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INTRODUCTION TO MODELS OF OPINION DYNAMICS AND THEIR EXAMPLES

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Abstract. This paper aims to provide a general review of Opinion Dynamics (OD) and its related models, along with application examples for special agents. We will discuss special classes of social actors, such as informed actors, opponents, and extremists, in the context of opinion dynamics. Our main objective is to determine the extent to which opinion dynamics, as a mathematical sociology, relates to social reality. To achieve this, we present key elements of mathematical sociology in Opinion Dynamics, which we then apply to real socioeconomic phenomena using modeling assumptions and mathematical formulations.

1. Introduction

Over the past few decades, there has been a significant increase in research on opinion dynamics, which provides mathematical models to simulate how opinions are diffused across social networks. Social networks are comprised of N agents that interact with one another through connections or links [30]. In the simplest scenario of an undirected network, there exists a single link with an influence weight w_{ij} between two communicating agents i and j. Earlier approaches to OD from the 1950s to the 1990s were mainly linear [1, 6, 7, 10, 11, 12, 20], which implies that communication patterns and interaction structures remained fixed. Two fundamental models in this context are the DeGroot model [7, 20] and the Friedkin-Jensen model [11, 12]. The DeGroot model assumes that agents' opinions are updated by taking a weighted average of their own and their neighbors' opinions. The Friedkin-Jensen model assumes that each agent has both an initial opinion and a stubbornness parameter, which measures how much an agent's opinion will change in response to others. In both models, the opinions of all agents converge to a consensus state over time. These models have been extensively studied, and their results have been compared with empirical

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observations to gain insights into how opinions are formed and how they change over time.

In recent literature, there has been a noticeable shift towards non-linear approaches. Non-linearity in this context refers to situations where the influence weights and/or communication interactions between agents are dependent on prior opinions expressed by them. Non-linear models can be classified into three major categories: continuous, discrete, and mixed. In continuous models, agents' opinions can be expressed within a defined interval, such as $[0, 1]$ or $[-1, 1]$. In discrete models, agents' opinions can only assume specific values, often binary, such as 0/1, yes/no, support/opposition, or buy/sell. In mixed models, agents' continuous opinions are expressed as discrete choices. Additionally, there are multi-dimensional (vector) representations available for all the above cases [3, 25]. These vector representations allow for more nuanced analysis of agent opinions and can provide insights into the relationships between different factors influencing those opinions. In the upcoming sections, we will delve into the intricate world of opinion dynamics models. Section 2 will be dedicated to a comprehensive exploration of Linear Models, with a detailed focus on the DeGroot model and Friedkin-Jensen model. These models play a pivotal role in comprehending the dynamics of linear opinions. Moving on to Section 3, we will embark on an in-depth study of Nonlinear Models, encompassing continuous models, discrete models, and mixed models. Section 4 will shine a spotlight on special classes of social agents, including informed agents, contrarian agents, and extremist agents, shedding light on their unique roles within opinion dynamics. Finally, in Section 5, the Conclusion, we will succinctly summarize the contents of this paper and present potential avenues for future research, along with their anticipated impact.

2. Linear Models of opinion dynamics

2.1. The DeGroot model

The DeGroot model is a mathematical framework used to study how opinions evolve in a group of communicating agents. In the model, each agent is assigned an initial opinion value, which represents their belief or attitude towards a particular issue. The opinion value is a number between 0 and 1, where 0 means complete disagreement and 1 means complete agreement with a particular viewpoint.

The model assumes that agents communicate with each other and that the opinions of agents can influence each other. The evolution of the opinion of each agent is determined by a linear combination of the opinions of all other agents in the system. This linear combination is a weighted average of the opinions of other agents, where each weight is a measure of the influence that the other agent has on the opinion of the agent in question.

The weight assigned to each agent i is proportional to the similarity between the opinion of the agent and the opinion of the other agent. In particular, agents with similar opinions are given higher weights than those with dissimilar opinions. The weights are normalized such that they sum up to 1,

$$\sum_{j=1}^{N} w_{ij} = 1,$$

which ensures that the opinion of each agent is always a valid probability distribution over the interval $[0, 1]$.

