5. [1] We denote by $\overline{N}(r,a;f|=p;g|=q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.

Definition 3.6. [1] Let f,g share a IM. Also let k be a positive integer or ∞ and $a\in\mathbb{C}\cup\{\infty\}$. If $\sum\limits_{p,q\leq k}\overline{N}(r,a;f|=p;g|=q)=S(r)$, the we say f and g share a with weight k in a relaxed manner. Here we write f and g share $(a,k)^*$ to mean that f and g share a with weight k relaxed manner.

4. Preliminary Lemmas

We state some lemmas which will be needed in the sequel. We denote H the following

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

where F and G are non-constant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 4.1 ([1]). *Let H be defined as above. If F and G share "(1, 2)" and $H\neq 0$, then*

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\big(r,F\big)+N_2\left(r,\frac{1}{G}\right)+N_2\big(r,G\big)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)$$
$$-\sum_{p=3}^{\infty}\overline{N}_{(p}\left(r,\frac{G}{G'}\right)+S(r,F)+S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 4.2 ([1]). *Let H be defined as above. If F and G share $(1,2)^*$ and $H\not\equiv 0$, then*

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\big(r,F\big)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+\overline{N}\left(r,\frac{1}{F}\right)$$

$$+\overline{N}\big(r,F\big)-m\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 4.3 ([18])**.** *Let* H *be defined as above. If* $H\equiv 0$ *and*

$$\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,F\right)+\overline{N}\big(r,\frac{1}{G}\big)+\overline{N}\big(r,G\big)}{T\left(r\right)}<1, r\in I,$$

where $T(r)=max\left\{T(r,F),T(r,G)\right\}$ *and* I *is a set with infinite linear measure, then* $F\equiv G$ *or* $FG\equiv 1$.

Lemma 4.4 ([2])**.** *Let* $f(z)$ *be a meromorphic function in the complex plane of finite order* $\sigma(f)$*, and* η *be a fixed nonzero complex number. Then for each* $\epsilon>0$*, one has*

$$T\big(r,f(z+\eta)\big)=T\big(r,f(z)\big)+O\big(r^{\sigma-1+\epsilon}\big)+O\big(logr\big)$$

Lemma 4.5 ([15])**.** *Let* $f(z)$ *be an entire function of finite order* $\sigma(f), c$ *a fixed non zero complex number, and* $P(z)=a_nf^n(z)+a_{n-1}f^{n-1})(z)+...+a_1f(z)+a_0$ *where* $a_j(j=0,1,...,n)$ *are constants. If* $F(z)=f^n\left(\alpha f^m+\beta\right)^s\mathscr{L}_cf(z)$*, and* $f(z), f(z+c)$ *share* 0 *CM. Then*

$$T(r,F)=(n+ms+1)T\big(r,f\big)+O\big(r^{\sigma(f)-1+\epsilon}\big)+O(logr)$$

and the same inequality is true for $T(r,G)$.

Lemma 4.6 ([20])**.** *Let* f *be nonconstant meromorphic function and* p,k *be integers. Then*

$$N_p\big(r,0;f^{(k)}\big)\le T\big(r,f^{(k)}\big)-T\big(r,f\big)+N_{p+k}\big(r,0;f\big)+S\big(r,f\big),$$
$$N_p\big(r,0;f^{(k)}\big)\le k\overline{N}\big(r,\infty;f\big)+N_{p+k}\big(r,0;f\big)+S\big(r,f\big)$$

Lemma 4.7 ([13])**.** *Let* F *and* G *be two nonconstant entire functions, and* $p\ge 2$ *an integer. If* $\overline{E}_{p)}\left(1,F\right)=\overline{E}_{p)}\left(1,G\right)$ *and* $H\ne 0$*, then*

$$\begin{aligned}&T\left(r,F\right)\\&\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S\left(r,F\right)+S\left(r,G\right)\end{aligned}$$

Lemma 4.8. *Let* f *and* g *be an entire functions* $n(\ge 1), m(\ge 1), s(\ge 1), k(\ge 0)$ *be integers, and* α,β *are two constants with* $|\alpha|+|\beta|\ne 0$,
let $F=\left(f^n\left(\alpha f^m+\beta\right)^s\mathscr{L}_c\left(f\right)\right)^{(k)}$ *and* $\left(g^n\left(\alpha g^m+\beta\right)^s\mathscr{L}_c\left(g\right)\right)^{(k)}$*. If there exists nonzero constants* c_1,c_2 *such that* $\overline{N}\big(r,c_1;F\big)=\overline{N}\big(r,0;G\big)$ *and* $\overline{N}\big(r,c_2;G\big)=\overline{N}\big(r,0;F\big)$*, then* $n\le 2k+ms+3$ *when* $ms>k+1$ *and* $n\le 4k-ms+5$ *when* $ms>k+1$.

