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THE SPLIT AND NON-SPLIT TREE (D, C)-NUMBER OF A GRAPH

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ABSTRACT. In this paper, we introduce the concept of split and non-split tree (D, C)- set of a connected graph G and its associated color variable, namely split tree (D, C) number and non-split tree (D, C) number of G. A subset $S \subseteq V$ of vertices in G is said to be a split tree (D, C) set of G if S is a tree (D, C) set and $\langle V - S \rangle$ is disconnected. The minimum size of the split tree (D, C) set of G is the split tree (D, C) number of G, $\gamma_{\chi_{ST}}(G) = \min\{|S| : S \text{ is a split tree } (D, C) \text{ set}\}$. A subset $S \subseteq V$ of vertices of G is said to be a non-split tree (D, C) set of G if S is a tree (D, C) set and $\langle V - S \rangle$ is connected and non-split tree (D, C) number of G is $\gamma_{\chi_{NST}}(G) = \min\{|S| : S \text{ is a non-split tree } (D, C) \text{ set of } G\}$. The split and non-split tree (D, C) number of some standard graphs and its compliments are identified.

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1. Introduction

A graph $G = (V, E)$ is an ordered pair of two sets V and E, where V is called vertex set and E is called an edge set of G. The cardinality of vertex set V and edge set E is called order and size of a graph G respectively. A graph is said to be acyclic if it has no cycles. A connected acyclic graph G is called a tree. A support vertex in a tree is defined as a vertex adjacent to a leaf and a leaf is a vertex of degree 1 in a tree. A vertex v is said to be a cut vertex of G if $G - v$ is disconnected. The set $Cut(G)$ is the set of all cut vertices of the graph G. For the basic terminology and graph theoretical notation, one can refer to [4].

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In this article, a graph G we means that an undirected, connected, finite and simple graph. A set $D \subseteq V$ in a connected graph G is said to be a dominating set if every vertex $V - D$ is adjacent to some vertex in D. That is, closed neighbourhood $N[D] = V$. The domination number $\gamma(G)$ is the minimum size of dominating sets of G, refer to [4, 5, 6].

A vertex coloring of a graph G is a function $f : V \to N$ such that $f(u) = f(v) \Rightarrow \{u, v\} \notin E(G)$ for all u, $v \in V$. A graph G is said to be k-colorable if it has a proper k-vertex coloring. The least such positive integer k such that G is k-colourable is called chromatic number $\chi(G)$ of a graph G, refer to [3, 6, 7]. A k-chromatic graph G means that G is a connected graph with $\chi(G) = k$. Next, we define the notion of chromatic sets of graphs by choosing the vertices from each colour classes of G, studied in [2]. Let G be k-chromatic graph. A set $C \subseteq V$ of vertices of G is said to be chromatic set of G if C is the set of all vertices of G having the different colors in G. The minimum size among all chromatic sets of G is known as the chromatic number $\chi(G)$ of G . That is,

$$\chi(G) = \min\{|C| : C \text{ is a chromatic set of } G\}.$$

A dominating set D of a graph G is said to be split dominating set if $\langle V - D\rangle$ is disconnected. The split domination number and related observations are found in [11].

As referred by [1, 10], the concept of (D, C)-set (Dominating Chromatic set and split (D, C) set of a connected graph G are defined as follows. A set $S \subseteq V$ of vertices in G is said to be a dominating chromatic set [(D, C)- set in short] of a connected graph G if it is both dominating and chromatic set of G. The minimum size among the (D, C) sets of G is called (D, C)-Number of G, it is denoted by $\gamma_\chi(G)$ and defined by $\gamma_\chi(G) = \min\{|S| : S \text{ is a } (D, C) - \text{set of } G\}$. In general, $1 \le \gamma_\chi(G) \le p$, where p is the order of the graph G. A (D, C) set $S \subseteq V$ of vertices of G is said to be a split (D, C) set of G if the induced subgraph $\langle V - S\rangle$ is disconnected. The split (D, C) set of minimum order is called minimum split (D, C) set of G and its cardinality is called split (D, C) number of G. It is denoted by $\gamma_{\chi S}(G)$. i.e.,

$$\gamma_{\chi S}(G) = \min\{|S| : S \text{ is a split } (D, C) \text{ set}\}.$$

Many observations on the parameters split domination number and Non-split domination number of a connected graph are found in [8, 9]

In this paper, we introduce the tree structure of split and non-split (D, C) sets in graphs. The split and non-split tree domination chromatic number of a graph G is a variant of the domination number and the chromatic number of G.