The DeGroot model has been widely used to study opinion dynamics in a variety of domains, including social networks, political science, and economics. It provides a useful tool for exploring how opinions change over time and how different factors, such as the structure of communication networks or the strength of social influence, affect the evolution of opinions in a group of agents:

$$o_i(t+1) = \sum_{j=1}^{N} w_{ij} o_j(t)$$

To update the opinion vector for all agents at time step $t+1$, a formula is used which involves multiplying the matrix W with the opinion vector at time step t. The resulting opinion vector, denoted as $\mathcal{O}(t+1)$, represents the collective opinion of all agents at the next time step. By applying this formula repeatedly, we can obtain the opinion vector at a later time step , which can be expressed as

$$\mathcal{O}(t+\tau) = W^{\tau}\mathcal{O}(t).$$

As per a reference cited in [7], consensus among the agents can be achieved if a specific condition is met. The condition states that if at least one column of the matrix Wτ contains solely positive elements, then consensus can be achieved. This means that all agents will eventually converge to the same opinion if the matrix W is designed in a way that satisfies this condition.

2.2. The Friedkin-Jensen model (FJ)

The FJ model is a modified version of the DeGroot model that accounts for an agent i's stubbornness when forming opinions. The parameter s_i represents the stubbornness level of agent i, indicating the tendency to adhere to their initial beliefs. The influence of other agents on agent i is determined by the value $(1-s_i)$, which is a measure of their openness to external opinions.

The evolution of opinions is modeled using the following formula:

$$o_i(t+1) = s_i o_i(0) + (1-s_i)\sum_{j=1}^{N} w_{ij} o_j(t),$$

where $o_i(t+1)$ represents the opinion of agent i at time $t+1$, $o_i(0)$ is the initial opinion of agent i, w_{ij} is the weight of the connection between agents i and j, and $o_j(t)$ is the opinion of agent j at time t.

This formula can be represented in matrix form as

$$\mathcal{O}(t+1) = S\mathcal{O}(0) + (I - S)W^{\tau}\mathcal{O}(t),$$

where S is a diagonal matrix of size $N \times N$ containing the stubbornness levels of each agent, and I is the identity matrix. The matrix W represents the connectivity between agents and the strength of their influence on each other.

Unlike the DeGroot model, the convergence of matrix W^{τ} does not necessarily lead to consensus. Consensus is conditional on the stubbornness level of the agents. If most of the agents have a high level of stubbornness, then consensus may not be reached. However, if the agents are relatively open-minded, then the system can converge to a consensus state.

3. Nonlinear Models of opinion dynamics

3.1. Continuous Models

In the OD literature, there are several non-linear continuous models that have been studied. The Bounded Confidence class of models [9, 16, 21, 29] is particularly noteworthy. These models are based on the assumption that social influence can only occur when the opinions $o_i(t)$ and $o_j(t)$ of two neighboring agents i and j are below a certain tolerance threshold ε and sufficiently close:

$$|o_j(t) - o_i(t)| < \varepsilon. \tag{1}$$

In other words, the Bounded Confidence model posits that individuals are more likely to be influenced by those who hold similar opinions to their own. If the difference between two individuals' opinions is greater than the tolerance threshold ε, then they are unlikely to be influenced by one another. However, if their opinions are within this threshold, then they may be influenced by each other's views.

Overall, the Bounded Confidence model provides a useful framework for understanding how social influence operates in a variety of contexts. By accounting for the role of tolerance thresholds in shaping social influence, this model can help shed light on a wide range of social phenomena.

The Bounded Confidence model is a popular framework used to study opinion dynamics in social networks. The Hegselmann-Krause (HK) model and the Deffuant-Weisbuch (DW) model are the most well-known among these models [31]. While they share some similarities, they differ primarily in their communication assumptions. In the HK model, each agent communicates simultaneously with all sufficiently like-minded neighbors. On the other hand, in the DW model, communication is realized between two random like-minded neighbors at a time. In the HK model, the evolution of an agent's opinion is influenced by the opinions of their like-minded neighbors and the degree of confidence they have in their own opinion:

$$o_i(t+1) = \frac{1}{|S_i(j,\varepsilon)|} \sum_{j \in S_i(j,\varepsilon)} o_j(t),$$

where $S_i(j, \varepsilon) = \{j : |o_j(t) - o_i(t)| < \varepsilon\}$ is the set containing those neighbors j of agent i with a sufficiently close opinion to his own. This leads to a convergence of opinions among agents with similar initial opinions.