Proof. We put $F = \left(f^n\left(\alpha f^m+\beta\right)^s \mathscr{L}_c\left(f\right)\right)^{(k)}$, $G = \left(g^n\left(\alpha g^m+\beta\right)^s \mathscr{L}_c\left(g\right)\right)^{(k)}$. Then from Lemma 4.5, we have

$$T\left(r,F_1\right) = \left(n+ms+1\right)T\left(r,f\right) + O\left\{r^{\rho(f)-1+\epsilon}\right\} + S\left(r,f\right) \tag{2}$$

$$T\left(r,G_1\right) = \left(n+ms+1\right)T\left(r,g\right) + O\left\{r^{\rho(g)-1+\epsilon}\right\} + S\left(r,g\right) \tag{3}$$

Now by the hypothesis and the second fundamental theorem of Nevanlinna, we have

$$T\left(r,F\right) \leq \overline{N}\left(r,0,F\right) + \overline{N}\left(r,C,F\right) + S\left(r,F\right) \tag{4}$$

$$T\left(r,G\right) \leq \overline{N}\left(r,0,G\right) + \overline{N}\left(r,C,G\right) + S\left(r,G\right) \tag{5}$$

using (2),(3),(4),((5)), we obtain

$$\begin{aligned}
\left(n+ms+1\right)T\left(r,f\right) &\leq T\left(r,F\right) - \overline{N}\left(r,0:F\right) + N_{k+1}\left(r,0:F_1\right) + S\left(r,f\right)\\
&\leq \overline{N}\left(r,0:G\right) + N_{k+1}\left(r,0:F_1\right) + S\left(r,f\right)\\
\left(n+ms+1\right)T\left(r,f\right) &\leq N_{k+1}\left(r,0:G_1\right) + N_{k+1}\left(r,0:F_1\right) + S\left(r,f\right) + S\left(r,g\right)\\
&\leq \left(k+ms+1\right)T\left(r,f\right) + S\left(r,f\right) + S\left(r,g\right)
\end{aligned} \tag{6}$$

Similarly

$$\left(n+ms+1\right)T\left(r,g\right) \leq \left(k+ms+1\right)T\left(r,g\right) + S\left(r,f\right) + S\left(r,g\right) \tag{7}$$

Case 1: When $ms \leq k+1$, using (6), (7) and Lemma 4.5 we see that,

$$\begin{aligned}
\left(n+ms+1\right)T\left(r,f\right) \leq \left(k+ms+2\right)\left[T\left(r,f\right)+T\left(r,g\right)\right] + O\left\{r^{\rho(f)-1+\epsilon}\right\} +\\
O\left\{r^{\rho(g)-1+\epsilon}\right\} + S\left(r,f\right) + S\left(r,g\right)
\end{aligned} \tag{8}$$

Similarly

$$\begin{aligned}
\left(n+ms+1\right)T\left(r,g\right) \leq \left(k+ms+2\right)\left[T\left(r,g\right)+T\left(r,f\right)\right] + O\left\{r^{\rho(g)-1+\epsilon}\right\} +\\
O\left\{r^{\rho(f)-1+\epsilon}\right\} + S\left(r,g\right) + S\left(r,f\right)
\end{aligned} \tag{9}$$

From (8), (9) we obtain,

$$\left(n-2k-ms-3\right)\left[T\left(r,f\right)+T\left(r,g\right)\right] \leq S\left(r,f\right) + S\left(r,g\right)$$

which gives $n \leq 2k + ms + 3$ when $ms \leq k + 1$.