2. The Split Tree (D, C)-Number of a Graph

Definition 2.1. *Let G be a connected graph. A subset $S \subseteq V$ of vertices in G is said to be a split tree (D, C) set of G if*

(i) *S is a (D, C) set*
(ii) *$\langle S\rangle$ is a tree*
(iii) *$\langle V - S\rangle$ is disconnected.*

The minimum cardinality of the split tree (D,C) set of G is called split tree (D,C) number of G, it is denoted by $\gamma_{\chi_{ST}}(G)$.

$$i.e, \quad \gamma_{\chi_{ST}}(G) = \min\{|S| : S \text{ is a split tree } (D,C) \text{ set}\}.$$

The $\gamma_{\chi_{ST}}$-set is a split tree (D,C) set with minimum cardinality. In fact, cardinality of the $\gamma_{\chi_{ST}}$-set is known as the split tree (D,C) number of G.

Next, define the non-split tree (D,C) sets in connected graphs,

Definition 2.2. *Let G be a connected graph. A subset $S \subseteq V$ of vertices in G is said to be a non-split tree (D,C) set of G if*

(i) *S is a (D,C) set*
(ii) *$\langle S\rangle$ is a tree*
(iii) *$\langle V-S\rangle$ is connected.*

The minimum cardinality of the non-split tree (D,C) set of G is called non-split tree (D,C) number of G, it is denoted by $\gamma_{\chi_{NST}}(G)$

$$i.e, \quad \gamma_{\chi_{NST}}(G) = \min\{|S| : S \text{ is a non-split tree } (D,C) \text{ set of } G\}.$$

The $\gamma_{\chi_{NST}}$-set is a non-split tree (D,C) set with minimum cardinality.
We establish these concepts by considering the following examples.

Example 1. *Consider a graph G with seven vertices with color pattern $v_1, v_5(red)$, $v_2, v_3, v_6(green)$ and $v_4, v_7(blue)$, which is in the Figure 1. The 3-element set $S = \{v_1, v_4, v_6\}$ is both dominating and chromatic set of G. Therefore $\gamma_{\chi_S}(G) = 3$. Also, induced Subgraph $\langle S\rangle$ is a tree and $\langle V-S\rangle$ is an empty graph on 4 vertices, which is disconnected. So 'S' is a split tree (D,C) set of G. Hence, split tree (D,C) number, $\gamma_{\chi_{ST}}(G) = 3$.*

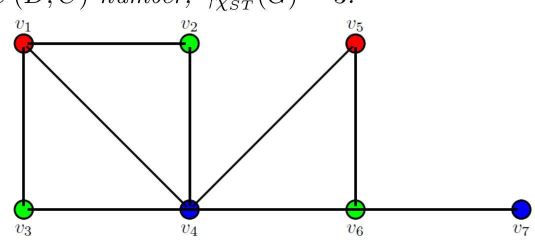


FIGURE 1. A graph G with $\gamma_{\chi_{ST}}(G) = 3$

Example 2. *Consider a graph G as five vertices with color pattern v_1, v_4(red), v_2, v_5(green) and v_3(blue), which is given in Figure 2. The 3-element set $S = \{v_2, v_3, v_4\}$ is a tree (D, C) set. Also, $\langle V - S\rangle$ is connected. Hence split tree (D, C) number, $\gamma_{\chi_{NST}}(G) = 3$.*

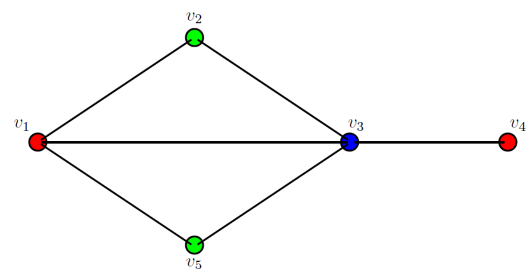


FIGURE 2. A graph G with $\gamma_{\chi_{NST}}(G) = 3$

Theorem 2.3. *Let G be a connected graph and $S \subseteq V$ be a (D, C) set of G. Then, S is a split (non-split) tree (D, C) set of G if and only if for any two vertices x, $y \in S$, there is an $x - y$ path contains at least one vertex of $V - S$.*

Proof. Assume S is a split tree (D, C)-set of G. Then clearly $\langle S\rangle$ is a tree. That is, for x, $y \in S$, $x - y$ path exist. Since $\langle V - S\rangle$ is disconnected (connected), the vertices x and y are connected by a path including at least one vertex from the complement $V - S$.