In the DW model, two agents communicate with each other and if a certain condition is satisfied, a mutual opinion update takes place. The condition that needs to be met is expressed by equation (1):

$$\begin{cases} o_i(t+1) = o_i(t) + \mu(o_j(t) - o_i(t)), \\ o_j(t+1) = o_j(t) + \mu(o_i(t) - o_j(t)), \end{cases} \tag{2}$$

The model includes a parameter called μ which determines the convergence rate between the two agents.

If the tolerance threshold ε is set to a high value, then there is a greater likelihood of opinion convergence between the two agents. This means that their opinions will become more similar to each other. On the other hand, if the tolerance threshold ε is set to a low value, then opinion clustering takes place. This means that groups of agents will have similar opinions and there will be less interaction between groups.

The DW model is designed to simulate social dynamics and how opinions are formed and changed over time based on interactions between individuals. By studying these dynamics, researchers can gain a better understanding of how social phenomena such as polarization and echo chambers arise.

3.2. Discrete Models

In the realm of discrete opinion dynamics (OD) modeling, several models have been developed to understand the behavior of agents and their opinions. Among these models, the voter model [5, 17], Sznajd model [4, 26, 27, 28], and majority rule model [13, 14, 15, 19] have received much attention. The voter model is typically implemented on a square lattice, where each agent has a binary opinion $o_i(t)$, which can be either 0 or 1. At each time step, an agent i randomly adopts the opinion of one of its neighbors. In other words, an agent updates its opinion $o_j(t)$ to that of a randomly selected neighbor:

$$o_i(t+1) = o_j(t).$$

On the other hand, the Sznajd model is implemented on a line with agents placed at regular intervals. Here, a pair of neighboring agents is randomly selected at each step. If their opinions are identical, then the two neighboring agents on either side of the pair (i.e., $i-1$ and $i+2$) will adopt the same opinion as the pair (i.e., i and $i+1$):

$$o_{i`1}(t+1) = o_{i+2}(t+1) = o_{i+1}(t) = o_i(t).$$

However, if their opinions differ, then each agent will influence only its adjacent neighbor. Specifically, the agent to the left of the pair (i.e., $i-1$) will adopt the opinion of i, while the agent to the right of the pair (i.e., $i+2$) will take on the opinion of $i+1$. These models help us understand the complex dynamics of opinion formation in diverse contexts:

$$o_{i`1}(t+1) = o_i(t) \quad \text{and} \quad o_{i+2}(t+1) = o_{i+1}(t).$$

The basic concept involves the random clustering of the population into groups of three agents at each hierarchical level. The majority rule model is a voting system that operates across different hierarchical levels. Within each group, the three members vote between two candidates, type A or type B. The voting result at the first hierarchical level influences the composition of the population at the next level. After clustering in groups of three at the next level, the same voting process is repeated until vote convergence is observed.

The probability of a type A candidate being elected at level $n+1$ is calculated using the following formula:

$$P_A(n+1) = P_A^3(n) + 3P_A^2(n)(1 - P_A(n)),$$

where $P_A(n)$ represents the probability of a type A candidate being elected at level n. It can be shown that when $P_A(0)$ is less than $\frac{1}{2}$, the probability sequence $P_A(n)$ eventually converges to zero, which results in the elimination of A. However, when $P_A(0)$ is greater than $\frac{1}{2}$, the sequence converges to 1, and the prevalence of A is certain.

The key finding from this voting system is that although it operates in a fully democratic manner, the condition $P_A(0) > P_B(0)$ (or the reverse) is sufficient for a totalitarian result to eventually emerge. This means that one of the two political positions (A or B) will be all but excluded, regardless of the amount of initial support for B or A. Therefore, the majority rule model has significant implications for democratic societies, as it highlights the potential for a highly polarized political environment.

