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# MULTIPLICITY OF POSITIVE SOLUTIONS OF A SCHRÖDINGER-TYPE ELLIPTIC EQUATION

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Abstract. We investigate the existence of multiple positive solutions of the following elliptic equation with a Schrödinger-type term:

$$\begin{cases} -\Delta u + V(x)u &= \lambda f(u) \quad x \in \Omega, \\ u &= 0, \quad\quad\;\; x \in \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$, with a smooth boundary $\partial\Omega$, $f \in C[0,\infty)$, $V \in L^\infty(\Omega)$ and $\lambda$ is a positive parameter. In particular, when $f(s) > 0$ for $0 \leq s < \sigma$ and $f(s) < 0$ for $s > \sigma$, we establish the existence of at least three positive solutions for a certain range of $\lambda$ by using the method of sub and supersolutions.

## 1. Introduction

We are concerned with positive solutions of the Schrödinger-type elliptic equation

$$(P_\lambda) \qquad \begin{cases} -\Delta u + V(x)u &= \lambda f(u), \quad x \in \Omega, \\ u &= 0, \qquad\quad\; x \in \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $V \in L^\infty(\Omega)$, $f \in C([0,\infty))$ and $\lambda$ is a positive parameter. We assume that the reaction term $f$ and Schrödinger term $V$ satisfy the following conditions,, respectively.

$(H1)$ There exists $\sigma > 0$ such that $f(s) > 0$ for $0 \leq s < \sigma$ and $f(s) < 0$ for $s > \sigma$.

$(H2)$ There exists $c_V > 0$ such that $V(x) \geq -c_V > -\frac{1}{\|e\|_\infty}$ for $x \in \Omega$, when $e$ is the positive solution of

$$\begin{cases} -\Delta e = 1, \text{ in } \Omega, \\ e = 0, \text{ on } \partial\Omega. \end{cases}$$

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The first equation in $(P_\lambda)$ is derived from the nonlinear Schrödinger equation (details in [15]). Nonlinear Schrödinger equations have been studied widely to demonstrate the existence of solutions which act on $V$ in the whole space $\mathbb{R}^N$ (see [2], [10] and [13]) or on bounded domains (see [8]).

In the case when $V \equiv 0$, investigation of existence of multiple positive solutions of $(P_\lambda)$ has a rich history for a long time (see [3], [5], [11], [12] and [16] and references therein). In this paper, when $V \in L^\infty(\Omega)$ satisfies $(H2)$, by using the method of sub and supersolutions, we establish the existence of positive solutions of $(P_\lambda)$ for each $\lambda > 0$. Further, under the assumption $(H3)$ of $f$ and $V$, we prove the existence of multiple positive solutions of $(P_\lambda)$ for a certain range of $\lambda$.

We first state our existence result:

**Theorem 1.1.** *Assume* $(H1)$ *and* $(H2)$*. Then the problem* $(P_\lambda)$ *has a positive solution for each* $\lambda > 0$.

*Remark* 1. Theorem 1.1 assures that there exists a positive solution of $(P_\lambda)$ for each $\lambda > 0$ even when $V$ is negative in $\Omega$. In the case when $\Omega = (0,1)$ and $V(x) \equiv -\mu$ with $\mu > 0$ in $(0,1)$, the one dimensional problem of $(P_\lambda)$ is written as

$$\begin{cases} \quad -u''(x) \;=\; \lambda f(u(x)) + \mu u(x), \quad x \in (0,1), \\ u(0) \;=\; 0 \;=\; u(1). \end{cases} \tag{1}$$

Here, we emphasize that (1) has a positive solution for each $\lambda > 0$, especially $\lambda > 0$ very small, if $0 < \mu < c_V$. Now, considering the case when $\lambda = 0$, the above equation is corresponding to the following eigenvalue problem

$$\begin{cases} \quad -u''(x) \;=\; \mu u(x), \quad x \in (0,1), \\ u(0) \;=\; 0 \;=\; u(1). \end{cases} \tag{2}$$

Observing the well-known result that (2) does not allow a positive solution for $\mu$ satisfying $0 < \mu < \sqrt{\pi}$, we might anticipate that when $\lambda \approx 0$, the problem (1) might have no positive solution for a very small $\mu > 0$ since the neglectable term $\lambda f(u)$ is added in the first equation in (2). However, from Theorem 1.1 we can see that for each $\mu \in (0, \min\{\sqrt{\pi}, c_V\})$ the problem (2) has a positive solution even for $\lambda \approx 0$.