Case 2: When $ms > k + 1$, by using (6) and lemma 4.5 we obtain,

$$
\begin{aligned}
(n+ms+1)\,T(r,f) &\leq (k+ms+2)\,[T(r,f)+T(r,g)] \\
&\quad + O\left\{r^{\rho(f)-1+\epsilon}\right\} + O\left\{r^{\rho(g)-1+\epsilon}\right\} + S(r,f) + S(r,g) \\
(n+ms+1)\,T(r,f) &\leq (2k+3)\,[T(r,f)+T(r,g)] + O\left\{r^{\rho(f)-1+\epsilon}\right\} \\
&\quad + O\left\{r^{\rho(g)-1+\epsilon}\right\} + S(r,f) + S(r,g)
\end{aligned}
\tag{10}
$$

Similarly,

$$
\begin{aligned}
(n+ms+1)\,T(r,g) &\leq (2k+3)\,[T(r,g)+T(r,f)] + O\left\{r^{\rho(g)-1+\epsilon}\right\} + \\
&\quad O\left\{r^{\rho(f)-1+\epsilon}\right\} + S(r,g) + S(r,f)
\end{aligned}
\tag{11}
$$

From (10), (11), we obtain

$$
\begin{aligned}
&(n+ms+1)\,[T(r,f)+T(r,g)] \leq (4k+6)\,[T(r,f)+T(r,g)] + S(r,f) + S(r,g) \\
&(n+ms-4k-5)\,[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g),
\end{aligned}
$$

which gives $n \leq 4k - ms + 5$ when $ms > k + 1$. This proves the lemma. □

Lemma 4.9 ([1]). *Let F and G be nonconstant meromrophic functions that share "$(1,2)$" and $H \neq 0$. Then*

$$
\begin{aligned}
T(r,F) \leq & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\
& - \sum_{3}^{\infty} \overline{N}\left(r,0;\frac{G'}{G}| \geq p\right) + S(r,F) + S(r,G),
\end{aligned}
$$

and the same inequality holds for $T(r,G)$.

Lemma 4.10 ([1]). *Let F, G be nonconstant meromorphic functions that share $(1,2)^*$ and $H \neq 0$. Then*

$$
\begin{aligned}
T(r,F) \leq & N_2(r,0;F) + N_2(r,0;G) + \overline{\overline{N}}(r,0;F) + \overline{\overline{N}}(r,\infty;G) \\
& - m(r,1;G) + S(r,F) + S(r,G),
\end{aligned}
$$

5. Proof of the Main Results

Theorem 2.1

Proof. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where
$F_1 = f^n(\alpha f^m + \beta)^s \mathscr{L}_c(f), G_1 = g^n(\alpha g^m + \beta)^s \mathscr{L}_c(g)$. Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles

of $\alpha(z)$. Then we have (2) and (3). If possible, we may assume that $H \neq 0$. Using (2), (3) and Lemma 4.3, we obtain

$$\begin{aligned} N_2\left(r,0;F\right) \leq & N_2\left(r,0;F_1^{(k)}\right)+S(r,f) \\ \leq & T\left(r,F_1^{(k)}\right)-(n+ms+1)T(r,f)+N_{k+2}\left(r,0;F_1\right)+S(r,f) \\ N_2\left(r,0;F\right) \leq & T\left(r,\alpha(z)\right)-(n+ms+1)T(r,f)+N_{k+2}\left(r,0;F_1\right)+S(r,f) \end{aligned}$$

From this we get

$$(n+ms+1)T(r,f) \leq T(r,F)-N_2(r,0;F)+N_{k+2}(r,0;F_1)+S(r,f) \tag{12}$$

Also by (3) we obtain,

$$\begin{aligned} N_2\left(r,0;F\right) \leq & N_2\left(r,0;(F_1)^{(k)}\right)+S(r,f) \\ \leq & T\left(r,(F_1)^{(k)}\right)-(n+ms+1)T(r,f)+N_{k+2}\left(r,0;F_1\right)+S(r,f) \\ \leq & T(r.F)-(n+ms+1)T(r,f)+N_{k+2}\left(r,0;F_1\right)+S(r,f) \end{aligned}$$

From this we get,

$$(n+ms+1)T(r,f) \leq T(r,F)-N_2(r,0;F)+N_{k+2}(r,0;F_1)+S(r,f) \tag{13}$$

Similarly,

$$N_2\left(r,0;G\right) \leq N_{k+2}(r,0;G_1)+S(r,g) \tag{14}$$

Using (14) and Lemma 4.5, we obtain (12)

$$\begin{aligned} (n+ms+1)T(r,f) \leq & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+N_{k+2}(r,0;F_1) \\ & +S(r,f)+S(r,g) \\ \leq & N_{k+2}(r,0;F_1)+N_2(r,0;G_1)+S(r,f)+S(r,g) \end{aligned} \tag{15}$$

