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A CHARACTERIZATION OF MAXIMAL SURFACES IN TERMS OF THE GEODESIC CURVATURES

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Abstract. Maximal surfaces have a prominent place in the field of differential geometry, captivating researchers with their intriguing properties. Bearing a direct analogy to the minimal surfaces in Euclidean space, investigating both their similarities and differences has long been an important issue. This paper is aimed to give a local characterization of maximal surfaces in $\mathbb{L}^3$ in terms of their geodesic curvatures, which is analogous to the minimal surface case presented in [8]. We present a classification of the maximal surfaces under some simple condition on the geodesic curvatures of the parameter curves in the line of curvature coordinates.

1. Introduction

A maximal surface is a spacelike surface in the three-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ with vanishing mean curvature. Here, a spacelike surface is defined to be a surface in $\mathbb{L}^3$ whose induced metric is positive definite.

Maximal surfaces share some analogous properties with minimal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. For example, maximal surfaces locally maximize the area in the variational sense, which is why they are named that way. Additionally, maximal surfaces have a Weierstrass-Enneper type representation formula [6], which is reminiscent of its Euclidean counterpart.

Nevertheless, substantial differences between maximal and minimal surfaces emerge both locally and globally. For instance, maximal surfaces may have non-isolated singularities, which lead them to have much

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more complicated local geometry. And, a Calabi-Bernstein type theorem for maximal surfaces [3] asserts that a complete regular maximal surface must be a spacelike plane.

Given these similarities and distinctions, constructing interesting maximal surfaces or investigating their rigidities becomes a meaningful pursuit. Various approaches, such as utilizing the Weierstrass-Enneper type representation formula and the Björling formula, have been commonly employed. Presented below are notable results in this context.

Alias, Chaves and Mira [1] solved the Björing problem in $\mathbb{L}^3$ to generate new examples. Kim and Yang [7] constructed complete higher genus maximal surfaces with singularities, having two catenoidal ends. F. López, R. López and Souam [10] classified the maximal surfaces foliated by peices of circles. I. Fernández and F. López [5] demonstrated a reflection principle for maximal surfaces with respect to an isolated singularity. Umehara and Yamada [12] proved that the complete maximal surfaces with singularities satisfy the Osserman-type inequality.

This paper focuses on characterizing maximal surfaces in terms of their geodesic curvatures of lines of curvature. Specifically, if the product of the conformal factor of the metric and the geodesic curvature of each line of curvature of a non-planar umbilic-free maximal surface is a single variable function, then up to isometries and homotheties of $\mathbb{L}^3$, the surface must be a piece of one and only one of Enneper surface of first or second kind, catenoid of first or second kind, or one surface in Bonnet family.

To that end, we first list up some properties of non-planar umbilic-free maximal surfaces in the line of curvature coordinates. We then compute the geodesic curvatures of lines of curvature to see that they can be expressed by the conformal factor of the metric. Finally, we observe that if the product of the geodesic curvature of each line of curvature and the conformal factor has single-variable, then our maximal surface must have planar lines of curvature. We note that this characterization is local in nature.

2. Preliminaries and Notations

The three dimensional Lorentz-Minkowski space $\mathbb{L}^3$ is the vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle := x_1 y_1 + x_2 y_2 - x_3 y_3$$

where (x_1, x_2, x_3) and (y_1, y_2, y_3) are the canonical coordinates in $\mathbb{R}^3$.

For $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$ in $\mathbb{L}^3$ the cross product $\vec{x} \times \vec{y}$ is defined as

$$\vec{x} \times \vec{y} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2)$$

so that the relation $\langle \vec{x} \times \vec{y}, \vec{z} \rangle = \det(\vec{x}, \vec{y}, \vec{z})$ holds for any $\vec{z} \in \mathbb{L}^3$.

A nonzero vector $\vec{x} \in \mathbb{R}^3$ is called spacelike, timelike or lightlike if and only if $\langle \vec{x}, \vec{x} \rangle$ is positive, negative or zero, respectively. A spacelike surface is a surface in $\mathbb{L}^3$ such that all tangent vectors to it are everywhere spacelike.