Next, to state the multiplicity result, we let

$$A = \frac{(N+1)^{N+1}}{N^N} \text{ and } B = \frac{R^2}{AN + \|V\|_\infty R^2},$$

where $R$ is the radius of the largest inscribed ball $B_R$ in $\Omega$, and $K > 0$ be a constant such that

$$\frac{1}{K} \leq 1 - c_V \|e\|_\infty. \tag{3}$$

We define $f^*(s) := \max_{t\in[0,s]} f(t)$ and for any $0 < a < d < b$,

$$Q(a,d,b) := \frac{\frac{d}{f(d)}\frac{1}{B}}{\min\left\{\frac{a}{f^*(a)}\frac{1}{K\|e\|_\infty}, \frac{2b}{f(d)AB}\right\}}.$$

We further assume that

$(H3)$ there exist $a, b$ and $d$ with $0 < a < d < b$ such that $Q(a,d,b) < 1$ and

$$\tilde{f}(s) := f(s) - \frac{f(d)}{d} B\|V\|_\infty s > 0, \ \forall s \in [0,b]$$

and is nondecreasing on $[a,b]$.

**Theorem 1.2.** *Assume $(H1), (H2)$ and $(H3)$. Then the problem $(P_\lambda)$ has at least three positive solutions $u_\lambda \in C^2(\Omega)\cap C(\bar{\Omega})$ for each $\lambda_* < \lambda < \lambda^*$, where*

$$\lambda_* = \frac{d}{f(d)}\frac{1}{B}, \ \lambda^* = \min\left\{\frac{a}{f^*(a)}\frac{1}{K\|e\|_\infty}, \frac{2b}{f(d)AB}\right\}.$$

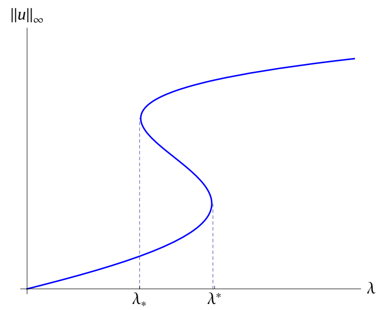


FIGURE 1. S-shaped bifurcation diagram showing the existence of multiple positive solutions for $(P_\lambda)$

*Remark* 2. The condition $Q(a,d,b) = \frac{\frac{d}{f(d)}\frac{1}{B}}{\min\left\{\frac{a}{f^*(a)}\frac{1}{K\|e\|_\infty}, \frac{2b}{f(d)AB}\right\}} < 1$ implies $\lambda_* < \lambda^*$. This shows that there exists a nonempty interval of $\lambda$ in which the problem $(P_\lambda)$ has at least three positive solutions.

In order to establish the existence of multiple positive solutions of $(P_\lambda)$, we employ three solution theorem ([1] and [17]). The typical way constructing a second subsolution of $(P_\lambda)$ when $V \equiv 0$ is not applicable since $V(x)$ acts on $u$. To obtain three positive solutions for a certain range of $\lambda$, it is important to construct two pairs of sub and supersolutions $(\psi_1, Z_1), (\psi_2, Z_2)$ of $(P_\lambda)$ with the property that $\psi_1 \le \psi_2 \le Z_1$, $\psi_1 \le Z_2 \le Z_1$ such that $\psi_2 \not\le Z_2$ so that three solution results in [1] can be applied. However, the term $V(x)$ acting on $u$ gives a nontrivial difficulty in the construction of the second pair of sub and supersolution $(\psi_2, Z_2)$. We overcome this difficulty by following the arguments used in [11] and [16], to an associated problem: $-\Delta u = \lambda \tilde{f}(u);\ \Omega, u = 0; \partial\Omega$, where $\tilde{f}(s) = f(s) - \frac{f(d)}{d} B\|V\|_\infty s$. To the best our knowledge, when $V \in L^\infty(\Omega)$, the existence and multiplicity of positive solutions of $(P_\lambda)$ have not been treated.