We suppose that $ms \leq k+1$. Then from (15) we get,

$$(n+ms+1)T(r,f) \leq (k+ms+2)\left\{T(r,f)+T(r,g)\right\}+S(r,f)+S(r,g) \tag{16}$$

Similarly,

$$(n+ms+1)T(r,g) \leq (k+ms+2)\left\{T(r,g)+T(r,f)\right\}+S(r,g)+S(r,f) \tag{17}$$

From (16) and (17) together yield,

$$\begin{aligned} (n-2k-ms-5)\left\{T(r,f)+T(r,g)\right\} \leq & O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\} \\ & +S(r,f)+S(r,g), \end{aligned}$$

a contradiction with the assumption that $n \leq 2k+ms+6$. Next we assume that $ms > k+1$. Then from (15) we obtain,

$$(n+ms+1)\left\{T(r,f)\right\} \leq (2k+5)\left\{T(r,f)+T(r,g)\right\}$$

$$O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,f)+S(r,g) \tag{18}$$

In a similar manner we obtain,

$$\begin{aligned}(n+ms+1)\{T(r,g)\} \leq &(2k+5)\{T(r,g)+T(r,f)\}\\ &O\left\{r^{\rho(g)-1+\epsilon}\right\}+O\left\{r^{\rho(f)-1+\epsilon}\right\}+S(r,g)+S(r,f)\end{aligned} \tag{19}$$

By (18) and (19) together give,

$$\begin{aligned}(n+ms-4k-9)\{T(r,f)+T(r,g)\} \leq &O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ &+S(r,f)+S(r,g),\end{aligned}$$

a contradiction with the assumption that $n \geq 4k-ms+10$. Therefore we must have $H \equiv 0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{GF'}{G-1}\right)=0$$

Integrating both sides twice we get,

$$\left(\frac{1}{F-1}\right)=\frac{A}{G-1}+B, \tag{20}$$

where $A(\neq 0)$ ad B are constants. From (20) it is clear that F,G share 1 CM and hence they share "$(1,2)$". Therefore $n \geq 2k+ms+6$ if $ms \leq k+1$ and $n \geq 4k-ms+10$ is $ms > k+1$. We now discuss the following cases separately.

Case 1: Let $B \neq 0$ and $A=B$. Then from (20) we get

$$\left(\frac{1}{F-1}\right)=\frac{BG}{G-1}, \tag{21}$$

If $B=-1$, then from (21) we obtain $FG=1$. Then

$$\left(f^n\left(\alpha f^m+\beta\right)\mathscr{L}_c(f)\right)^{(k)}\left(g^n\left(\alpha g^m+\beta\right)\mathscr{L}_c(g)\right)^{(k)}=\alpha^2$$

It can be easily seen from the above that $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus we obtain,

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

which is not possible.

If $B \neq -1$, from (21), we see that $\frac{1}{F}=\frac{BG}{(1+B)G-1}$ and so $\overline{N}\left(r,\frac{1}{1+B};G\right)=\overline{N}(r,0;F)$.

Using (2), (3), (5) and the second fundamental theorem of Nevanlinna, we deduce that

$$T(r,G) \leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{1+B};G\right)+\overline{N}(r,\infty;G)+S(r,G)$$

$$\begin{aligned}
&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\
&\leq N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)-(n+ms+1)T(r,g)\\
&\quad +S(r,g) \qquad (22)
\end{aligned}$$

If $ms \leq k+1$, from (22) we get,

$$\begin{aligned}
(n+ms+1)T(r,g) \leq &(k+ms+2)\{T(r,f)+T(r,g)\}+O\left\{r^{\rho(f)-1+\epsilon}\right\}\\
&+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,f)+S(r,g)
\end{aligned}$$

Thus we obtain,

$$\begin{aligned}
(n-2k-ms-3)T(r,g)\{T(r,f)+T(r,g)\} \leq &O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\
&+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction since $n \geq 2k+ms+6$. If $ms > k+1$, from (21) we get

$$\begin{aligned}
(n+ms+1)T(r,g) \leq &(2k+3)\{T(r,f)+T(r,g)\}+O\left\{r^{\rho(f)-1+\epsilon}\right\}\\
&+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,g)
\end{aligned}$$

Therefore

$$\begin{aligned}
(n+ms-4k-5)T(r,g)\{T(r,f)+T(r,g)\} \leq &O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\
&+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction since $n \geq 4k-ms+10$.

