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WEAK MEASURE-EXPANSIVE TRANSITIVE SETS

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Abstract. In this paper, we prove that if a transitive set Λ is robustly weak measure-expansive then Λ is hyperbolic.

1. Introduction

In this paper, we assume that M is a compact smooth manifold without boundary (dim$M \geq 2$). A dffeomorphism $f : M \to M$ is *expansive* if there is a constant $\delta > 0$(called an expansive constant) such that if for any $x, y \in M$, $d(f^i(x), f^i(y)) < \delta \ \forall i \in \mathbb{Z}$ then $x = y$.

About expansivity, Morales and Sirvent [9] have introduced measure-expansive, which is a viewpoint of measure theory.

For any $\delta > 0$ and $x \in M$, we define $\Gamma(\delta, x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \in \mathbb{Z}\}$, which is called a *dynamic δ-ball.* If a diffeomorphism f is expansive then $\Gamma(\delta, x) = \{x\}$ for some $\delta > 0$. Let μ be a Borel probability measure on M. Denote by $\mathcal{M}(M)$ the set of all Borel probability measures on M endowed with weak* topology. We say that a diffeomorphism f is *measure-expansive* if there is a constant $\delta > 0$ such that $\mu(\Gamma(\delta, x)) = 0$ for all $x \in M$. If μ is non-atomic then measure-expansive is a general notion of expansive.

After that, various measure-theoretic expansivities were introduced for [1] and [2], etc. About the notions, we deal with weak measure-expansive([1]). A collection $\mathcal{P} = \{A_1, A_2, \ldots, A_n : A_i \subset M\}$ is a *finite measurable partition* of M if

(a) $A_i \cap A_j = \emptyset$ if $i \neq j$,
(b) $\bigcup_{i=0}^{n} A_i = M$,
(c) each A_i is measurable and $intA_i \neq \emptyset$ for all $i = 1, \ldots, n$.

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Moreover, by compactness for any $\delta > 0$, we can construct $\mathcal{P} = \{A_1, A_2, \dots, A_n : A_i \subset M\}$ with $\text{diam}A_i \le \delta(i = 1, \dots, n)$.

For a measure $\mu \in \mathcal{M}(M)$, a diffeomorphism $f : M \to M$ is *weak measure-expansive* if there is a finite measurable partition $\mathcal{P} = \{A_i \subset M : i = 1, \dots, n\}$ such that $\mu(\{y \in M : f^i(y) \in \mathcal{P}(f^i(x))$ for all $i \in \mathbb{Z}\}$. Here, $\mathcal{P}(x)$ is the element of $\mathcal{P}$ having x. If μ is an invariant measure on M and non-atomic, then it is a general notion of expansive and measure-expansive.

In particular, we know that various expansivities are close to hyperbolic structure(quasi-Anosov, Axiom A, etc). For example, Mañé proved in [8] that if a diffeomorphism f belongs to the set of expansive diffeomorphisms of M, then it is qausi-Anosov. Here a diffeomorphism f is *quasi-Anosov* if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

For a measure-theoretic expansivity, Sakai, Sumi and Yamamoto proved in [10] that if a diffeomorphism f belongs to the set of measure-expansive diffeomorphisms of M, then it is qausi-Anosov.

A closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|D_x f^n(v)\| \le C\lambda^n \|v\| (v \in E^s_x \backslash \{0\}) \text{ and } \|D_x f^{-n}(v)\| \le C\lambda^n (v \in E^u_x \backslash \{0\}),$$

for all $x \in \Lambda$ and $n \ge 0$.

Let $Per(f) = \{x \in M : f^k(x) = x$ for some $k \in \mathbb{Z}\}$, and $\Omega(f) = \{x \in M :$ a neighborhood U_x of x there is $n \ge 0$ such that $f^n(U_x) \cap U_x \neq \emptyset\}$. A diffeomorphism f is *Axiom A* if $\overline{Per(f)} = \Omega(f)$ is hyperbolic.

Ahn and Kim proved in [1] that if a diffeomorphism f belongs to the set of weak measure-expansive diffeomorphisms of M, then it is Axiom A and has no-cycles.

The previous results, a main research topic is related to a closed set of a diffeomorphism $f : M \to M$ and various expansivities. To prove this, we use the property of C^1 robust. That is, a closed f-invariant set $\Lambda \subset M$ is called *robustly* $\mathcal{S}$ if there are a C^1 neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$ and for any $g \in \mathcal{U}$, $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is $\mathcal{S}$, where Λ_g is the continuation of Λ. In the above, $\mathcal{S}$ is replaced by expansive, measure-expansive, continuum-wise expansive, expanding measure, and weak measure-expansive, etc.