The text is organized as follows: In Section 2, we analyze in detail the phosphorous cycling model which is applicable to our results. In Section 3, we recall a method of sub and supersolutions for $(P_\lambda)$ and a three solution theorem for the problem $(P_\lambda)$. Section 4 is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

## 2. Example : A phosphorous cycling model

A simple model satisfying the conclusions of Theorem 1.1 and Theorem 1.2 is the problem $(P_\lambda)$ when $f(u) = \tau - u + c\frac{u^4}{1+u^4}$ with $c \gg 1$, namely,

$$\begin{cases} -\Delta u + V(x)u &=& \lambda\left(\tau - u + c\frac{u^4}{1+u^4}\right), & x \in \Omega, \\ u &=& 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

This model describes phosphorous cycling in stratified lake and the colonization of barren soils in drylands by vegetation. It also describes the colonization of barren soils in drylands by vegetation (more details in [5] and [6]). We recall some results from [5] that for large $c$ there exist some values of $\tau$ for which $f$ satisfies $(H1)$.

**Proposition 2.1.** *([5]) If $c > \frac{16}{5\sqrt[4]{135}} =: c_0$, then there exist $0 < m < M < \infty$ such that $f'(m) = f'(M) = 0$.*

**Proposition 2.2.** *([5]) If $\tau > \frac{3}{4}\sqrt[4]{\frac{3}{5}} - \frac{1}{4}\left(\sqrt[4]{\frac{3}{5}}\right)^5 =: \tau_0$, then there exists a unique $\sigma > 0$ such that $f(\sigma) = 0$.*

Hence, for $\tau > \tau_0$ and $c$ large, $f(s)$ satisfies $(H1)$ (see the shape of the graph $f$ given in Figure 2).

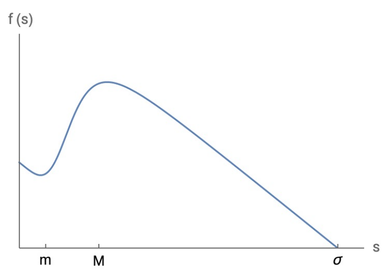


FIGURE 2. Graph of $f(s)$ satisfies $(H1)$ for $c > c_0$ and $\tau > \tau_0$.

**Proposition 2.3.** *([5]) If $\tau < \frac{9}{16}c$, then $\frac{s}{f(s)}$ has the shape given in Figure 3.*

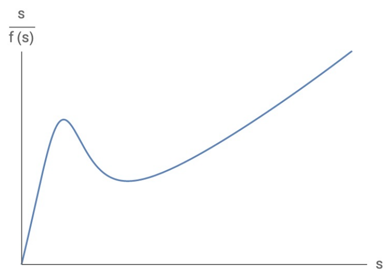


FIGURE 3. Graph of $s/f(s)$ has a decreasing interval.

Now for each $c > c_0$ we let $\tau \in (\tau_0, \frac{9c}{16})$ be arbitrary fixed and choose $\alpha \in (0, M)$ such that $f(\alpha) = f^*(\alpha) = f(0)$ (see Figure 4). We recall again the following additional results from [5].

**Proposition 2.4.** *([5]) For $c \gg 1$,*

(1) $\sqrt[5]{c} < M < \sqrt[4]{c}$
(2) $\frac{1}{\sqrt[3]{c}} < \alpha < \frac{1}{\sqrt[4]{c}}$.

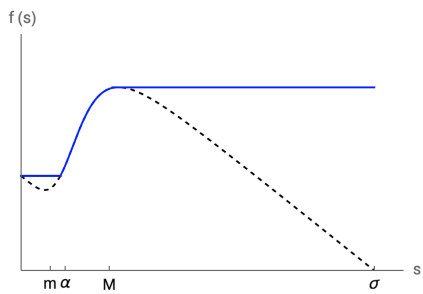


FIGURE 4. Graph of $f^*(s) := \max_{t\in[0,s]} f(t)$

Here, we verify that $f(u) = \tau - u + c\frac{u^4}{1+u^4}$ satisfies the conclusions of Theorem 1.2, which means that there exist the values of $c$ and $\tau$ satisfying all assumptions in Theorem 1.2. First, let us take $d = \sqrt[8]{c}$. Observing the calculation