Case 2: Let $b \neq 0$ and $A \neq 0$. Then from (20) we see that ,

$$F=\frac{(B+1)G-(B-A+1)}{BG+(A-B)}$$

and hence

$$\overline{N}\left(r,\frac{B-A+1}{B+1};G\right)=\overline{N}(r,0;F),$$

Proceeding in manner similar to Case 1 we can get a contradiction.

Case 3: Let $B=0$ and $A \neq 0$. Then from (20) we have $F=\dfrac{G+A-1}{A}$ and $G=AF-(A-1)$. If $A \neq 1$, it follows that $\overline{N}\left(r,\dfrac{A-1}{A};F\right)=\overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$

Then applying Lemma 4.4 we arrive at a contradiction. Therefore $A=1$ and then $F=G$. That is

$$\left(f^n\left(\alpha f^m+\beta\right)\mathscr{L}_c(f)\right)^{(k)}=\left(g^n\left(\alpha g^m+\beta\right)\mathscr{L}_c(g)\right)^{(k)}$$

Integrating once we obtain,

$$\left(f^n\left(\alpha f^m+\beta\right)\mathscr{L}_c(f)\right)^{(k)}=\left(g^n\left(\alpha g^m+\beta\right)\mathscr{L}_c(g)\right)^{(k)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, by Lemma 4.4 it follows that $n \leq 4k-ms+1$ when $ms > k+1$, a contradiction to the hypothesis. Hence $c_{k-1} = 0$. Repeating the process k times, we deduce that

$$\left(f^n\left(\alpha f^m+\beta\right)\mathscr{L}_c(f)\right) = \left(g^n\left(\alpha g^m+\beta\right)\mathscr{L}_c(g)\right) \tag{23}$$

Set $h = \dfrac{f}{L}$. If h is a constant, then substituting $f = Lh$ in (23), we deduce that

$$\mathscr{L}_c(g)\Big(\left((\alpha g^m)^s\left(h^{n+ms+1}-1\right) + \binom{s}{1}(\alpha g^m)^{s-1}\beta^{s-1}(h^{n+m(s-1)+1}) + \ldots + \right.$$
$$\left.\binom{s}{s}\beta^s(h^{n+1}-1)\right)\Big)$$

which implies $h = 1$ and hence $f \equiv g$. If h is not a constant, then it follows from (23) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g)$ is given by (1). This completes the proof of Theorem 2.1 □

Theorem 2.2

Proof. Let F, G, F_1, G_1 be defined as in Theorem 2.1. Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. Let $H \neq 0$. The, using (3) for $p = 1$, (14) and Lemma 4.6, we obtain from (12)

$$\begin{aligned}(n+ms+1)T(r,f) \leq & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F)\\ & + \overline{N}(r,\infty;F) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g),\\ \leq & N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + N_{k+1}(r,0;F_1)\\ & + S(r,f) + S(r,g)\end{aligned} \tag{24}$$

If $ms \geq k+1$, from (24) we obtain,

$$\begin{aligned}(n+ms+1)T(r,f) \leq & (2k+2ms+5)T(r,f) + (k+ms+3)T(r,g)\\ & + O\left\{r^{\rho(f)-1+\epsilon}\right\} + O\left\{r^{\rho(g)-1+\epsilon}\right\} + S(r,f) + S(r,g)\end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned}(n+ms+1)T(r,g) \leq & (2k+2ms+5)T(r,g) + (k+ms+3)T(r,f)\\ & + O\left\{r^{\rho(g)-1+\epsilon}\right\} + O\left\{r^{\rho(f)-1+\epsilon}\right\} + S(r,g) + S(r,f)\end{aligned} \tag{26}$$

From (25) and (26) we get

$$\begin{aligned}(n+ms+1)\left\{T(r,f)+T(r,g)\right\} \leq & O\left\{r^{\rho(f)-1+\epsilon}\right\} + O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ & + S(r,f) + S(r,g)\end{aligned}$$

Contradicting the fact that $n \geq 3k+2ms+8$.
If $ms \geq k+1$, from (24) we get,

$$\begin{aligned}(n+ms+1)T(r,f) \leq &(4k+8)T(r,f)+(2k+5)T(r,g)+O\left\{r^{\rho(f)-1+\epsilon}\right\}\\ &+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,f)+S(r,g)\end{aligned} \tag{27}$$

In a similar manner we obtain,

$$\begin{aligned}(n+ms+1)T(r,g) \leq &(4k+8)T(r,g)+(2k+5)T(r,f)+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ &+O\left\{r^{\rho(f)-1+\epsilon}\right\}+S(r,g)+S(r,f)\end{aligned} \tag{28}$$