For a spacelike surface we define the Gauss map $G : \Sigma \to \mathbb{H}^2$ to be a map which assigns to each point of the surface the unit normal vector at that point, where $\mathbb{H}^2 = \{\vec{x} \in \mathbb{L}^3 \,|\, \langle \vec{x}, \vec{x} \rangle = -1\}$ is the two sheeted hyperboloid. It is easy to check that the Gauss map of a maximal surface is conformal. This leads us to the following local representation formula for maximal surfaces which is analogous to minimal surfaces in $\mathbb{E}^3$.

Weierstrass-Enneper type representation formula. *Any conformal maximal surface $X : \Sigma \subset \mathbb{C} \to \mathbb{L}^3$ is represented as*

$$X(z) = \frac{1}{2}\, Re \int (f(1+g^2), if(1-g^2), -2fg)\, dz$$

over a simply-connected domain Σ where g is meromorphic and f and fg^2 are holomorphic.

We refer [6] for the proof and further investigation regarding the Weierstrass-Enneper type representation formula.

Lastly, for a non-planar umbilic-free maximal surface in a simply connected domain, a pair of lines of curvature at each point can be introduced as the parameter curves. See Chapter 2 in [2].

3. Main Results

The following proposition is analogous to the Euclidean case demonstrated in [8], which can be proved through a straightforward computation.

Proposition 3.1. *Let $X : \Sigma \subset \mathbb{R}^2 \to \mathbb{L}^3$ be a non-planar umbilic-free maximal immersion of a simply connected domain Σ. Then, there always exists a change of variables such that in the new parameter $z = u + iv \in \mathbb{C}$ we have $\langle X_{zz}, \vec{N} \rangle = -\frac{1}{2}$ for the timelike unit normal vector field $\vec{N} : \Sigma \to \mathbb{H}^2$ of X. Furthermore, the new coordinates*

(u, v), which will be called the conformal lines of curvature coordinates, have the following properties.

(i) Coordinate curves are lines of curvatures.
(ii) The Gauss-Weingarten equations are

$$(3.1)\qquad \begin{cases} X_{uu} = \frac{\rho_u}{\rho} X_u - \frac{\rho_v}{\rho} X_v + \vec{N}, \\ X_{uv} = \frac{\rho_v}{\rho} X_u + \frac{\rho_u}{\rho} X_v, \\ X_{vv} = -\frac{\rho_u}{\rho} X_u + \frac{\rho_v}{\rho} X_v - \vec{N}, \\ \vec{N}_u = \frac{1}{\rho^2} X_u, \\ \vec{N}_v = -\frac{1}{\rho^2} X_v. \end{cases}$$

Let Σ be a non-planar umbilic-free maximal surface in the conformal lines of curvature coordinates (u, v). Then, Proposition 3.1 tells us that the first and second fundamental form of Σ become $I = \rho^2(\mathrm{d}u^2 + \mathrm{d}v^2)$ and $II = -\mathrm{d}u^2 + \mathrm{d}v^2$, respectively.

Now let us proceed to show that, at each point of Σ, the geodesic curvatures of lines of curvature–i.e., the coordinate curves in our setting– are expressed by the conformal factor of the metric.

Lemma 3.2 (Geodesic curvatures of lines of curvature). *Let $X : \Sigma \to \mathbb{L}^3$ be a non-planar umbilic-free spacelike maximal surface in with the conformal lines of curvature coordinates (u, v). Let l_1 and l_2 be the u-parameter curve and the v-parameter curve, respectively. Then the geodesic curvature $(\kappa_g)_{l_i}$ of l_i $(i = 1, 2)$ and the conformal factor ρ^2 have the following relation.*

$$(\kappa_g)_{l_1} = -\frac{\rho_v}{\rho^2} \text{ and } (\kappa_g)_{l_2} = \frac{\rho_u}{\rho^2}$$

Proof. Let $C(t)$ be a curve on $X(u, v)$ in the neighborhood of a non-umbilic point $p = X(u_0, v_0)$ and let ϕ be the angle between $C'(t)$ and the velocity vector $X_u(u, v_0)$ of the u-parameter curve at p. Here we consider that ϕ is independent on t. In other words, $C(t) = X(t\cos\phi, t\sin\phi)$. Denote by s the arclength parameter. From