$$\begin{aligned}\tilde{f}'(s) = f'(s) - \frac{f(\sqrt[8]{c})}{\sqrt[8]{c}}B\|V\|_\infty &= -1 + c\frac{4s^3}{(1+s^4)^2} - \frac{\tau - \sqrt[8]{c} + \frac{c\sqrt{c}}{1+\sqrt{c}}}{\sqrt[8]{c}}B\|V\|_\infty \\ &= c\left[\frac{4s^3}{(1+s^4)^2} - \frac{1}{c} - \frac{\tau - \sqrt[8]{c} + \frac{c\sqrt{c}}{1+\sqrt{c}}}{c\sqrt[8]{c}}B\|V\|_\infty\right],\end{aligned}$$

it is easy to see that $\tilde{f}'(s) \to \infty$ for all $s \in (0, \infty)$ as $c \to \infty$. Hence, there exists $c_1 > 0$ large enough such that $\tilde{f}'(s) > 0$ in $[\frac{1}{\sqrt[4]{c}}, \sqrt[5]{c}]$ for all $c \geq c_1$. Further, since

$$\tilde{Q}\left(\frac{1}{\sqrt[4]{c}}, \sqrt[8]{c}\right) := \frac{\frac{1}{\sqrt[4]{c}}}{f(\frac{1}{\sqrt[4]{c}})} \Big/ \frac{\sqrt[8]{c}}{f(\sqrt[8]{c})} = \frac{\tau c^{-\frac{3}{8}} - c^{-\frac{1}{4}} + \frac{c^{\frac{9}{8}}}{1+\sqrt{c}}}{\tau - \frac{1}{\sqrt[4]{c}} + \frac{c}{1+c}} \to \infty \text{ as } c \to \infty,$$

there exists $c_2 > 0$ sufficiently large such that $\tilde{Q}(\frac{1}{\sqrt[4]{c}}, \sqrt[8]{c}) > \frac{K\|e\|_\infty}{B}$ for all $c > c_2$. Since $\alpha < \frac{1}{\sqrt[4]{c}}$ by Proposition 2.4, we have $f^*(\frac{1}{\sqrt[4]{c}}) = f(\frac{1}{\sqrt[4]{c}})$, which implies that

$$\frac{\sqrt[8]{c}}{f(\sqrt[8]{c})} \frac{1}{B} \Big/ \frac{\frac{1}{\sqrt[4]{c}}}{f^*(\frac{1}{\sqrt[4]{c}})} \frac{1}{K\|e\|_\infty} < 1 \tag{5}$$

for all $c > c_2$. It also follows that there exists $c_3 > 0$ such that $\sqrt[8]{c} < \frac{2}{A}\sqrt[5]{c}$ for all $c > c_3$, and we obtain

$$\frac{\sqrt[8]{c}}{f(\sqrt[8]{c})} \frac{1}{B} \Big/ \frac{2\sqrt[5]{c}}{f(\sqrt[8]{c})AB} < 1 \tag{6}$$

for all $c > c_3$. Hence, by (5) and (6), we show that $Q(\frac{1}{\sqrt[4]{c}}, \sqrt[8]{c}, \sqrt[5]{c}) < 1$ for all $c \geq \max\{c_2, c_3\}$. Now, letting $c^* = \max\{c_1, c_2, c_3\}$ and choosing

$$a = \frac{1}{\sqrt[4]{c^*}}, \; d = \sqrt[8]{c^*} \text{ and } b = \sqrt[5]{c^*},$$

we have $\tilde{f}'(s) \geq 0$ in $[a, b]$, $Q(a, d, b) < 1$ and $\tilde{f}(s) > 0$ in $[0, b]$ for a sufficiently small value of $\|V\|_\infty$.

## 3. Preliminaries

In this section, we define a subsolution and supersoluton of $(P_\lambda)$ and recall the method of obtaining sub and supersolutions and three solution theorem for $(P_\lambda)$.

A subsolution of $(P_\lambda)$ is defined as a function $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta\psi + V(x)\psi \leq \lambda f(\psi), & x \in \Omega, \\ \psi \leq 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

while a supersolution of $(P_\lambda)$ is defined as a function $Z \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta Z + V(x)Z \geq \lambda f(Z), & x \in \Omega, \\ Z \geq 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

Now we introduce the theorem of sub and supersolution and three solution theorem.