From (27) and (28) we get

$$\begin{aligned}(n+ms-6k-12)\left\{T(r,f)+T(r,g)\right\} \leq &O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ &+S(r,f)+S(r,g)\end{aligned}$$

Contradicting to the fact that $n \geq 6k-ms+13$. Thus $h \equiv 0$ and proof of the theorem follows from Theorem 2.1. This completes the proof of Theorem 2.2. □

Theorem 2.3

Proof. Let F, G, F_1, G_1 be as defined in Theorem 2.1. Then F and G are the transcendental meromorphic functions such that $\overline{E}_{2)}(1,F)=\overline{E}_{2)}(1,G)$ except for zeros and poles of $\alpha(z)$. Let $H \neq 0$. Then by (3), (4) and Lemma 4.7 we deduce from (12),

$$\begin{aligned}(n+ms+1)T(r,f) \leq &N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1)\\ &+S(r,f)+S(r,g),\\ \leq &N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\ &+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\end{aligned} \tag{29}$$

If $ms \leq k+1$, from (29) we get,

$$\begin{aligned}(n+ms+1)T(r,f) \leq &(3k+3ms+7)T(r,f)+(2k+2ms+5)T(r,g)\\ &+O\left\{r^{\rho(g)-1+\epsilon}\right\}+O\left\{r^{\rho(f)-1+\epsilon}\right\}+S(r,g)+S(r,f)\end{aligned} \tag{30}$$

Similarly,

$$\begin{aligned}(n+ms+1)T(r,g) \leq &(3k+3ms+7)T(r,g)+(2k+2ms+5)T(r,f)\\ &+O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,f)+S(r,g)\end{aligned} \tag{31}$$

Combining (30) and (31) we obtain

$$\begin{aligned}(n-5k-4ms-11)\{T(r,f)+T(r,g)\} \leq & O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ & +S(r,f)+S(r,g),\end{aligned}$$

a contradiction with the assumption that $n \geq 5k+4ms+12$.
If $ms \geq k+1$, from (29) we get

$$\begin{aligned}(n+ms+1)T(r,f) \leq & (6k+11)T(r,f)+(4k+8)T(r,g)\\ & +O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}+S(r,f)+S(r,g)\end{aligned} \tag{32}$$

In similar manner we obtain

$$\begin{aligned}(n+ms+1)T(r,g) \leq & (6k+11)T(r,g)+(4k+8)T(r,f)\\ & +O\left\{r^{\rho(g)-1+\epsilon}\right\}+O\left\{r^{\rho(f)-1+\epsilon}\right\}+S(r,g)+S(r,f)\end{aligned} \tag{33}$$

From (32) and (33) we get

$$\begin{aligned}(n+ms-10k-18)\{T(r,f)+T(r,g)\} \leq & O\left\{r^{\rho(f)-1+\epsilon}\right\}+O\left\{r^{\rho(g)-1+\epsilon}\right\}\\ & +S(r,f)+S(r,g),\end{aligned}$$

contradicting the fact that $n \geq 10k-ms+19$. Thus $H \equiv 0$ and the rest of the theorem follows from the proof of Theorem 2.1. This completes the proof of Theorem 2.3. □

6. Conclusion

The two non constant transcendental entire functions of the form $\left(f^n\left(\alpha f^m+\beta\right)^s \mathscr{L}_c(f)\right)^{(k)}$ and $\left(g^n\left(\alpha g^m+\beta\right)^s \mathscr{L}_c(g)\right)^{(k)}$ sharing a small function, there exists uniqueness between the entire functions in terms of weighted sharing and relaxed weighted sharing, as well as there conditions.

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Harina P. Waghamore received M.Sc. from Darwad University and Ph.D. from Darwad University. She is currently a professor at Bangalore University since 2003. Her research interests are Nevanlinna Theory, Functional Analysis and Complex analysis.

Department of Mathematics, Bangalore University, Bangalore 560-560, Karnataka, India.
e-mail: harinapw@gmail.com

M. Roopa received M.Sc. from Mysore University and Research scholar at University of Bangalore. Since her inception into research to 2021, her research interests include Complex analysis and Nevanlinna theory.

Department of Mathematics, Bangalore University, Bangalore 560-560, Karnataka, India.
e-mail: mroopapakash@gmail.com