The findings discussed above, which may seem counter-intuitive at first, actually demonstrate the usefulness of Sociophysics in identifying underlying issues that qualitative studies are often unable to detect. Specifically, multi-stage electoral systems that operate in a bottom-up manner, as found in socialist or liberal democratic states such as the US presidential electoral system, bear certain similarities to the majority rule model. Sociophysics can reveal hidden holes in these seemingly robust social or political processes, which may not be immediately apparent through traditional qualitative analysis. By using a quantitative approach to study these complex systems, Sociophysics can provide valuable insights into the workings of these systems and help identify potential areas for improvement.

3.3. Mixed Models

Sophisticated mixed models have been developed to study the dynamics of opinion formation and diffusion in a social network. One such model is the Continuous Opinion and Discrete Actions (CODA) model [22, 23, 24], which has gained significant prominence in this area of research.

In the CODA model, each agent i in a social network internalizes his preference at a given time step t in the form of a continuous probability $P_i(t)$, which represents his level of agreement or disagreement with a particular issue. The agent then expresses his opinion to other agents in the form of a quantized binary opinion $o_i(t)$, taking values of either -1 or 1. This is done by calculating $o_i(t)$ as

$$o_i(t) = sign(P_i(t) - 0.5).$$

When an agent i is influenced by another agent j in the network, his internalized preference is modified as

$$u_i(t) = \log\Big(\frac{P_i(t)}{1 - P_i(t)}\Big),$$

and shifts to

$$u_i(t+1) = u_i(t) \pm a_i = u_i(t) + o_j(t)a_i,$$

where a_i is the susceptibility parameter of agent i to the opinion $o_j(t)$ of his neighbor j. This means that the level of agreement or disagreement of agent i with a particular issue shifts based on the opinion of his neighbor j. As a result, $P_i(t+1)$ is no longer equal to $P_i(t)$, $P_i(t+1) \neq P_i(t)$, and it is probable that agent i will change his professed opinion $o_i(t+1)$ in response to the influence of his neighbor:

$$o_i(t+1) \neq o_i(t).$$

This way, opinion diffusion can take place in the social network.

4. Special Classes of Social Agents

4.1. Informed Agents

It has been observed that social influence is not only dependent on the existence of key opinion leaders or influencers who spread their opinions to the rest of the population, but also on the presence of a critical mass of easily influenced individuals. This critical mass is essential for effective social influence, as it initiates a cascading effect that leads to widespread social influence. Much of the available literature on organizational development focuses on the impact of informed agents, also known as individuals with hidden agendas who act as secret advertisers of ideas and norms to the rest of the population. Studies on informed agents are grounded on findings derived from studies on animal population dynamics, which suggest that the collective behavior of social groups can be guided by a small fraction of purposeful agents. Therefore, it is imperative to understand the role of informed agents in social influence and how they can impact the behavior of a population.

In reference [2], it is observed that when considering the assumption of bounded confidence (1), the process of opinion diffusion from a regular agent j to another regular agent i is carried out in the following manner:

$$o_i(t+1) = w_{ii}(t)o_i(t) + w_{ji}(t)o_j(t).$$

The opinion $o_i(t+1)$ of agent i at time $t+1$ is influenced by two factors: the tendency $w_{ii}(t)$ of agent i to adhere to his opinion at time t and the interpersonal social influence $w_{ji}(t)$, which is dependent on their previous opinions at time t, of agent j on agent i.

However, when an informed agent k is present, a third parameter $w_k^g(t)$ is added to the opinion diffusion process. This additional parameter indicates the hidden devotion of agent k towards the pursuit of a pre-specified goal o^g. Therefore, when opinion diffusion occurs from a regular agent λ to an informed agent k, the process is realized as follows:

$$o_k(t+1) = w_{kk}(t)o_k(t) + w_{\lambda k}(t)o_\lambda(t) + w_k^g(t)o^g.$$

The opinion of informed agent k at time $t+1$ is influenced by the tendency $w_{kk}(t)$ of agent k to adhere to his opinion at time t. The interpersonal influence $w_{\lambda k}(t)$ of regular agent λ on informed agent k, and the hidden devotion $w_k^g(t)$ of informed agent k towards the pursuit of a pre-specified goal.