Conversely, clearly $\langle S\rangle$ is a tree, then there exist a path connected by two vertices x, y of S contains at least one vertex from $V - S$. This shows that induced graph $\langle V - S\rangle$ is disconnected (connected) and hence S is a split tree (D, C) set of the connected graph G. □

Remark 2.4.

1. Every split(non-split) tree (D, C) set of any connected graph G is always a (D, C) set of G. But, the converse need not be true.
2. The split (non-split) tree (D, C) set of a connected graph G exists when G is non-complete.
3. For any connected graph G, if split (non-split) tree (D, C) set does not exist, then we keep split (non-split) tree (D, C) number as zero. That is,

$$\gamma_{\chi_{ST}}(G) = 0\,(\,\gamma_{\chi_{NST}}(G) = 0)$$

3. Main Results

The chromatic set of complete graph on n vertices is its entire vertex set. So the entire vertex set doesn't form a tree structure. Therefore,

Theorem 3.1. *For the Complete graph $G = K_n$, $\gamma_{\chi_{ST}}(G) = 0$*

Theorem 3.2. *For the Complete graph $G = K_n$, $\gamma_{\chi_{NST}}(G) = 0$.*

Theorem 3.3. *For the path graph $G = P_n$,*

$$\gamma_{\chi_{ST}}(G) = \begin{cases} 2 & \textit{if } n = 4 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $G = P_n$

Case (i) When $n = 4$, we can find (D, C) sets in two dimensions, one is the collection of pendant vertices and other is the collection of internal vertices. In the case of set of pendant vertices, not possible to induce a tree structure. So the set of internal vertices say $S = \{v_2, v_3\}$ induce a tree and its complement clearly disconnected. This lead to $\gamma_{\chi_{ST}}(P_4) = |S| = 2$.

Case (ii) When $n < 4$, the possible (D, C) sets are induced a tree but its complement never disconnected. So $\gamma_{\chi_{ST}}(G) = 0$.

Case (iii) When $n > 4$, we cannot induce tree structures from the possible (D, C) set of vertices. Therefore, the result follows. □

Corollary 3.4. *For a k-colourable path graph $G = P_n$, $(k \leq n \;\&\; n \geq 4)$,*

$$\gamma_{\chi_{ST}}(G) = n - 2$$

Proof. Let Pend $(G) = \{$all pendant vertices of $G\}$. That is, $|\text{Pend } (G)| = 2$. If we fix a color pattern based on the number of internal vertices (say k) where $k = |V - \text{Pend } (G)|$.

Clearly $S = V - \text{Pend}(G)$ is a tree (D, C) set of G and $\langle V - S\rangle$ is disconnected. Therefore, $\gamma_{\chi_{ST}}(G) = n - 2$. □

Theorem 3.5. *For a path graph $G = P_n$,*

$$\gamma_{\chi_{NST}}(G) = \begin{cases} 1 & \textit{if } n = 1 \\ 2 & \textit{if } n = 2, 3 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $G = P_n$

Case (i) If $n = 1, 2$, the case is obvious. If $n = 3$, then any two adjacent vertices form a (D, C) sets which induce a tree and the induced subgraph of its complement is connected. Therefore, $\gamma_{\chi_{NST}}(P_3) = 2$.

Case (ii) If $n \geq 4$, we cannot induce tree structures from the possible (D, C) sets which consists non-adjacent internal vertices. If $n = 4$, the internal vertices form a tree structure, but its complement never induced a connected graph. Therefore, the result follows. □

Theorem 3.6. *For a connected path complement graph $\overline{P}_n (n \geq 4)$,*

$$\gamma_{\chi_{ST}}(\overline{P}_n) = \begin{cases} 2 & \textit{if } n = 4 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Due to Theorem 3.3, the result follows. □

Theorem 3.7. *For a connected path complement graph $\overline{P}_n (n \geq 4)$,*