**Lemma 3.1.** *(Theorem for sub and supersolution in* [1]*). If a subsolution $\psi$ and a supersolution $Z$ of $(P_\lambda)$ exist such that $\psi \leq Z$ on $\Omega$, then $(P_\lambda)$ has at least one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \leq u \leq Z$ on $\Omega$.*

**Lemma 3.2.** *(Three solution Theorem in* [1] *and* [17]*). Suppose there exists two pairs of ordered sub and supersolutions* $(\psi_1, Z_1)$ *and* $(\psi_2, Z_2)$ *of* $(P_\lambda)$ *with the property that* $\psi_1 \le \psi_2 \le Z_1$, $\psi_1 \le Z_2 \le Z_1$ *and* $\psi_2 \not\le Z_2$*. Additionally assume that* $\psi_2, Z_2$ *are not solutions of* $(P_\lambda)$*. Then there exists at least three solutions* $u_i, i = 1, 2, 3$ *for* $(P_\lambda)$ *where* $u_1 \in [\psi_1, Z_2], u_2 \in [\psi_2, Z_1]$ *and* $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$*.*

*Remark* 3. Note that the set $[\psi, Z]$ is defined by

$$[\psi, Z] := \{u \in C^2(\Omega) \cap C(\bar{\Omega}) : \psi(x) \le u(x) \le Z(x) \text{ for all } \ x \in \Omega\}.$$

We emphasize that the set $[\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$ is not empty if two pairs of ordered sub and supersolutions $(\psi_1, Z_1)$ and $(\psi_2, Z_2)$ satisfying $\psi_1 \le \psi_2 \le Z_1$, $\psi_1 \le Z_2 \le Z_1$ and $\psi_2 \not\le Z_2$ as in in Lemma 3.2.

**Lemma 3.3.** *Assume* $(H2)$*. Then the problem*

$$\begin{cases} -\Delta w + V(x)w = 1 \ \textit{in} \ \Omega, \\ w = 0 \ \textit{on} \ \partial\Omega \end{cases} \tag{W}$$

*has a solution* $w$ *such that* $w(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial w}{\partial \eta} < 0$ *on* $\partial\Omega$ *where* $\eta$ *is the outward normal vector on* $\partial\Omega$*.*

*Proof.* Let $Z = Ke$. Then from $(H2)$ we have in $\Omega$

$$-\Delta Z + V(x)Z = K(-\Delta e + V(x)e) = K(1 + V(x)e) \ge K(1 - c_1 e) \ge K(1 - c_1\|e\|_\infty) \ge 1,$$

where $K$ is the constant in (3) and $Z = 0$ on $\partial\Omega$. Hence, $Z$ is a supersolution of $(W)$. Since $\psi \equiv 0$ is a subsolution but not a solution of $(W)$, there exists a nontrival solution $w$ of $(W)$ such that $0 \le w(x) \le Z(x)$ by Lemma 3.1.

Next, we claim that $w(x) \ge \bar{e}(x)$ where $\bar{e}$ is the positive solution of

$$\begin{cases} -\Delta \bar{e} + \|V\|_\infty \bar{e} = 1 \text{ in } \Omega, \\ \bar{e} = 0 \text{ on } \partial\Omega. \end{cases}$$

Notice that $\bar{e} > 0$ in $\Omega$ and $\frac{\partial \bar{e}}{\partial \eta} < 0$ on $\partial\Omega$. Let $\tilde{\Omega} := \{x \in \Omega | w(x) < \bar{e}(x)\}$ and suppose $\tilde{\Omega} \not\equiv \emptyset$. Then we find

$$-\Delta(w - \bar{e}) = 1 - V(x)w + \|V\|\bar{e} - 1 = (\|V\|_\infty - V(x))w + \|V\|_\infty(\bar{e} - w) > 0$$

in $\tilde{\Omega}$ and $w - \bar{e} = 0$ on $\partial\tilde{\Omega}$. By the Maximum principle, we know $w - \bar{e} \ge 0$ on $\tilde{\Omega}$, which is a contradiction. Hence, $\tilde{\Omega} = \emptyset$. Since $w \ge \bar{e}$ in $\Omega$, $w > 0$ in $\Omega$ and $\frac{\partial w}{\partial \eta} < 0$ on $\partial\Omega$. □