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\cos\phi\, X_u + \sin\phi\, X_v| = \rho$$

we have

$$\frac{\mathrm{d}C}{\mathrm{d}s} = \rho^{-1} C'(t) = \rho^{-1}\left(\cos\phi\, X_u + \sin\phi\, X_v\right).$$

Consequently,

$$
\begin{aligned}
\frac{\mathrm{d}^2C}{\mathrm{d}s^2} &= \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho^{-1}\left(\cos\phi\, X_u + \sin\phi\, X_v\right)\big)\\
&= -\rho^{-3}\big(\left(\rho_u\cos\phi + \rho_v\sin\phi\right)\left(\cos\phi\, X_u + \sin\phi\, X_v\right)\\
&\qquad\qquad + \rho^{-2}\left(\cos^2\phi\, X_{uu} + 2\cos\phi\sin\phi\, X_{uv} + \sin^2\phi\, X_{vv}\right)\\
&= \rho^{-3}\Big((\rho_v\cos\phi\sin\phi - \rho_u\sin^2\phi)X_u + (\rho_u\cos\phi\sin\phi - \rho_v\cos^2\phi)X_v\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \rho(\cos^2\phi - \sin^2\phi)\vec{N}\Big)
\end{aligned}
$$

by Proposition 3.1(ii). Hence, the tangential component of $\frac{\mathrm{d}^2C}{\mathrm{d}s^2}$ becomes

$$
\begin{aligned}
\frac{\mathrm{d}^2C}{\mathrm{d}s^2}\bigg|_{T\Sigma} &= \rho^{-3}(\sin\phi\,\rho_u - \cos\phi\,\rho_v)(-\sin\phi\, X_u + \cos\phi\, X_v)\\
&= \rho^{-2}\left(\sin\phi\,\rho_u - \cos\phi\,\rho_v\right)\,\vec{N}\times C'(s).
\end{aligned}
$$

Thus, $\kappa_g = \rho^{-2}(\sin\phi\,\rho_u - \cos\phi\,\rho_v)$. Therefore, the geodesic curvatures of parameter curves satisfy

$$
(\kappa_g)_{l_1} = \kappa_g\,\big|_{\phi=0} = -\frac{\rho_v}{\rho^2},\ (\kappa_g)_{l_2} = \kappa_g\,\big|_{\phi=\pi/2} = \frac{\rho_u}{\rho^2}.
$$

□

THEOREM 3.3. *Let Σ be a non-planar umbilic-free maximal surface in $\mathbb{L}^3$ with $ds^2 = \rho^2(du^2 + dv^2)$. Let $(\kappa_g)_{l_1}$ and $(\kappa_g)_{l_2}$ be the geodesic curvatures of lines of curvature l_1 and l_2 at each point of Σ. If $\rho^2(\kappa_g)_{l_1}$ is a function of v-variable and $\rho^2(\kappa_g)_{l_2}$ is a function of u-variable only, then, up to isometries and homotheties of $\mathbb{L}^3$, Σ must be a piece of one and only one of the following surfaces:*
- *Enneper surface of first or second kind,*
- *catenoid of first or second kind,*
- *one surface in Bonnet family.*

Proof. Since $\rho^2(\kappa_g)_{l_1}$ is a function of v and $\rho^2(\kappa_g)_{l_2}$ is a function of u, Lemma 3.2 implies that $\rho_{uv} = \rho_{vu} = 0$. Using (3.1), we compute $X_u\times X_{uu}$ and X_{uuu} to derive that $\rho_{uv} = 0$ is equivalent to $\det(X_u, X_{uu}, X_{uuu}) =$

0 as follows.