In the context of social network analysis, an informed agent is an individual who has access to relevant information and can make informed decisions based on it. However, it is important to note that even an informed agent can be influenced by other individuals in their social network. Despite this, it is possible to steer public opinion by using a small set of informed agents who have an unchanging pre-set goal, particularly if these agents are well-connected individuals.

It is worth mentioning that informed agents do not necessarily need to be prominent members of society; they can be anyone who possesses relevant information and has the ability to disseminate it within their social network.

A more comprehensive approach to understanding the processes of social change induced by informed agents has been presented in a research paper (reference [18]). This research introduces the concept of change agents, who not only mimic the opinions of their neighbors but also attempt to divert them towards a preferred direction. Change agents gradually shift their neighbors' opinions towards a pre-set goal, which can initiate a cascading diffusion of social influence.

To achieve this goal, change agents employ the strategy of salami slicing, which involves slicing a large goal into smaller, more achievable pieces. By doing so, change agents can gradually shift their neighbors' opinions towards the pre-set goal without raising any red flags or causing any significant resistance. Ultimately, this approach can lead to a significant change in public opinion, as well as an overall shift in societal norms and values.

The process of opinion diffusion for regular agents in the context of social networks is often modeled using various implementations of the DeGroot model. However, for a change agent, identified as k, the opinion update works in a slightly different way. Specifically, the opinion of the change agent at time $t+1$, denoted as $o_k(t+1)$, is updated as a sum of two terms:

$$o_k(t+1) = \bar{o}_k^{out}(t) + s_k(t)(o^g - \bar{o}_k^{out}(t)),$$

The first term, $\bar{o}_k^{out}(t)$, denoted by

$$\bar{o}_k^{out}(t) = \frac{1}{|S_k^+|} \sum_{\lambda \in S_k^+} o_\lambda(t),$$

is the average opinion of all out-neighbors of k to whom k is connected with a positive outgoing (directed) link weight $w_{k\to\lambda}$. This set of out-neighbors is represented by

$$S_k^+ = \lambda : w_k = w_{k\to\lambda} > 0.$$

The second term, $s_k(t)(o^g - \bar{o}_k^{out}(t))$, is a time-dependent slicing parameter lying in the interval $[0, 1]$, which represents a gradual shift in the change agent's professed opinion over time.

Essentially, the change agent k is equipped with the ability to induce opinion change in its out-neighbors through this mechanism. The average opinion of the out-neighbors is gradually incorporated into the change agent's public stance, allowing it to steer the direction of the network's opinion dynamics. This process represents an extension of the DeGroot model that is particularly useful in analyzing the impact of change agents within a social network.

4.2. Contrarian Agents

Contrarians are individuals who tend to hold views that are opposed to those held by the majority. To account for the influence of such individuals, a modification of the majority rule model has been proposed. This modification introduces a density parameter a to represent the presence of contrarian agents in the population. The aim is to investigate the impact of contrarians on electoral outcomes of lower-level groups. The modified model can be expressed as follows:

$$P_A(t+1) = (1-a)\{P_A^3(t) + 3P_A^2(t)(1-P_A(t))\} + a\{P_B^3(t) + 3P_B^2(t)(1-P_B(t))\},$$

where $P_A(n)$ and $P_B(n)$ are the probabilities of a candidate of type A or B being elected at level n. The introduction of contrarian agents in the model helps to avoid the totalitarian outcomes that are typical of the majority rule model, especially for small values of a. For instance, when $a = 0.1$, the opinion convergence is more balanced, with a shift of 0.85 to 0.15 in favor of outcome A, instead of a complete convergence of 1 to 0.

4.3. Extremist Agents

The term extremist agents is used to describe individuals whose opinions fall on the far ends of a given spectrum. For instance, if agent opinions are rated on a scale of -1 to 1, extremist agents would have opinions that are very close to -1 or 1. Studies have shown that the presence of extremist agents in a social network can have a significant impact on the diffusion of opinions within the network. In particular, the views of extremist agents can eventually become widely accepted, especially if there is a degree of uncertainty among regular agents.