*Remark* 4. The function $w$ can be used to construct a large supersolution $Z_1$ of $(P_\lambda)$ in the proof of Theorem 1.1. Here, the property of $w$ obtained in Lemma 3.3, $w(x) > 0$ for $x \in \Omega$ and $\frac{\partial w}{\partial \eta} < 0$ on $\partial\Omega$, is necessary to construct a supersolution $Z_1$ of $(P_\lambda)$ so that $\psi_1(x) \le Z_1(x)$ for all $x \in \Omega$ for any subsolution $\psi_1$ of $(P_\lambda)$.

## 4. Proof of Main Theorems

### 4.1. Proof of Theorem 1.1

*Proof.* We first construct a positive subsolution $\psi_1$ of $(P_\lambda)$ for each $\lambda > 0$. Let $\lambda_1$ be the principal eigenvalue of $-\Delta + V(x)$ with Dirichlet boundary condition and $\phi_1$ be a corresponding eigenfunction. Hence $\phi_1$ and $\lambda_1$ satisfy:

$$\begin{cases} -\Delta\phi_1 + V(x)\phi_1 = \lambda_1\phi_1, \text{ in } \Omega \\ \phi_1 = 0, \text{ on } \partial\Omega. \end{cases}$$

It is known that $\phi_1 > 0$ in $\Omega$ and $\frac{\partial\phi_1}{\partial\eta} < 0$ on $\partial\Omega$ (see [7], [9] and [14]). From $(H1)$ there exists $\epsilon_\lambda > 0$ sufficiently small such that

$$\lambda_1\epsilon_\lambda\phi_1 \le \lambda f(\epsilon_\lambda\phi_1).$$

Let $\psi_1 = \epsilon_\lambda\phi_1$. Then we have

$$-\Delta\psi_1 + V(x)\psi_1 = \epsilon_\lambda\lambda_1\phi_1 \le \lambda f(\epsilon_\lambda\phi_1) \text{ in } \Omega$$

and $\psi_1 = 0$ on $\partial\Omega$. Hence, $\psi_1$ is a positive solution of $(P_\lambda)$ for all $\lambda > 0$. From $(H1)$, for each $\lambda > 0$ there exists $M_\lambda > 0$ such that $\lambda f(s) \le M_\lambda$ for all $s \in [0, \infty)$. Now we define $Z_1 = M_\lambda w$. It follows that

$$-\Delta Z_1 + V(x)Z_1 = M_\lambda \ge \lambda f(M_\lambda w) = \lambda f(Z_1) \text{ in } \Omega$$

and $Z_1 = 0$ on $\partial\Omega$, which means that $Z_1$ is a positive supersolution of $(P_\lambda)$ for each $\lambda > 0$. Since $\psi_1$ is a subsolution which is not a solution of $(P_\lambda)$ and $\psi_1 \le Z_1$ in $\Omega$, there exists a positive solution $u_\lambda$ of $(P_\lambda)$ such that $\psi_1 \le u_\lambda \le Z_1$ for each $\lambda > 0$ based on Lemma 3.1. □

### 4.2. Proof of Theorem 1.2

*Proof.* We construct a supersolution for $\lambda < \lambda^* = \min\left\{\frac{a}{f^*(a)}\frac{1}{K\|e\|_\infty}, \frac{2b}{f(d)AB}\right\}$. First we notice that $f(s) \le f^*(s)$ for all $s \ge 0$ and $f^*$ is nondecreasing in $[0, .\infty)$ from the definition of $f^*(s)$. Let $Z_2 = a\frac{e}{\|e\|_\infty}$. Then we evaluate

$$-\Delta Z_2 + V(x)Z_2 = \frac{a}{\|e\|_\infty}(-\Delta e + V(x)e) = \frac{a}{\|e\|_\infty}(1 + V(x)e) > \lambda K f^*(a)(1 + V(x)e)$$

where we used the definition of $e$ in $(H2)$ and the fact $\lambda < \frac{a}{f^*(a)K\|e\|_\infty}$. Now by the condition (3) at the last inequality, we have

$$\begin{aligned} -\Delta Z_2 + V(x)Z_2 &> \lambda K f^*(a)(1 + V(x)e) \\ &\ge \lambda K f^*(a\frac{e}{\|e\|_\infty})(1 - c_V\|e\|_\infty) \ge \lambda f(Z_2) \text{ in } \ \Omega. \end{aligned}$$

Clearly, $Z_2 = 0$ on $\partial\Omega$. Hence, $Z_2$ is supersolution for $\lambda < \lambda^*$.