$$
\begin{aligned}
X_u \times X_{uu} =& X_u \times \left(\frac{\rho_u}{\rho} X_u - \frac{\rho_v}{\rho} X_v + \vec{N} \right) = -\frac{\rho_v}{\rho} X_u \times X_v + X_v, \\
X_{uuu} =& \frac{\rho_{uu}\rho - \rho_u^2}{\rho^2} X_u + \frac{\rho_u}{\rho} X_{uu} - \frac{\rho_{vu}\rho - \rho_v\rho_u}{\rho^2} X_v - \frac{\rho_v}{\rho} X_{vu} + \vec{N}_u \\
=& \frac{\rho_{uu}\rho - \rho_u^2}{\rho^2} X_u + \frac{\rho_u}{\rho} \left(\frac{\rho_u}{\rho} X_u - \frac{\rho_v}{\rho} X_v + \vec{N} \right) - \frac{\rho_{vu}\rho - \rho_v\rho_u}{\rho^2} X_v \\
& - \frac{\rho_v}{\rho} \left(\frac{\rho_v}{\rho} X_u + \frac{\rho_u}{\rho} X_v \right) + \vec{N}_u \\
=& \frac{1}{\rho^2}(\rho_{uu}\rho - \rho_v^2 + 1) X_u - \frac{1}{\rho^2}(\rho_{uv}\rho + \rho_u\rho_v) X_v + \frac{\rho_u}{\rho}\vec{N}.
\end{aligned}
$$

Since $\vec{N}$ is timelike,

$$
\begin{aligned}
\det(X_u, X_{uu}, X_{uuu}) &= \langle X_u \times X_{uu}, X_{uuu} \rangle = \langle -\frac{\rho_v}{\rho} X_u \times X_v + X_v, X_{uuu} \rangle \\
&= -\frac{\rho_v\rho_u}{\rho^2} \langle X_u \times X_v, \vec{N} \rangle - \frac{1}{\rho^2}(-\rho_{uv}\rho + \rho_u\rho_v)\langle X_v, X_v \rangle \\
&= \rho_{uv}\rho.
\end{aligned}
$$

This implies that $\rho_{uv} = 0$ if and only if $\det(X_u, X_{uu}, X_{uuu}) = 0$. In the same manner, it is straightforward to prove that $\rho_{uv} = 0$ if and only if $\det(X_v, X_{vv}, X_{vvv}) = 0$. Therefore, u-parameter curves and v-parameter curves are plane curves. Since the parameter curves of Σ are lines of curvature, the classification result of the maximal surfaces with planar lines of curvature obtained in [9] leads us to conclude that our non-planar maximal surface $\Sigma \subset \mathbb{L}^3$ must be a piece of one and only one of Enneper surface of first or second kind, Catenoid of first or second kind, or one surface in Bonnet family. □

Remark 3.4. Concerning the last part of our proof, we note that the maximal surfaces with planar lines of curvature, in fact, have not been fully classified until Leite [9] proposed a method analogous to Nitsche's approach about the classification of the minimal surfaces with planar lines of curvature in the Euclidean space $\mathbb{E}^3$. Namely, Nitsche proved that a connected orientable minimal surface with planar lines of curvature in $\mathbb{E}^3$ must be a piece of one and only one of plane, Enneper surface, catenoid or one surface in the Bonnet family [11]. The main ingredient of his proof was to check that the planar lines of curvature are transformed by the Gauss map of the given minimal surface into the orthogonal families of circles on $\mathbb{S}^2$. Leite showed that this approach can be applicable to the maximal surface case to give an anaologous classification.

And, an alternative method was developed in [4] to reach the same classification result. The authors first solved a system of partial differential equations for the metric function and found some axial directions to recover the Weierstrass data and parametrizations of maximal surfaces with planar lines of curvature. More specifically, they solved the equation $\rho_{uv} = 0$ together with the Gauss equation, deriving that $\rho_u = f(u)$ and $\rho_v = g(v)$ for some single valued functions $f(u)$ and $g(v)$. And then they recover $f(u)$ and $g(v)$ to get the classification and singularities of the surface. Our theorem implies that such $f(u)$ and $g(v)$ are actually the geodesic curvatures of lines of curvature multiplied by the conformal factor. In fact, this property remains valid in the Euclidean case [8]. In this respect, the geodesic curvatures of lines of curvature can be a key factor to classify the surfaces with planar lines of curvature.

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