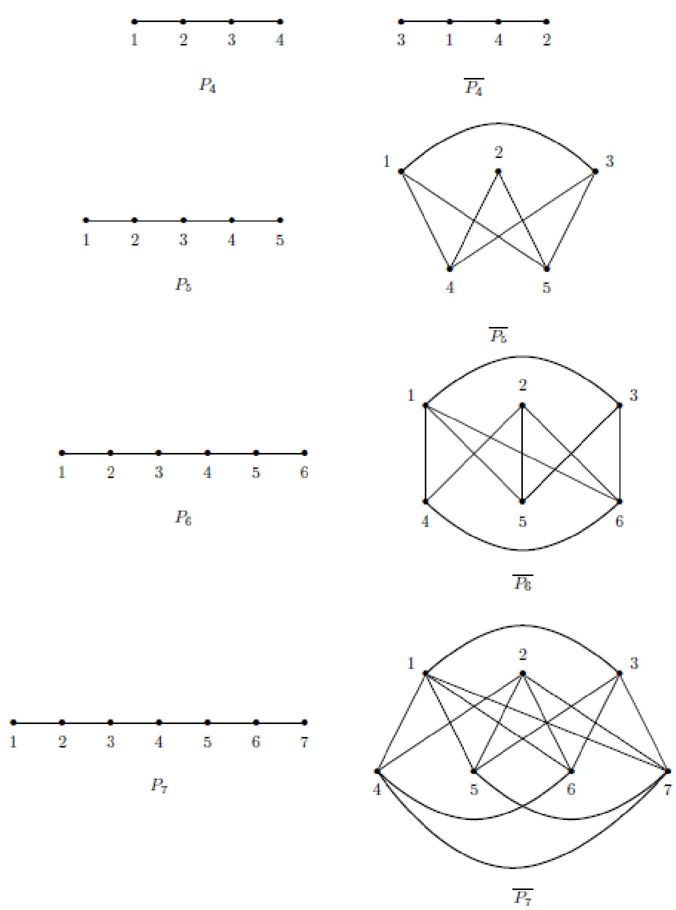
$$\gamma_{\chi_{NST}}(\overline{P}_n) = \begin{cases} 3 & \textit{if} \quad n = 5 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The vertex set of the path complement graph $\overline{P}_n$ for order $n \geq 4$ is $V(\overline{P}_n) = \{v_1, v_2, \ldots, v_n\}$. See the Figure 3,

Case (i) If $n = 4$, by Theorem 3.5, the result follows.

Case (ii) If $n = 5$, then $S = \{v_{n-4}, v_{n-2}, v_{n-1}\}$ is a tree (D, C) set. Since $\langle V - S \rangle$ is connected. Hence $\gamma_{\chi_{NST}}(G) = 3$.

Case (iii) If $n > 5$, we cannot find a tree (D, C) set S, see the Figures 3.



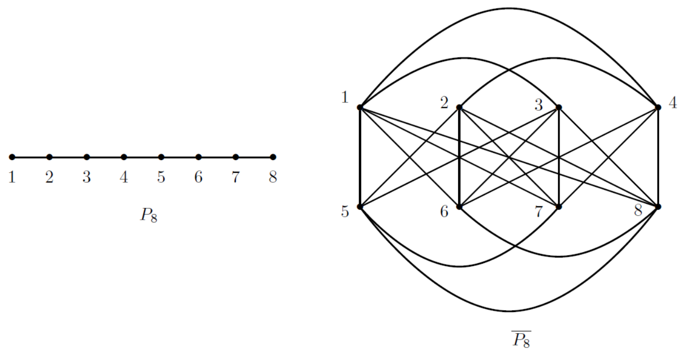


FIGURE 3. Path graph P_n and its complements for $n = 4, 5, 6, 7, 8$

Hence, we conclude that,

$$\gamma_{\chi_{NST}}(\overline{P}_n) = \begin{cases} 3 & if \quad n = 5 \\ 0 & \text{otherwise} \end{cases}$$

□

Theorem 3.8. *For a cycle graph $G = C_n$, $\gamma_{\chi_{ST}}(G) = 0$ for all $n \geq 3$.*

Proof. Let $G = C_n$ be a cycle graph on n vertices.

Case (i) If $n = 3$, by Theorem 3.1, the result follows.

Case (ii) If $n = 4, 5$, we can form a tree (D, C) set by any two adjacent vertices and any three adjacent vertices of C_n respectively. Also, the induced subgraph of its complement is connected in all cases.

Case (iii) If $n > 5$, the minimum (D, C) set never leads to a tree (D, C) set.
So we conclude that $\gamma_{\chi_{ST}}(C_n) = 0$. □

Theorem 3.9. *For a cycle graph $G = C_n$ for all $n \geq 3$,*

$$\gamma_{\chi_{NST}}(G) = \begin{cases} 2 & if\, n = 4 \\ 3 & if\, n = 5 \\ 0 & otherwise. \end{cases}$$

Proof. Let $G = C_n$ be a cycle graph on n vertices.

Case (i) If $n = 3$, due to Theorem 3.1, the result follows.

Case (ii) If $n = 4$, the any two adjacent vertices of C_n is a tree (D, C) set. Hence the result follows.

Case (iii) If $n = 5$, the any three adjacent vertices of C_n form a tree (D, C) set, which leads to the non-split tree (D, C) number.