Now we construct a positive subsolution $\psi_2$ of the following problem

$$\begin{cases} -\Delta u + \|V\|_\infty u = \lambda f(u), \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{9}$$

when $\lambda > \lambda_*$. Then, $\psi_2$ is a subsolution of $(P_\lambda)$ since

$$-\Delta\psi_2 + V(x)\psi_2 \le -\Delta\psi_2 + \|V\|_\infty\psi_2 \le \lambda f(\psi_2).$$

We recall $\tilde{f}(u) = \frac{f(u)}{u^\beta} - \frac{f(d)}{d^\beta}B\|V\|_\infty u$ and note that $\tilde{f}(u)$ is nondecreasing on $[a,b]$. Let $a^* \in (0,a]$ be such that $\tilde{f}(a^*) = \min_{0<x\le a}\tilde{f}(x)$ and define $g \in C([0,\infty))$ such that

$$g(u) = \begin{cases} \tilde{f}(a^*), & u \le a^*, \\ \tilde{f}(u), & u \ge a, \end{cases}$$

so that $g$ is nondecreasing on $(0,a]$ and $g(u) \le \tilde{f}(u)$ for all $u \ge 0$. Consider the following problem:

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

For $0 < \epsilon < R$ and $\delta, \mu > 1$, let us define $\rho : [0,R] \to [0,1]$ by

$$\rho(r) = \begin{cases} 1, \ 0 \le r \le \epsilon, \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^\mu)^\delta, \ \epsilon < r \le R, \end{cases}$$

where $R$ is the radius of the biggest inscribed ball in $\Omega$. Then we have

$$\rho'(r) = \begin{cases} 0, \ 0 \le r \le \epsilon, \\ -\frac{\delta\mu}{R-\epsilon}(1 - (\frac{R-r}{R-\epsilon})^\mu)^{\delta-1}(\frac{R-r}{R-\epsilon})^{\mu-1}, \ \epsilon < r \le R. \end{cases}$$

Let $v(r) = d\rho(r)$. Note that $|v'(r)| \le d\frac{\delta\mu}{R-\epsilon}$. Define $\psi$ as the radially symmetric solution of

$$\begin{cases} -\Delta\psi = \lambda g(v(|x|)), \text{ in } B_R(0), \\ \psi = 0, \text{ on } \partial B_R(0). \end{cases}$$

Then $\psi$ satisfies

$$\begin{cases} -(r^{N-1}(\psi'(r))' = \lambda r^{N-1} g(v(r)), \\ \psi'(0) = 0, \ \psi(R) = 0. \end{cases}$$

Integrating (4.2) for $0 < r < R$, we have

$$-\psi'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} g(v(s))\, ds. \tag{11}$$

Now we claim that

$$\psi(r) \ge v(r), \ \forall\ 0 \le r \le R \tag{12}$$

and

$$\|\psi\|_\infty \le b \tag{13}$$

when $\frac{d}{f(d)}\frac{1}{B} < \lambda < \frac{2b}{f(d)AB}$.

In order to prove (12), it is enough to show that

$$-\psi'(r) \ge -v'(r), \ \forall\ 0 \le r \le R \tag{14}$$

as $\psi(R) = 0 = v(R)$. Notice that for $0 \le r \le \epsilon$, $\psi'(r) \le 0 = v'(r)$. Hence, for $r > \epsilon$ we get from (11)

$$\begin{aligned} -\psi'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} g(v(s))\, ds \\ &> \frac{\lambda}{R^{N-1}} \int_0^\epsilon s^{N-1} g(v(s))\, ds \\ &= \frac{\lambda}{R^{N-1}} \frac{\epsilon^N}{N} g(d) = \frac{\lambda}{R^{N-1}} \frac{\epsilon^N}{N} \tilde{f}(d), \end{aligned}$$