A closed f-invariant set $\Lambda \subset M$ is called *transitive set* if there is a point $x \in \Lambda$ such that the omega limit set of x, $\omega_f(x)$, is Λ. Using the notion and a transitive set Λ, Lee and Park [7] proved that if Λ is robustly expansive then Λ is hyperbolic for f. Lee [4] proved that if Λ is robustly continuum-wise expansive then Λ is hyperbolic for f and Lee [6] proved that if Λ is robustly expanding measure then Λ is hyperbolic. The result is a motivation about this paper. The following is a main result of this paper.

THEOREM 1.1. *Let $\Lambda \subset M$ be a transitive set of f. If Λ is robustly weak measure-expansive for f, then Λ is hyperbolic for f.*

2. Proof of Theorem 1.1

Let M be as before, and let $f : M \to M$ be a diffeomorphism.

REMARK 2.1. Let $f : M \to M$ be a diffeomorphism. Suppose that f is weak measure-expansive. Then we have the following (see [1]):

(a) If f is the identity map then f is not weak measure-expansive.
(b) A f is weak measure-expansive if and only if f^n is weak measure-expansive, for any $n \in \mathbb{Z} \setminus \{0\}$.
(c) If f is weak measure-expansive then $\Lambda \subset M$ is weak measure-expansive for f.

The following lemma is called Franks' lemma [3].

LEMMA 2.2. *Let $\mathcal{U}(f)$ be a C^1 neighborhood of a diffeomorphism $f : M \to M$. Then there exist a $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that if $g \in \mathcal{U}_0(f)$, a finite set $A = \{x_1, x_2, \cdots, x_N\}$, a neighborhood W of A and $L_i (i = 1, \ldots, N)$ are linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $\widehat{g} \in \mathcal{U}(f)$ satisfying $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus W)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \le i \le N$.*

LEMMA 2.3. *Let $\Lambda \subset M$ be a closed set. If Λ is robustly weak measure-expansive then for any $g \in \mathcal{U}$, every periodic point $p \in \Lambda_g$ is hyperbolic, where $\mathcal{U}$ and U are in the notion of robustly weak measure-expansive.*

Proof. Let $\mathcal{U}$ and U be the definition of robustly weak measure-expansivity. Suppose that by contradiction there exist a diffeomorphism $g \in \mathcal{U}$ and a periodic point $p \in \Lambda_g$ is not hyperbolic. Then $D_pg^{\pi(p)}(g^{\pi(p)}(p) = p)$ has an eigenvalue λ with $|\lambda| = 1$. For simplicity, we assume that $g(p) = p$. As Lemma 2.2, we also assume that D_pg has only one eigenvalue λ with $|\lambda| = 1$. Then we have $T_pM = E^s_p \oplus E^u_p \oplus E^c_p$, where E_p^s corresponds to the eigenvalues less than 1, E_p^u to the eigenvalues greater than 1, and E_p^c to λ. Note that if $\dim E_p^c = 1$ then $\lambda \in \mathbb{R}$, and if $\dim E_p^c = 2$ then $\lambda \in \mathbb{C}$. In this Lemma, we prove the case of $\dim E_p^c = 1$ (other case is similar(see [1])). Using Lemma 2.2, there are $r > 0$ and $g_1 \in \mathcal{U}$ such that

(a) $g_1(p) = g(p) = p$,
(b) $g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$, if $x \in B(x, r) \subset U$, and
(c) $g_1(x) = g(x)$, if $x \notin B(x, 4r)$.

Take a non-zero vector $v \in E_p^c$. Then we have

$$g_1(\exp_p(v)) = \exp_p(v).$$

Let $I_v = \{\tau \|v\| : -r/4 \le \tau \le r/4\}$. From the fact, we have an closed arc $\mathcal{J}_p \subset U$ such that

(i) $\exp_p(I_v) = \mathcal{J}_p$,
(ii) $g_1(\mathcal{J}_p) = \mathcal{J}_p$ and
(iii) $g_1|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map.

Let m be the Lebesgue measure on $\mathcal{J}_p$. For a Borel set $C \subset M$, we define $\nu \in \mathcal{M}(M)$ by

$$\nu(C) = m(C \cap \mathcal{J}_p).$$

Then ν is an invariant measure. Since $\mathcal{J}_p \subset U$ and $g_1|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map, by Remark 2.1, g_1 is not weak measure-expansive. This is a contradiction. □

For a closed f-invariant set $\Lambda \subset M$, a diffeomorphism f satisfies a *local star* on Λ if there are a C^1 neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}$, every $p \in Per(f) \cap \Lambda_g (= \bigcap_{n\in\mathbb{Z}} g^n(U))$ is hyperbolic.

Lemma 2.4. *Let $\Lambda \subset M$ be a closed set. If Λ is robustly weak measure-expansive then f satisfies a local star on Λ.*

Proof. Since Λ is robustly weak measure-expansive, by Lemma 2.3, every periodic point in Λ is hyperbolic. This means that f satisfies a local star on Λ. □

End of Proof of Theorem 1.1 Since Λ is robustly weak measure-expansive, by Lemma 2.4, f satisfies a local star on Λ. As the result of [5], Λ is hyperbolic. □

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