Case (iv) If $n > 5$, the minimum (D, C) set never leads to a tree (D, C) set. Hence the result. □

Theorem 3.10. *For a cycle compliment graph $G = \overline{C_n}$ for all $n \geq 5$,*

$$\gamma_{\chi_{ST}}(G) = 0$$

Proof. When $n = 5$, 6, we can form a tree (D, C)-set S of cardinality 3. But $\langle V - S \rangle$ is not disconnected. So, the result follows. For $n \geq 7$, the minimum (D, C)-set never landed as a tree (D, C)-set of G. □

Theorem 3.11. *For a cycle graph $G = \overline{C_n}$ for all $n \geq 5$,*

$$\gamma_{\chi_{NST}}(G) = \begin{cases} 3 & \text{if } n = 5,\, 6 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. When $n = 5$, 6, we can form a tree (D, C)-set S of cardinality 3 and $\langle V - S \rangle$ is connected. So, the result follows. For $n \geq 7$, the minimum (D, C)-set does not converted to a tree (D, C)-set of G. □

Theorem 3.12. *Let G be a connected graph. Then, $\gamma_{\chi_{ST}}(G) = 2$ if and only if there exist a vertex $u \in \mathrm{Cut}(G)$ such that $d(u)$ is almost $n - 1$.*

Proof. Assume $\gamma_{\chi_{ST}}(G) = 2$, there exist a tree (D, C) set $S = \{u, v\}$ such that $\langle V - S \rangle$ is disconnected. It follows that one of the elements of S must a cut vertex say 'u', that is, $u \in \mathrm{Cut}(G)$. We cannot find a vertex $v \in \mathrm{Cut}(G)$, so that $d(u)$ is at most $n - 1$.

Conversely assume that $u \in \mathrm{Cut}(G)$ and $d(u) \leq n - 1$. Then we can find a vertex v such that $\{u, v\}$ is a split tree (D, C) set because $\langle V - \{u, v\} \rangle$ is disconnected. Therefore, the result follows. □

Theorem 3.13. *Let $G = K_{m,n}$ be a complete bipartite graph. Then*

$$\gamma_{\chi_{ST}}(G) = \begin{cases} m + 1 & \text{if } m = n \geq 2 \\ \min\{n + 1, m + 1\} & \text{if } m \neq n \geq 2 \end{cases}$$

Proof. Let $G = K_{m,n}$.

Case (i) When $m = n \geq 2$

Let U & V be a vertex partition of G such that $|U| = m = |V|$ and $U \cup V = V(G)$. If we choose one element from each partition such that $u \in U$ and $v \in V$. It follows that the $\{u, v\}$ is a tree (D, C) set of G. But the induced subgraph $\langle V - \{u, v\}\rangle$ is always connected. It follows that $\{u, v\}$ is not a split tree (D, C) set of G. The set $S = U \cup \{v\}$ or $V \cup \{u\}$ is a split tree (D, C) set and the induced subgraph $\langle S\rangle \cong$ either $K_{m,1}$ or $K_{1,m}$. It shows that $\langle V - S\rangle \cong m$ copies of K_1. Therefore,

$$\gamma_{\chi_{ST}}(G) = m + 1 \quad \text{if}\, m = n \geq 2.$$

Case (ii) When $m \neq n \geq 2$

Let U & V be a vertex partition of G such that $|U| = m$ and $|V| = n$. If we choose $u \in U$ and $v \in V$ such that $\{u, v\}$ a tree (D, C) set of G. But the induced sub graph $\langle V - \{u, v\}\rangle$ is always connected. Therefore, $\{u, v\}$ is not a split tree (D, C) set of G. The set $S = U \cup \{v\}$ or $V \cup \{u\}$ is a split tree (D, C) set and the induced subgraph $\langle S\rangle \cong$ either $K_{m,1}$ or $K_{1,n}$. It shows that $\langle V - S\rangle \cong mK_1$ or nK_1. Therefore, $\gamma_{\chi_{ST}}(G) = \min |S| = \min\{n + 1 \text{ or } m + 1\}$. □

4. Conclusion

The split tree domination chromatic number of a connected network has applications in various areas such as network design, facility location, and social network analysis. The graph parameter, the split tree domination chromatic number is an important variable in the graph theory as it has been shown to be related to other well-studied parameters such as the domination number and the chromatic number of a graph. This will help to develop the algorithmic and complexity aspects in graph coloring problems. It can also be used to study the complexity of optimization problems in science and technology.

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Data availability : Not applicable

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