To model opinion uncertainty, it is assumed that each agent has an opinion segment, $s_i = [o_i - u_i, o_i + u_i]$, where o_i represents the agent's opinion and u_i represents the degree of uncertainty. The degree of overlap between two agents i and j is measured by the segment overlap,

$$h_{ji} = \min(o_j + u_j, o_i + u_i) - \max(o_j - u_j, o_i - u_i),$$

while the non-overlapping part is $2u_j - h_{ji}$. The relative agreement term, $\frac{h_{ji}}{u_j} - 1$, is obtained by subtracting the two quantities and dividing by u_j.

If the condition $h_{ji} > u_j$ is met, then opinion diffusion from agent j to agent i is possible. In this case, the DW Equation (2) for agent i takes the form:

$$o_i(t+1) = o_i(t) + \mu\Big(\frac{h_{ji}(t)}{u_j(t)} - 1\Big)(o_j(t) - o_i(t)),$$

$$u_i(t+1) = u_i(t) + \mu\Big(\frac{h_{ji}(t)}{u_j(t)} - 1\Big)(u_j(t) - u_i(t)),$$

where μ is a convergence parameter that determines the speed of the diffusion process. Given that extremist agents have a narrower opinion uncertainty than regular agents, the condition $h_{ji} > u_j$ is usually met when regular agents interact with extremists. This leads to the diffusion and, in some cases, the prevalence of extremist opinions within the social network.

5. Conclusion

The study of the social world through a mathematics-based quantitative approach can be a fruitful endeavor. By presenting key methods and applications with a more specified socio-economic character, we can focus on the interaction between agents. A wealth of available time series data, such as financial transactions, instances of collaboration, or common participation in social events, can be used to explore these topics. However, we believe that a more detailed mathematical depiction of the co-evolution of agent and structure is necessary. This approach is relevant to all categories of social diffusion processes analyzed herein.

In the case of opinion diffusion (OD), it is essential for firms, organizations, and governmental agencies to adopt current norms and prevent the spread of disruptive disinformation/misinformation and fake news in relevant social networks. Opinion diffusion modeling can be considered a must-have tool to prevent the spread of such information. By incorporating this tool, we can monitor the spread of false information and help maintain social stability.

Mathematical sociology is a field that uses mathematical modeling, statistical analysis, and computational simulations to study complex social phenomena. It offers a quantitative approach to the study of social systems, which enables researchers to move beyond purely qualitative or descriptive approaches and gain a deeper understanding of social systems. Recent developments and studies in mathematical sociology have contributed significantly to our understanding of social networks, collective behavior, social dynamics, and modeling approaches.

One of the key areas of research in mathematical sociology is social network analysis. This field allows researchers to explore network structures, identify influential nodes, and analyze information diffusion. Social network analysis has become a fundamental area of research within mathematical sociology, and it has contributed to our understanding of how social networks function and how they influence the behavior of individuals within them.

Another important area of research in mathematical sociology is the study of collective behavior and social dynamics. This field has provided insights into opinion formation, social influence, coordination, cooperation, and the spread of innovations or cultural traits. Mathematical models, such as agent-based models, game theory, and mathematical formalisms, have played a central role in capturing these dynamics and uncovering the underlying mechanisms of social phenomena.

Mathematical sociology has applications in various social science domains, including economics, political science, anthropology, and urban studies. By incorporating mathematical rigor into these disciplines, researchers can analyze complex social systems and make informed policy decisions.

Future research in mathematical sociology should focus on integrating big data and computational advances into research methodologies, enabling the analysis of large-scale social systems and testing more complex models. Interdisciplinary collaborations between sociologists, mathematicians, computer scientists, and other social scientists will be crucial for addressing these challenges and advancing the field.

To further enhance the applicability and relevance of mathematical models in sociology, researchers should also address ethical considerations, account for long-term dynamics and historical context, and incorporate a nuanced understanding of human agency. These factors are important in ensuring that mathematical models are valid and useful tools for understanding complex social systems.

In summary, mathematical sociology offers a quantitative foundation for studying and understanding social phenomena. By combining mathematical modeling, statistical analysis, and computational simulations, it provides valuable insights into social networks, collective behavior, social dynamics, and decision-making processes. Although the field faces new challenges, its interdisciplinary nature and the potential to integrate emerging technologies and methodologies ensure its continued relevance and impact in the study of complex social systems.

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