where the last equality is obtained from the fact that $g(s) = \tilde{f}(s)$ for all $s \ge a$. If $\lambda > \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$, then we conclude (14). Note that

$$\inf_\epsilon \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu = \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}} \delta\mu$$

and is achieved at $\epsilon = \frac{NR}{N+1}$. Hence, if $\lambda > \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}}$, then in the definition of $\rho$ we can choose $\epsilon = \frac{NR}{N+1}$ and the values of $\delta$ and $\mu$ so that $\lambda \ge \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$, and hence (14) holds. Note that

$$\tilde{f}(d) = (1 - B\|V\|_\infty) f(d). \tag{15}$$

Hence we obtain

$$\psi(r) \ge v(r),\ \forall\ 0 \le r \le R$$

when $\lambda > \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}} = \frac{d}{f(d)} \frac{1}{B}$.

Now to show (13), we integrate (11) from $t$ to $R$, we obtain that for $0 \le r \le R$

$$\begin{aligned} \psi(t) &= \int_t^R \frac{\lambda}{r^{N-1}} \left( \int_0^r s^{N-1} g(v(s))\, ds \right) dr \\ &\le \int_t^R \frac{\lambda}{r^{N-1}} g(d) \left( \int_0^r s^{N-1}\, ds \right) dr \\ &\le \lambda \frac{g(d)}{N} \int_0^R r\, dr = \lambda \frac{\tilde{f}(d)}{2N} R^2, \end{aligned}$$

where the last equality is again obtained from the fact that $g(s) = \tilde{f}(s)$ for all $s \ge a$. Hence, if $\lambda < \frac{b}{\tilde{f}(d)} \frac{2N}{R^2} = \frac{2b}{f(d)AB}$, then we get $\|\psi\|_\infty \le b$. Hence, we find that $v(r) \le \psi(r) \le b, \forall\ 0 \le r \le R$ when $\frac{d}{f(d)} \frac{1}{B} < \lambda < \frac{2b}{f(d)AB}$. Now, since $v(r) \le \psi(r) \le b, \forall\ 0 \le r \le R$ and $g$ is nondecreasing on $[0, b]$, we have

$$-\Delta\psi = \lambda g(v) \le \lambda g(\psi), \quad \text{in } B_R(0) \text{ and } \psi = 0 \text{ on } \partial B_R(0).$$

Here we let $\psi_2(x) = \psi(x)$ if $x \in B_R(0)$ and $\psi_2(x) = 0$ if $x \in \Omega \setminus B_R(0)$. Then $\psi_2$ is a nonnegative subsolution of (10) for $\frac{d}{f(d)} \frac{1}{B} < \lambda < \frac{2b}{f(d)AB}$. Hence, we have that for $\lambda > \frac{d}{f(d)}\frac{1}{B}$

$$\begin{aligned} -\Delta\psi_2 \leq \lambda g(\psi_2) \leq \lambda \tilde{f}(\psi_2) &= \lambda\left(f(\psi_2) - \frac{f(d)}{d}B\|V\|_\infty\psi_2\right) \\ &< \lambda\left(f(\psi_2) - \frac{1}{\lambda}\|V\|_\infty\psi_2\right) \\ &= \lambda f(\psi_2) - \|V\|_\infty\psi_2, \end{aligned}$$

which implies that $\psi_2$ is a nonnegative subsolution of (9). Finally, we obtain the subsolution $\psi_2$ of $(P_\lambda)$ satisfying $\psi_2 \not\leq Z_2$ for $\lambda_* < \lambda < \lambda^*$.

From the proof of Theorem 1.1 we construct a sufficiently small subsolution $\psi_1 = \epsilon_\lambda\phi_1$ so that $\psi_1 \leq Z_2$ and a sufficiently large supersolution $Z_1 = M_\lambda w$ so that $\psi_2 \leq Z_1$. Hence, there exist a positive solutions $u_1$ and $u_2$ of $(P_\lambda)$ such that $\psi_1 \leq u_1 \leq Z_2$ and $\psi_2 \leq u_2 \leq Z_1$. Note that $u_1 \neq u_2$ as $\psi_2 \not\leq Z_2$. By three solution theorem ([1] and [17]), there exists a positive solution $u_3$ such that $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$. □

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